Radiation by a Time-Dependent Current Loop

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1 Problem

Discuss the electromagnetic radiation by a time-dependent current loop that is electrically neutral. The loop is not necessarily circular, but it has a characteristic length scale a (which can be taken as the radius in case of a circular loop). The medium surrounding the loop can be regarded as vacuum.

Consider the particular examples:

1. The current I_0 is independent of time.
2. The current rises linearly with time, but changes very little during time interval a/c, where c is the speed of light in vacuum.
3. The current is zero until time $t = 0$, after which it rises linearly to I_0 at time $t_0 \gg a/c$, after which it is again constant in time.
4. The current is zero except for the interval $[-t_0, t_0]$ during which it has a triangular waveform with peak current I_0 at time $t = 0$.
5. The current is periodic with period $t_0 \gg a/c$ with a triangular waveform.
6. The current is constant for $t < 0$, and decays exponentially for $t > 0$.
7. The current is periodic with period $t_0 \gg a/c$ with a sinusoidal waveform.

2 Solution

In this note I will use the term electromagnetic "radiation" to mean the Poynting vector [1],

$$\mathbf{S} = \frac{c}{4\pi}\mathbf{E} \times \mathbf{B}, \tag{1}$$

(in Gaussian units and in vacuum) that is associated with electric and magnetic fields $\mathbf{E}$ and $\mathbf{B}$, as discussed in [2]. If the power,

$$P(r,t) = \int_{\text{large sphere of radius } r} \mathbf{S}(\mathbf{x},t) \cdot d\mathbf{Area}, \tag{2}$$

intercepted at time t by a large sphere of radius r that surrounds the source has a term that is independent of the radius, then I say that the radiation includes "radiation to infinity".[1] For there to be "radiation to infinity", the Poynting vector must have a term that varies as

[1] Many people simply use the term "radiation" to mean what I call "radiation to infinity".

$1/r^2$, and the electric and magnetic fields must have terms that vary as $1/r$, for large r. In this case I will write,

$$\mathbf{E} = \mathbf{E}_{1/r} + \mathbf{E}_{\rm other}\,, \qquad \mathbf{B} = \mathbf{B}_{1/r} + \mathbf{B}_{\rm other}\,, \tag{3}$$

and,

$$\mathbf{S} = \mathbf{S}_{1/r^2} + \mathbf{S}_{\rm other}\,, \qquad \text{where} \qquad \mathbf{S}_{1/r^2} = \frac{c}{4\pi}\mathbf{E}_{1/r} \times \mathbf{B}_{1/r}\,. \tag{4}$$

The presence of the "other" terms in the fields $\mathbf{E}$, $\mathbf{B}$ and $\mathbf{S}$ means that not all energy radiated by sources is "radiated to infinity". Some of the energy radiated by the sources can become stored in the electromagnetic fields, and in general this stored energy is later transfered back to (absorbed by) the sources/sinks, or later becomes "radiation to infinity" (as illustrated in secs. 2.4-7). The latter process is a kind of classical vacuum laser (light amplification by stimulated emission of radiation) effect!

2.1 The Current I_0 Is Independent of Time

If the current is time-independent, and the electric charge density is everywhere zero, then there is no electric field, no Poynting vector and no radiation.

However, the circulating current in the loop involves charges that are accelerating (centripetal acceleration).

The lesson is that **not all accelerating charges are associated with radiation**.[2]

Of course, if a single electric charge were constrained to move around the loop at constant velocity, then there would be an electric field, a nonzero Poynting vector, and hence radiation, including "radiation to infinity".

But, when a collection of moving charges constitutes a steady current, there is no "radiation to infinity" (even if there the charge density ρ is nonzero such that the electric field and the Poynting vector are nonzero and radiation in the broad sense is present).

The superposition/interference of the $\mathbf{E}$ and $\mathbf{B}$ fields of the moving charges in a steady current distribution leads to a cancelation of the $1/r$ components of these fields.

To see this in more detail, consider the relations, given by Panofsky and Phillips [5] (and further popularized by Jefimenko [6, 7]), between observed fields and source charge and current distributions ρ and $\mathbf{J}$,

$$\begin{aligned}\mathbf{E}(\mathbf{x},t) &= \int \frac{[\rho]\,\hat{\mathbf{R}}}{R^2}\,d\mathbf{x}' + \frac{1}{c}\int \frac{[\dot\rho]\,\hat{\mathbf{R}}}{R}\,d\mathbf{x}' - \frac{1}{c^2}\int \frac{[\dot{\mathbf{J}}]}{R}\,d\mathbf{x}' &(5)\\ &= \int \frac{[\rho]\,\hat{\mathbf{R}}}{R^2}\,d\mathbf{x}' + \frac{1}{c}\int \frac{([\mathbf{J}]\cdot\hat{\mathbf{R}})\hat{\mathbf{R}} + ([\mathbf{J}]\times\hat{\mathbf{R}})\times\hat{\mathbf{n}}}{R^2}\,d\mathbf{x}' + \frac{1}{c^2}\int \frac{([\dot{\mathbf{J}}]\times\hat{\mathbf{n}})\times\hat{\mathbf{R}}}{R}\,d\mathbf{x}' &(6)\end{aligned}$$

where $\dot{\mathbf{J}} = \partial\mathbf{J}/\partial t$, $\hat{\mathbf{R}} = \mathbf{R}/R = (\mathbf{x} - \mathbf{x}')/\left|\mathbf{x} - \mathbf{x}'\right|$, quantities inside the brackets [ ] are evaluated at the retarded time $t' = t - R/c$, and,

$$\mathbf{B}(\mathbf{x},t) = \frac{1}{c}\int \frac{[\mathbf{J}]\times\hat{\mathbf{R}}}{R^2}\,d\mathbf{x}' + \frac{1}{c^2}\int \frac{[\dot{\mathbf{J}}]\times\hat{\mathbf{R}}}{R}\,d\mathbf{x}'. \tag{7}$$

[2]See, for example, [3] for discussion of the case of N electrons moving uniformly around a circle. For a review of the topic of "nonradiating sources", see [4].

In general (for an exception see sec. 2.2), the fields that fall off as $1/r$ at large distance from their sources are given by,

$$\mathbf{E}_{1/r}(\mathbf{x},t) = \frac{1}{c^2}\int \frac{([\dot{\mathbf{J}}]\times\hat{\mathbf{n}})\times\hat{\mathbf{R}}}{R}\,d\mathbf{x}', \qquad \mathbf{B}_{1/r}(\mathbf{x},t) = \frac{1}{c^2}\int \frac{[\dot{\mathbf{J}}]\times\hat{\mathbf{R}}}{R}\,d\mathbf{x}'. \tag{8}$$

For steady currents, $\dot{\mathbf{J}} = 0$, the $1/r$ electric and magnetic fields vanish whether or not the (steady) charge density ρ is zero. The Poynting vector falls off as $1/r^4$ at large r, and we say that there is no "radiation to infinity" by a steady current.

2.2 The Current Rises Linearly with Time

We now consider the case that the source current density $\mathbf{J}$ obeys,

$$\mathbf{J}(\mathbf{x}, t_{\rm source}) = \mathbf{J}_0(\mathbf{x}) + \dot{\mathbf{J}}_0(\mathbf{x})t_{\rm source}\,, \tag{9}$$

for some time interval $[t_1, t_2]$, where $\mathbf{J}_0(\mathbf{x})$ and $\dot{\mathbf{J}}_0(\mathbf{x})$ are independent of time. Then, for an observer at $\mathbf{x}$ and time t such that the retarded time $t' = t - R/c$ falls within the interval $[t_1, t_2]$ everywhere in the source distribution,

$$\left[\dot{\mathbf{J}}\right] = \dot{\mathbf{J}}_0, \qquad \text{and} \qquad [\mathbf{J}] = \mathbf{J}_0 + \dot{\mathbf{J}}_0(t - R/c), \tag{10}$$

and the observed magnetic field (7) is,

$$\mathbf{B}(\mathbf{x},t) = \frac{1}{c}\int \frac{(\mathbf{J}_0 + \dot{\mathbf{J}}_0 t)\times\hat{\mathbf{R}}}{R^2}\,d\mathbf{x}'. \tag{11}$$

In the special case of a linearly rising current, the $1/r$ magnetic field is zero even though $\dot{\mathbf{J}}$ is nonzero. According to eq. (6) the $1/r$ electric field is nonzero, so the Poynting vector falls off as $1/r^3$ at large r, and we again say that there is no "radiation to infinity".

The charges that make up the linearly rising current are accelerating along the direction of the current, as well as towards the local center of curvature, and yet there still is no "radiation to infinity".

The magnetic field (11) has the same value as that given by the instantaneous Biot-Savart law, supposing that the linear rise in the current continues from the retarded time t' up to the present time t. In general, the current does not remain linear in t for large time intervals, and the result (11) should not be interpreted as implying that the magnetic field of a linearly rising current propagates instantaneously.

However, we should note that if during a more general time variation the current density had linear time dependence at some time(s) $t_{\rm source}$, then the sources would not emit any energy at these times which propagates to "infinity". A distant observer finds no "radiation to infinity" at times $t_{\rm observer} = t_{\rm source} + R/c$ such that the source current density varied linearly at time $t_{\rm source}$.[3]

[3]However, in general "radiation to inifinity" occurs at times close to the interval over which the current is observed to be rising linearly. See, for example, [8], and sec. 2.5 below.

2.3 The Current Has Nonlinear Time Dependence

When the current in the loop has nonlinear time dependence both the $1/r$ electric and magnetic fields (8) are nonzero, and there is nonzero "radiation to infinity". The latter can be characterized in terms of multipole moments of the source distributions, and the leading term for current loops is magnetic dipole radiation.[4] As discussed in sec. 71 of [9], the $1/r$ electric and magnetic fields at the observer can be written in terms of the source magnetic moment,

$$\mathbf{m}(t) = \frac{1}{2c}\int \mathbf{x}\times\mathbf{J}(\mathbf{x},t)\, d^3\mathbf{x}, \tag{12}$$

as,

$$\mathbf{E}_{1/r}(\mathbf{x},t) = \frac{\hat{\mathbf{r}}\times[\ddot{\mathbf{m}}]}{c^2 r}, \qquad \mathbf{B}_{1/r}(\mathbf{x},t) = \hat{\mathbf{r}}\times\mathbf{E}_{1/r}, \tag{13}$$

where $r = |\mathbf{x}|$ and $\hat{\mathbf{r}} = \hat{\mathbf{x}}$. Then, the power intercepted on a large sphere has a term independent of its radius,

$$P = \frac{2\,|[\ddot{\mathbf{m}}]|^2}{3c^3}. \tag{14}$$

The form (14) contains the insight noted at the end of sec. 2.2, that the "radiation to infinity" vanishes at times (at the observer) for which the current density had linear time dependence at the corresponding retarded times, since $[\ddot{\mathbf{m}}] = 0$ then.[5]

In addition to the expressions (13) for the $1/r$ fields of a time-dependent magnetic dipole, we can give expressions valid at any distance r large compared to the characteristic length scale a of the current distribution by taking the dual ($\mathbf{E}_\mathbf{m} = -\mathbf{B}_\mathbf{p}$, $\mathbf{B}_\mathbf{m} = \mathbf{E}_\mathbf{p}$) of the corresponding expressions for a time-dependent electric dipole,[6] to obtain

$$\mathbf{E}(\mathbf{r},t;r\gg a) = \frac{\hat{\mathbf{r}}\times[\ddot{\mathbf{m}}]}{c^2 r} + \frac{\hat{\mathbf{r}}\times[\dot{\mathbf{m}}]}{c r^2}, \tag{15}$$

$$\mathbf{B}(\mathbf{r},t;r\gg a) = \frac{\hat{\mathbf{r}}\times(\hat{\mathbf{r}}\times[\ddot{\mathbf{m}}])}{c^2 r} + \frac{3([\dot{\mathbf{m}}]\cdot\hat{\mathbf{r}})\hat{\mathbf{r}} - [\dot{\mathbf{m}}]}{c r^2} + \frac{3([\mathbf{m}]\cdot\hat{\mathbf{r}})\hat{\mathbf{r}} - [\mathbf{m}]}{r^3}. \tag{16}$$

In the rest of this note we will consider the fields only at positions where $r \gg a$.

2.4 The Current Rises from 0 to I_0 during Time $t_0 \gg a/c$

If the current is constant in time, except during the interval $[-t_0, 0]$ when it has a linear rise, then the magnetic moment $\mathbf{m}$ has a nonzero second time derivative only at times $t = -t_0$ and 0. According to eq. (14), only at these times might radiation be emitted that will eventually flow to "infinity". We now show that this statement requires qualification.

[4] An exception is a counterwound helix [10], which has an "anapole" moment rather than a magnetic dipole moment.

[5] Heras [12] has noted that an ideal point (Hertzian) electric dipole whose moment $\mathbf{p}$ varies quadratically with time has a current density that varies linearly with time. Equation (11) holds for the magnetic field in this case, so there is no "radiation to infinity" here. This provides an exception to the relation $P = 2\,|[\ddot{\mathbf{p}}]|^2/3c^3$ for the power intercepted by a large sphere from a time-dependent electric dipole (eq. (67.8) of [9]). For further discussion of this case, see secs. 2.4-5 of [11].

[6] See, for example, sec. 2.2.3 of [13], sec. 7.1 of [14], [15] and the Appendix of [11].

At large times t the electric field is zero and the magnetic field is static at any finite radius r, with the value,

$$\mathbf{B}_0(\mathbf{r}) = \frac{3(\mathbf{m}_0 \cdot \hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{m}_0}{r^3} = \frac{m_0(2\cos\theta\,\hat{\mathbf{r}} + \sin\theta\,\hat{\boldsymbol{\theta}})}{r^3} \qquad (r \gg a,\ t > (r+a)/c), \tag{17}$$

where the static magnetic moment for a loop in the x-y plane is $\mathbf{m}_0 = m_0\,\hat{\mathbf{z}}$, and we adopt a spherical coordinate system (r, θ, ϕ) in which $\hat{\mathbf{z}} = \cos\theta\,\hat{\mathbf{r}} - \sin\theta\,\hat{\boldsymbol{\theta}}$. The magnetic energy stored outside a sphere of radius $r_0 \gg a$ at large times is,

$$U_0 = \int_{r>r_0} \frac{B_0^2}{8\pi}\, d^3\mathbf{x} = 2\pi \int_{r_0}^{\infty} r^2\, dr \int_{-1}^{1} d\cos\theta\, \frac{m_0^2(3\cos^2\theta + 1)}{8\pi r^6} = \frac{m_0^2}{3r_0^3}\,. \tag{18}$$

This stored energy is, in principle, due to contributions from the pulse at $t = -t_0$, from the linear rise of the current between times $t = -t_0$ and 0, and from the pulse at $t = 0$. The stored energy can also be calculated as the time integral over the surface of the sphere of radius r_0 of the radial component of the part of the Poynting vector,

$$\begin{aligned} \mathbf{S}_{\rm other}(r_0, t) &= \mathbf{S}(r_0, t) - \mathbf{S}_{1/r^2}(r_0, t) \\ &= \frac{c}{4\pi}\{\mathbf{E}(r_0, t) \times \mathbf{B}(r_0, t) - \mathbf{E}_{1/r}(r_0, t) \times \mathbf{B}_{1/r}(r_0, t)\}, \end{aligned} \tag{19}$$

which does not eventually flow to "infinity". That is,

$$\begin{aligned} U_0 &= \int dt \int_{\rm surface\ of\ sphere\ of\ radius\ \mathit{r}_0} \mathbf{S}_{\rm other}(r_0, t) \cdot d\mathbf{Area} \\ &= \int_{{\rm pulse\ at\ }t=-t_0} + \int_{{\rm linear\ rise\ of\ }\mathbf{m}} + \int_{{\rm pulse\ at\ }t=0} = U_1 + U_2 + U_3\,. \end{aligned} \tag{20}$$

The pulse emitted at time $t = -t_0$ arrives at the sphere of radius r_0 at time $t = r_0/c - t_0$, the fields associated with the linear rise of $\mathbf{m}$ arrive between $t = r_0/c - t_0$ and $t = r_0/c$, and the pulse emitted at $t = 0$ arrives at $t = r_0/c$.

2.4.1 Effect of the Pulse at $t_{\rm source} = -t_0$

During the pulse at time $t_{\rm source} = -t_0$, the magnetic moment $\mathbf{m}$ of the loop remains zero, while the time derivative $\dot{\mathbf{m}}$ rises from zero to the constant value,

$$\dot{\mathbf{m}} = \frac{\mathbf{m}_0}{t_0} = \int_{\rm pulse} \ddot{\mathbf{m}}\, dt, \tag{21}$$

which holds for the subsequent interval $-t_0 < t_{\rm source} < 0$. During the pulse the average value of $\dot{\mathbf{m}}$ is $\mathbf{m}_0/2t_0$. According to eq. (21), we can represent the second time derivative of the magnetic moment during the pulse as,

$$\ddot{\mathbf{m}} = \frac{\mathbf{m}_0\,\delta(t + t_0)}{t_0} = \frac{m_0\,\delta(t_{\rm source} + t_0)}{t_0}\,\hat{\mathbf{z}} = \frac{m_0\,\delta(t_{\rm source} + t_0)}{t_0}(\cos\theta\,\hat{\mathbf{r}} - \sin\theta\,\hat{\boldsymbol{\theta}}). \tag{22}$$

When discussing the fields on the sphere of radius r_0 at time t there, the relevant time at the source is the retarded time $t_{\rm source} = t - r_0/c$, so we can write,

$$[\ddot{\mathbf{m}}] = \frac{m_0\,\delta(t + t_0 - r_0/c)}{t_0}(\cos\theta\,\hat{\mathbf{r}} - \sin\theta\,\hat{\boldsymbol{\theta}}). \qquad (23)$$

Then, from eqs. (15)-(16), the fields at the sphere of radius r_0 due to the pulse at $t = -t_0$ can be written as,

$$\mathbf{E}(r_0, t = -t_0 + r_0/c) = -\left(\frac{m_0\,\delta(t + t_0 - r_0/c)}{c^2 r_0 t_0} + \frac{m_0}{2cr_0^2 t_0}\right)\sin\theta\,\hat{\boldsymbol{\phi}}, \qquad (24)$$

$$\mathbf{B}(r_0, t = -t_0 + r_0/c) = \frac{m_0\,\delta(t + t_0 - r_0/c)}{c^2 r_0 t_0}\sin\theta\,\hat{\boldsymbol{\theta}} + \frac{m_0}{2cr_0^2 t_0}(2\cos\theta\,\hat{\mathbf{r}} + \sin\theta\,\hat{\boldsymbol{\theta}}). \qquad (25)$$

The first terms on the right sides of eqs. (24) and (25) are the $1/r$ fields from the pulse, so the radial component of the "other" Poynting vector (19) is zero, and hence $U_1 = 0$. The pulse at $t = -t_0$ does not contribute to the stored energy in the later static magnetic field.

2.4.2 Effect of the Linear Rise of the Current on the Interval $-t_0 < t_{\rm source} < 0$

For an observer at $r = r_0$ the retarded moment associated with the linear rise of the current on the interval $-t_0 < t_{\rm source} < 0$ has magnitude $[m] = m_0(t + t_0 - r_0/c)/t_0$ during the time interval $[r_0/c - t_0, r_0/c]$, while $[\dot{m}] = m_0/t_0$ and $[\ddot{m}] = 0$, so the electric and magnetic fields (15)-(16) are,

$$\mathbf{E}(r_0, t) = -\frac{m_0}{cr_0^2 t_0}\sin\theta\,\hat{\boldsymbol{\phi}}, \qquad (26)$$

$$\mathbf{B}(r_0, t) = \frac{m_0(t + t_0)}{r_0^3 t_0}(2\cos\theta\,\hat{\mathbf{r}} + \sin\theta\,\hat{\boldsymbol{\theta}}). \qquad (27)$$

The magnetic field at time t equals the static field associated with the current that would exist at that time, if it continued to grow linearly after time $t = 0$. The $1/r^2$ Poynting vector vanishes during this interval, so the radial component of the "other" Poynting vector at $r = r_0$ is,

$$S_{{\rm other},r}(r_0, t) = S_r(r_0, t) = \frac{c}{4\pi}\hat{\mathbf{r}}\cdot\mathbf{E}(r_0, t)\times\mathbf{B}(r_0, t) = \frac{m_0^2(t + t_0)}{4\pi r_0^5 t_0^2}\sin^2\theta, \qquad (28)$$

This energy flow is positive for all r_0, but the quantity $r_0^2 S_{{\rm other},r}(r_0, t)$ decreases with r_0. which implies that as the energy flows outwards some is left behind in the stored energy of the fields. Indeed, all the energy flow $S_{{\rm other},r}(r_0, t)$ during the interval $[r_0/c - t_0, r_0/c]$ becomes stored in the magnetic field at some $r > r_0$,[7] totaling

$$U_2 = \int_{r_0/c - t_0}^{r_0/c} dt \int_{-1}^{1} 2\pi r_0^2\, d\cos\theta\; S_{r,{\rm other}}(r_0, t) = \frac{m_0^2}{3r_0^3} + \frac{2m_0^2}{3cr_0^2 t_0}\,. \qquad (29)$$

Note that the stored energy U_2 is larger than the energy U_0 of eq. (18)that is expected to be stored at $r > r_0$ at large times.

[7]The flow of energy (at the speed of light) that becomes stored at a finite distance from its source, rather than eventually reaching "infinity", is not typically called "radiation," although it is this author's view that it should be [2].

2.4.3 Effect of the Pulse at $t_{\rm source} = 0$

For an observer at $r = r_0$ the pulse emitted at $t_{\rm source} = 0$ arrives at $t = r_0/c$, the retarded moment is $m_0\,\hat{\mathbf z}$, its retarded time derivative has the average value $[\dot{\mathbf m}] = m_0\,\hat{\mathbf z}/2t_0$, and its retarded second time derivative can be written as $[\ddot{\mathbf m}] = -m_0\delta(t - r_0/c)\,\hat{\mathbf z}/t_0$, so the fields of this pulse are,

$$
\begin{aligned}
\mathbf{E}(r_0, t = r_0/c) &= -\left(-\frac{m_0\delta(t - r_0/c)}{c^2 r_0 t_0} + \frac{m_0}{2c r_0^2 t_0}\right)\sin\theta\,\hat{\boldsymbol\phi}, &(30)\\
\mathbf{B}(r_0, t = r_0/c) &= -\frac{m_0\delta(t - r_0/c)}{c^2 r_0 t_0}\sin\theta\,\hat{\boldsymbol\theta} + \left(\frac{m_0}{2c r_0^2 t_0} + \frac{m_0}{r_0^3}\right)(2\cos\theta\,\hat{\mathbf r} + \sin\theta\,\hat{\boldsymbol\theta}). &(31)
\end{aligned}
$$

The first terms on the right sides of eqs. (30) and (31) are the $1/r$ fields from the pulse, so the radial component of the "other" Poynting vector (19) is,

$$
\begin{aligned}
S_{{\rm other},r}(r_0, t = r_0/c) &= \frac{m_0^2\sin^2\theta}{4\pi}\left(-\frac{\delta(t - r_0/c)}{c r_0^4 t_0} + \frac{1}{4c r_0^4 t_0^2} + \frac{1}{2 r_0^5 t_0}\right)\\
&\approx -\frac{m_0^2\,\delta(t - r_0/c)\sin^2\theta}{4\pi c r_0^4 t_0}\,, &(32)
\end{aligned}
$$

noting that only the term with a delta function will contribute to the time integral over the pulse.

Although $S_{{\rm other},r}$ is negative, the radial component S_r of the total Poynting vector of the pulse is positive, as the radial component $S_{1/r^2,r}$ is much larger than $S_{{\rm other},r}$. We then infer from eq. (32) that less energy is emitted by the source during the pulse than eventually flows to "infinity" in this pulse, since $S_r(r_0, t = r_0/c)$ is less than $S_{1/r^2,r}(r_0, t = r_0/c)$. As the pulse passes through the electromagnetic fields, which contain stored energy from the linear rise of the source current, these fields "radiate" some of that energy into the pulse, which then carries it away to "infinity".[8] Although the fields $\mathbf E$ and $\mathbf B$ can be traced entirely to the source currents via eqs. (15)-(16), the Poynting vector cannot be similarly traced only to the source currents, but rather includes contributions for which the fields have acted as sources. See also sec. 2.3 of [2].

Using eq. (32), we find the contribution to the energy stored at $r > r_0$ from the pulse at time $t = 0$ to be,

$$
U_3 = \int dt \int_{-1}^{1} 2\pi r_0^2\; d\cos\theta\;\frac{-m_0^2\sin^2\theta\delta(t - r_0/c))}{4\pi c r_0^4 t_0} = -\frac{2m_0^2}{3c r_0^2 t_0}\,. \qquad (33)
$$

Thus,

$$
U_1 + U_2 + U_3 = \frac{m_0^2}{3r_0^3} = U_0. \qquad (34)
$$

A lesson here is that a source which radiates energy to "infinity", such as the pulse at time $t = 0$, can also radiate energy that does not flow to "infinity", but rather adds (or

[8]In the quantum view, where photons are bosons, this is an effect of Bose statistics. See, for example, chap. 4, vol. III of [16].

subtracts, as in the present example) to/from the energy stored in the near field. In other words, some of the energy that has been stored in the electromagnetic fields can be later released to flow to "infinity".[9]

2.5 A Triangular Current Pulse

The current is zero except on the time interval $[-t_0, t_0]$, during which is rises linearly to I_0 at $t = 0$ and then falls linearly to zero at $t = t_0$.

At large times no energy remains stored at any finite distance from the current loop. As in sec. 2.4, we consider the energy stored outside a sphere of radius r_0 from the "other" component (19) of the Poynting vector. The emission of energy by the current loop occurs in five steps: the pulse at time $t = -t_0$, the linear rise on the interval $[-t_0, 0]$, the pulse at $t = 0$, the linear fall on the interval $[0, t_0]$, and the pulse at $t = t_0$. The contributions to the stored energy from these five steps sum to zero,

$$U_1 + U_2 + U_3 + U_4 + U_5 = 0. \tag{35}$$

The energies U_1 and U_2 are the same as those found in sec. 2.4, $U_1 = 0$, and U_2 is given by eq. (29).

2.5.1 Effect of the Pulse at $t_{\rm source} = 0$

For an observer at $r = r_0$ the pulse emitted at $t_{\rm source} = 0$ arrives at $t = r_0/c$, the retarded moment is $m_0\,\hat{\mathbf{z}}$, its retarded time derivative drops from $m_0\,\hat{\mathbf{z}}/t_0$ to $-m_0\,\hat{\mathbf{z}}/t_0$ such that the average value is $[\dot{\mathbf{m}}] = 0$, and its retarded second time derivative can be written as, $[\ddot{\mathbf{m}}] = -2m_0\delta(t - r_0/c)\,\hat{\mathbf{z}}/t_0$, so the fields of this pulse on the sphere of radius r_0 at time $t = r_0/c$ are

$$\begin{aligned} \mathbf{E}(r_0, t = r_0/c) &= \frac{2m_0\delta(t - r_0/c)}{c^2 r_0 t_0}\sin\theta\,\hat{\boldsymbol{\phi}}, & (36) \\ \mathbf{B}(r_0, t = r_0/c) &= -\frac{2m_0\delta(t - r_0/c)}{c^2 r_0 t_0}\sin\theta\,\hat{\boldsymbol{\theta}} + \frac{m_0}{r_0^3}(2\cos\theta\,\hat{\mathbf{r}} + \sin\theta\,\hat{\boldsymbol{\theta}}). & (37) \end{aligned}$$

The first terms on the right sides of eqs. (36) and (37) are the $1/r$ fields from the pulse, so the radial component of the "other" Poynting vector (19) is,

$$S_{{\rm other},r}(r_0, t = r_0/c) = -\frac{m_0^2\delta(t - r_0/c)\sin^2\theta}{2\pi c r_0^4 t_0}. \tag{38}$$

The contribution to the energy stored at $r > r_0$ from the pulse at time $t = 0$ is,

$$U_3 = \int dt \int_{-1}^{1} 2\pi r_0^2\; d\cos\theta\; \frac{-m_0^2\sin^2\theta\delta(t - r_0/c))}{2\pi c r_0^4 t_0} = -\frac{4m_0^2}{3cr_0^2 t_0}. \tag{39}$$

As discussed in sec. 2.4 the negative values of $S_{{\rm other},r}(r_0, t = r_0/c)$ and U_3 mean that as the pulse passes through the electromagnetic fields left from the linear rise in the current,

[9] This point has also been argued by Mandel [17] and by Schantz [18], and is discussed in sec. 2.7 below.

those fields "radiate" energy into the pulse, which carries more energy to "infinity" that was emitted into the pulse by the current source. In the present case, $U_1 + U_2 + U_3 = m_0^2/3r_0^3 - 2m_0^2/3cr_0^2 t_0$, which is negative for $r_0 > ct_0/2$. This does not mean that the field energy is negative (an impossibility), because the very large positive field energy associated with the pulse emitted at $t = 0$ must be considered also. Rather, it means that the energy flow associated with the waveform for $t > 0$ must involve an outward flow of $S_{\rm other}$ at $r_0 > ct_0/2$ to result in zero energy stored in the vacuum after the effect of the waveform has ceased.

2.5.2 Effect of the Linear Fall of the Current on the Interval $0 < t_{\rm source} < t_0$

The linear fall of the source current during the interval $0 < t_{\rm source} < t_0$ implies that for an observer on the sphere of radius r_0 and during the time interval $[r_0/c, r_0/c+t_0]$ the magnitude of the retarded magnetic moment is $[m] = -m_0(t - t_0 - r_0/c)/t_0$, while $[\dot m] = -m_0/t_0$ and $[\ddot m] = 0$, so the electric and magnetic fields (15)-(16) are,

$$\mathbf{E}(r_0, t) = \frac{m_0}{c r_0^2 t_0} \sin\theta\, \hat{\boldsymbol{\phi}}, \tag{40}$$

$$\mathbf{B}(r_0, t) = -\frac{m_0(t - t_0)}{r_0^3 t_0}(2\cos\theta\,\hat{\mathbf{r}} + \sin\theta\,\hat{\boldsymbol{\theta}}). \tag{41}$$

The magnetic field at time t equals the static field associated with the current that would exist at that time, if it continued to fall linearly after time t_0. The $1/r^2$ Poynting vector vanishes during this interval, so the radial component of the "other" Poynting vector at $r = r_0$ during the time interval $[r_0/c, r_0/c + t_0]$ is,

$$S_{{\rm other},r}(r_0, t) = S_r(r_0, t) = \frac{c}{4\pi}\hat{\mathbf{r}} \cdot \mathbf{E}(r_0, t) \times \mathbf{B}(r_0, t) = \frac{m_0^2 (t - t_0)}{4\pi r_0^5 t_0^2} \sin^2\theta. \tag{42}$$

The energy flow at radius r_0 during the interval $[r_0/c, r_0/c + t_0]$ is always outwards for $r_0 > ct_0$, but the flow is inwards until time t_0 for $r_0 < ct_0$. The flow of energy at the current loop itself, $r_0 \approx a \ll ct_0$ is always negative on this interval; the loop absorbs rather than emits energy during this time. The inward flow of energy occurs only up to distance ct_0 from the source, which is the farthest distance from the source from which energy can propagate at the speed of light back to the source. Since the current loops acts as a sink, rather than a source of energy during the interval $[0, t_0]$, we must regard the fields at $r > 0$ as the source of the flow of energy, either inwards or outwards, that occurs at $r = r_0$ during the interval $[r_0/c, r_0/c + t_0]$.

The contribution to the energy stored in the electromagnetic fields for $r > r_0$ is

$$U_4 = \int_{r_0/c}^{r_0/c+t_0} dt \int_{-1}^{1} 2\pi r_0^2 \; d\cos\theta \; S_{r,\text{near zone}}(r_0, t) = -\frac{m_0^2}{3r_0^3} + \frac{2m_0^2}{3cr_0^2 t_0}. \tag{43}$$

We note that $U_1 + U_2 + U_3 + U_4 = 0$, so we anticipate that the final pulse at source time $t = t_0$ should not affect the stored energy; all of the energy of that pulse should be "radiated to infinity".

2.5.3 Effect of the Pulse at $t_{\rm source} = t_0$

For an observer at $r = r_0$ the pulse emitted at $t_{\rm source} = t_0$ arrives at $t = t_0 + r_0/c$, the retarded moment is $[\mathbf{m}] = 0$, its retarded time derivative increases from $-m_0\,\hat{\mathbf{z}}/t_0$ to 0 such that the average value is $[\dot{\mathbf{m}}] = -m_0\,\hat{\mathbf{z}}/2t_0$, and its retarded second time derivative can be written as $[\ddot{\mathbf{m}}] = m_0\delta(t - t_0 - r_0/c)\,\hat{\mathbf{z}}/t_0$. Hence, the fields of this pulse on the sphere of radius r_0 at time $t = r_0/c + t_0$ are,

$$\mathbf{E}(r_0, t = t_0 + r_0/c) = -\left(\frac{m_0\delta(t - t_0 - r_0/c)}{c^2 r_0 t_0} - \frac{m_0}{2cr_0^2 t_0}\right)\sin\theta\,\hat{\boldsymbol{\phi}}, \tag{44}$$

$$\mathbf{B}(r_0, t = t_0 + r_0/c) = \frac{m_0\delta(t - t_0 - r_0/c)}{c^2 r_0 t_0}\sin\theta\,\hat{\boldsymbol{\theta}} - \frac{m_0}{2cr_0^2 t_0}(2\cos\theta\,\hat{\mathbf{r}} + \sin\theta\,\hat{\boldsymbol{\theta}}). \tag{45}$$

The first terms on the right sides of eqs. (44) and (45) are the $1/r$ fields from the pulse, so the radial component of the "other" Poynting vector (19) is zero, and the energy U_5 is zero, as anticipated above. This completes the verification of eq. (35) that the total energy stored in the near field at large times is zero.

A lesson of this example beyond that of sec. 2.4 is that when electromagnetic fields "radiate" some of their stored energy, the resulting energy flow can be in any direction.

2.6 The Current Waveform is Repetitive and Triangular

We now consider the case of a repetitive, triangular current waveform of period $t_0 \gg a/c$, such that the current and magnetic moment obey,

$$\begin{pmatrix} I(t) \\ \mathbf{m}(t) \end{pmatrix} = \begin{pmatrix} I_0 \\ m_0\,\hat{\mathbf{z}} \end{pmatrix} \begin{cases} 1 + 4(t' - nt_0)/t_0 & (-t_0/2 < t' - nt_0 < 0), \\ 1 - 4(t' - nt_0)/t_0 & (0 < t' - nt_0 < t_0/2), \end{cases} \tag{46}$$

for any integer n, where t' is the time as the source. An observer at distance r from the source at time t detects fields that were generated by the source at the retarded time $t' = t - r/c$.

2.6.1 $-t_0/2 < t - nt_0 - r/c < 0$ for some n

When the observer's time t obeys $-t_0/2 < t - nt_0 - r/c < 0$ for some n the observed fields are due to the linearly rising current, the retarded moment has magnitude $[m] = m_0\{1 + 4(t - nt_0 - r_0/c)/t_0\}$, while $[\dot{m}] = 4m_0/t_0$ and $[\ddot{m}] = 0$, so the electric and magnetic fields (15)-(16) are,[10]

$$\mathbf{E}(r, t) = -\frac{4m_0}{cr^2 t_0}\sin\theta\,\hat{\boldsymbol{\phi}}, \tag{47}$$

$$\mathbf{B}(r, t) = \frac{m_0}{r^3}\left(1 + \frac{4(t - nt_0)}{t_0}\right)(2\cos\theta\,\hat{\mathbf{r}} + \sin\theta\,\hat{\boldsymbol{\theta}}). \tag{48}$$

[10]The magnetic field at such times equals the static field associated with the current that would exist at that time, if it continued to grow linearly after time $t = nt_0$. The value of r can be such that the source current at the observer's time t is $I_0(1 - 4(t - nt_0)/t_0)$, which emphasizes that the field (48) is not the "instantaneous" magnetic field.

The $1/r^2$ Poynting vector vanishes during this interval, so the radial component of the "other" Poynting vector is,

$$S_{\text{other},r}(r,t) = S_r(r,t) = \frac{c}{4\pi}\hat{\mathbf{r}} \cdot \mathbf{E}(r,t) \times \mathbf{B}(r,t) = \frac{m_0^2}{\pi r^5 t_0}\left(1 + \frac{4(t - nt_0)}{t_0}\right)\sin^2\theta, \qquad (49)$$

2.6.2 $t - nt_0 - r/c = 0$ for some n

When the observer's time t obeys $t - nt_0 - r/c = t_0/2$ for some n the observer fields are due to the pulse of current at source time $nt_0 + t_0/2$, the retarded moment has magnitude $[m] = m_0$, the average time derivative is $[\dot{m}] = 0$ and $[\ddot{m}]$ can be written as $-8m_0\delta(t - nt_0 - r/c)/t_0$, so the observed electric and magnetic fields (15)-(16) are,

$$\mathbf{E}(r, t = nt_0 + r/c) = \frac{8m_0\delta(t - nt_0 - r/c)}{c^2 r t_0}\sin\theta\,\hat{\boldsymbol{\phi}}, \qquad (50)$$

$$\mathbf{B}(r, t = nt_0 + r/c) = -\frac{8m_0\delta(t - nt_0 - r/c)}{c^2 r t_0}\sin\theta\,\hat{\boldsymbol{\theta}} + \frac{m_0}{r^3}(2\cos\theta\,\hat{\mathbf{r}} + \sin\theta\,\hat{\boldsymbol{\theta}}). \qquad (51)$$

The first terms on the right sides of eqs. (50) and (51) are the $1/r$ fields from the pulse, so the radial component of the "other" Poynting vector (19) is,

$$S_{\text{other},r}(r,t) = -\frac{2m_0^2\delta(t - nt_0 - r/c)\sin^2\theta}{\pi c r^4 t_0}. \qquad (52)$$

2.6.3 $0 < t - nt_0 - r/c < t_0/2$ for some n

When the observer's time t obeys $0 < t - nt_0 - r/c < t_0/2$ for some n the observed fields are due to the linearly falling current, the retarded moment has magnitude $[m] = m_0\{1 - 4(t - nt_0 - r_0/c)/t_0\}$, while $[\dot{m}] = -4m_0/t_0$ and $[\ddot{m}] = 0$. The observed electric and magnetic fields (15)-(16) are,

$$\mathbf{E}(r,t) = \frac{4m_0}{c r^2 t_0}\sin\theta\,\hat{\boldsymbol{\phi}}, \qquad (53)$$

$$\mathbf{B}(r,t) = \frac{m_0}{r^3}\left(1 - \frac{4(t - nt_0)}{t_0}\right)(2\cos\theta\,\hat{\mathbf{r}} + \sin\theta\,\hat{\boldsymbol{\theta}}). \qquad (54)$$

The $1/r^2$ Poynting vector vanishes during this interval, so the radial component of the "other" Poynting vector is,

$$S_{\text{other},r}(r,t) = S_r(r,t) = \frac{c}{4\pi}\hat{\mathbf{r}} \cdot \mathbf{E}(r,t) \times \mathbf{B}(r,t) = -\frac{m_0^2}{\pi r^5 t_0}\left(1 - \frac{4(t - nt_0)}{t_0}\right)\sin^2\theta, \qquad (55)$$

2.6.4 $t - nt_0 - r/c = t_0/2$ for some n

When the observer's time t obeys $t - nt_0 - r/c = t_0/2$ for some n the observer fields are due to the pulse of current at source time nt_0, the retarded moment has magnitude $[m] = -m_0$, the average time derivative is $[\dot{m}] = 0$ and $[\ddot{m}]$ can be written as $8m_0\delta(t - nt_0 - t_0/2 - r/c)$, so the observed electric and magnetic fields (15)-(16) are,

$$\mathbf{E}(r, t = nt_0 + t_0/2 + r/c) = -\frac{8m_0\delta(t - nt_0 - t_0/2 - r/c)}{c^2 r t_0} \sin\theta\, \hat{\boldsymbol{\phi}}, \tag{56}$$

$$\begin{aligned} \mathbf{B}(r, t = nt_0 + t_0/2 + r/c) &= \frac{8m_0\delta(t - nt_0 - t_0/2 - r/c)}{c^2 r t_0} \sin\theta\, \hat{\boldsymbol{\theta}} \\ &\quad - \frac{m_0}{r^3}(2\cos\theta\, \hat{\mathbf{r}} + \sin\theta\, \hat{\boldsymbol{\theta}}). \end{aligned} \tag{57}$$

The first terms on the right sides of eqs. (56) and (57) are the $1/r$ fields from the pulse, so the radial component of the "other" Poynting vector (19) is,

$$S_{\mathrm{other},r}(r, t) = -\frac{2m_0^2\delta(t - nt_0 - t_0/2 - r/c)\sin^2\theta}{\pi c r^4 t_0}. \tag{58}$$

2.6.5 Discussion

Radiation to "infinity" is associated only with the pulses generated at the transition times between the intervals of current with a linear time dependence. Both the pulses associated with source times nt_0 and $nt_0 + t_0/2$ have $S_{\mathrm{other},r} < 0$, which means that these pulses sweep up energy stored in the fields at $r > 0$ and carry it to "infinity." The radial flow of energy per unit area at radius r during each interval of length $t_0/2$ in which the current varies linearly is $2m_0^2 \sin^2\theta/\pi c r^4 t_0$, which equals the energy "swept away to "infinity" by the pulse at the end of that interval.

As for the example in sec. 2.5, the current pulse is not the immediate source of all the energy that is "radiated to infinity". Some of this radiated energy had been previously stored in the fields throughout all space surrounding the current source.

2.7 The Current Decays Exponentially

We now consider the case of exponential decay of the current and magnetic moment from their constant values at $t < 0$,

$$\begin{pmatrix} I(t) \\ \mathbf{m}(t) \end{pmatrix} = \begin{pmatrix} I_0 \\ m_0\, \hat{\mathbf{z}} \end{pmatrix} \begin{cases} 1 & (t < 0), \\ e^{-t/t_0} & (t > 0). \end{cases} \tag{59}$$

Similarly to the discussion at the beginning of sec. 2.4, at distance r from the current loop (of small radius a) and for times $t < r/c$ the electric field is zero, the magnetic field is,

$$\mathbf{B}_0(\mathbf{r}) = \frac{3(\mathbf{m}_0 \cdot \hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{m}_0}{r^3} = \frac{m_0(2\cos\theta\, \hat{\mathbf{r}} + \sin\theta\, \hat{\boldsymbol{\theta}})}{r^3} \qquad (r \gg a,\ t < r/c), \tag{60}$$

and the magnetic energy stored outside a sphere of radius $r_0 \gg a$ at times $t < r_0/c$ is,

$$U_0 = \int_{r > r_0} \frac{B_0^2}{8\pi}\, d^3\mathbf{x} = 2\pi \int_{r_0}^{\infty} r^2\, dr \int_{-1}^{1} d\cos\theta\, \frac{m_0^2(3\cos^2\theta + 1)}{8\pi r^6} = \frac{m_0^2}{3r_0^3}. \tag{61}$$

After time $t = r/c$ the field and the stored energy at radius r change, and at large times they are both zero. Some of the stored energy returns to the source, and some is "radiated to infinity".

To clarify this, we note that according to an observer at radius r at time $t > r/c$ the retarded magnetic moment are its derivatives are,

$$[\mathbf{m}] = m_0\, e^{-(t-r/c)/t_0}\,\hat{\mathbf{z}}, \qquad [\dot{\mathbf{m}}] = -\frac{m_0}{t_0}\, e^{-(t-r/c)/t_0}\,\hat{\mathbf{z}}, \qquad [\ddot{\mathbf{m}}] = \frac{m_0}{t_0^2}\, e^{-(t-r/c)/t_0}\,\hat{\mathbf{z}}, \tag{62}$$

so the fields for $t > r/c$ follow from eqs. (15)-(16) as

$$\mathbf{E}(\mathbf{r},t) = -m_0\, e^{-(t-r/c)/t_0}\left(\frac{1}{c^2 t_0^2 r} - \frac{1}{c t_0 r^2}\right)\sin\theta\,\hat{\boldsymbol{\phi}}, \tag{63}$$

$$\mathbf{B}(\mathbf{r},t) = m_0\, e^{-(t-r/c)/t_0}\left(\frac{\sin\theta\,\hat{\boldsymbol{\theta}}}{c^2 t_0^2 r} + (2\cos\theta\,\hat{\mathbf{r}} + \sin\theta\,\hat{\boldsymbol{\theta}})\left(\frac{1}{r^3} - \frac{1}{c t_0 r^2}\right)\right). \tag{64}$$

The radial component of the Poynting vector for $t > r/c$ is,

$$S_r(\mathbf{r},t) = \frac{m_0^2\, e^{-2(t-r/c)/t_0}\sin^2\theta}{4\pi t_0 r^2}\left(\frac{1}{c^3 t_0^3} - \frac{2}{c^2 t_0^2 r} + \frac{2}{c t_0 r^2} - \frac{1}{r^3}\right). \tag{65}$$

The sign of S_r is independent of time, is positive for large r and negative for small r, and vanishes when,

$$r^3 - 2ct_0 r^2 + 2c^2 t_0^2 - c^3 t_0^3 = (r - ct_0)\left(r^2 - ct_0 r + c^2 t_0^2\right) = 0, \tag{66}$$

i.e., when $r = ct_0$.

We are led to say that the time-dependent current for $t > 0$ does not emit any "radiation", but rather absorbs all the energy stored at $r < ct_0$, which energy is "radiated" by the vacuum back towards the "source" currents. Meanwhile, all of the energy $m_0^2/3c^3t_0^3$ stored at $r > ct_0$ is "radiated to infinity". The "source" of this "radiation" is not the physical current, but the electromagnetic field energy stored at $r > ct_0$. Of course, as seen in sec. 2.4, this stored energy is due to emission of energy by the source currents at a much earlier time, when they rose from zero to I_0. Hence, we could say that the "radiation to infinity" is ultimately due to the much earlier currents that established the static field $\mathbf{B}_0$ that existed before the decay of the currents back to zero.

This example was first discussed by Mandel [17]. See also [18].

2.7.1 The Current Rises Exponentially to I_0

It may be of interest to reconsider the example of sec. 2.4 now supposing that the current rises exponentially from zero at $t = 0$ such that,

$$\begin{pmatrix} I(t) \\ \mathbf{m}(t) \end{pmatrix} = \begin{pmatrix} I_0 \\ m_0\,\hat{\mathbf{z}} \end{pmatrix} \begin{cases} 0 & (t < 0), \\ 1 - e^{-t/t_0} & (t > 0). \end{cases} \tag{67}$$

Similarly to the discussion at the beginning of sec. 2.4, after time $t = r/c$ the field and the stored energy at radius r rise from zero, and at large times they are both constant. Some of the energy emitted by the currents becomes stored in the magnetic field, and some is "radiated to infinity". For times $t \gg r/c$ the electric field is zero, the magnetic field is,

$$\mathbf{B}_0(\mathbf{r}) = \frac{3(\mathbf{m}_0 \cdot \hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{m}_0}{r^3} = \frac{m_0(2\cos\theta\,\hat{\mathbf{r}} + \sin\theta\,\hat{\boldsymbol{\theta}})}{r^3} \qquad (r \gg a,\ t \gg r/c), \tag{68}$$

and the magnetic energy stored outside a sphere of radius $r_0 \gg a$ at times $t \gg r_0/c$ is,

$$U_0 = \int_{r>r_0} \frac{B_0^2}{8\pi}\, d^3\mathbf{x} = 2\pi \int_{r_0}^{\infty} r^2\, dr \int_{-1}^{1} d\cos\theta\, \frac{m_0^2(3\cos^2\theta + 1)}{8\pi r^6} = \frac{m_0^2}{3r_0^3}\,. \tag{69}$$

In more detail, we note that according to an observer at radius r at time $t > r/c$ the retarded magnetic moment are its derivatives are,

$$[\mathbf{m}] = m_0\,(1 - e^{-(t-r/c)/t_0})\,\hat{\mathbf{z}}, \qquad [\dot{\mathbf{m}}] = \frac{m_0}{t_0}\, e^{-(t-r/c)/t_0}\,\hat{\mathbf{z}}, \qquad [\ddot{\mathbf{m}}] = -\frac{m_0}{t_0^2}\, e^{-(t-r/c)/t_0}\,\hat{\mathbf{z}}, \tag{70}$$

so the fields for $t > r/c$ follow from eqs. (15)-(16) as,

$$\mathbf{E}(\mathbf{r},t) = m_0\, e^{-(t-r/c)/t_0}\left(\frac{1}{c^2 t_0^2 r} - \frac{1}{c t_0 r^2}\right)\sin\theta\,\hat{\boldsymbol{\phi}}, \tag{71}$$

$$\begin{aligned}\mathbf{B}(\mathbf{r},t) &= -m_0\, e^{-(t-r/c)/t_0}\left(\frac{\sin\theta\,\hat{\boldsymbol{\theta}}}{c^2 t_0^2 r} - \frac{(2\cos\theta\,\hat{\mathbf{r}} + \sin\theta\,\hat{\boldsymbol{\theta}})}{c t_0 r^2}\right) \\ &\quad + \frac{m_0\,(1 - e^{-(t-r/c)/t_0})}{r^3}(2\cos\theta\,\hat{\mathbf{r}} + \sin\theta\,\hat{\boldsymbol{\theta}}).\end{aligned} \tag{72}$$

The radial component of the Poynting vector for $t > r/c$ is,

$$\begin{aligned}S_r(\mathbf{r},t) &= \frac{m_0^2\, e^{-2(t-r/c)/t_0}\sin^2\theta}{4\pi t_0 r^2}\left(\frac{1}{c^3 t_0^3} - \frac{2}{c^2 t_0^2 r} + \frac{1}{c t_0 r^2}\right) \\ &\quad + \frac{m_0^2\, e^{-(t-r/c)/t_0}\,(1 - e^{-(t-r/c)/t_0})\sin^2\theta}{4\pi t_0 r^2}\left(\frac{1}{r^3} - \frac{1}{c t_0 r^2}\right),\end{aligned} \tag{73}$$

which is positive for all r. The total energy that crosses a sphere of radius r is,

$$U_r = 2\pi \int_{-1}^{1} d\cos\theta \int_{r/c}^{\infty} r^2 S_r(\mathbf{r},t)\, dt = \frac{m_0^2}{3}\left(\frac{1}{r^3} + \frac{1}{c^3 t_0^3} - \frac{2}{c^2 t_0^2 r}\right). \tag{74}$$

Of this, the first term is the energy U_0 of eq. (69) the remains stored in the fields at finite r and large times. Hence, we infer that the "radiation to infinity" at radius r is,

$$U_{\infty,r} = \frac{m_0^2}{3}\left(\frac{1}{c^3 t_0^3} - \frac{2}{c^2 t_0^2 r}\right). \tag{75}$$

However, according to eq. (14) the total energy "radiated to infinity" is,

$$U_\infty = \int \frac{2\,|[\ddot{\mathbf{m}}]|^2}{3c^3}\, dt = \int_0^\infty \frac{2m_0^2 e^{-2t_{\rm source}/t_0}}{3c^3 t_0^4}\, dt_{\rm source} = \frac{m_0^2}{3c^3 t_0^3}\,. \tag{76}$$

Thus, the amount of "radiation to infinity" that crosses a sphere of radius r is less than the total "radiation to infinity". As in the examples of secs. 2.4-6, part of the "radiation to infinity" comes from energy stored in the fields at large r, and which energy did not arrive there via the Poynting vector $\mathbf{S}_{1/r^2}$ associated with "radiation to infinity" at earlier times.

2.8 The Current Oscillates Sinusoidally

The examples in secs. 2.4-7 have shown how the energy that is "radiated to infinity" does not all flow directly from the source to "infinity" at the speed of light, but some of this energy was temporarily stored in the so-called "reactive near zone". Further, some of this temporarily stored energy flowed from the source via the "other" part of the Poynting vector defined in eq. (18). This behavior is also present in the case of harmonic time dependence of the sources, but for sources that are small compared to a wavelength the two pieces of the Poynting vector, $\mathbf{S}_{1/r^2}$ and $\mathbf{S}_{\rm other}$, separately obey the continuity equations,

$$\boldsymbol{\nabla}\cdot\mathbf{S}_{1/r^2} = -\frac{\partial u_{1/r^2}}{\partial t} = -\frac{\partial}{\partial t}\left(\frac{E_{1/r}^2 + B_{1/r}^2}{8\pi}\right), \tag{77}$$

$$\text{and}\qquad \boldsymbol{\nabla}\cdot\mathbf{S}_{\rm other} = -\frac{\partial u_{\rm other}}{\partial t} = -\frac{\partial}{\partial t}\left(\frac{E_{\rm other}^2 + B_{\rm other}^2}{8\pi}\right) \tag{78}$$

outside the source currents, where $\mathbf{E}_{1/r}$ and $\mathbf{B}_{1/r}$ are the parts of the fields that vary as $1/r$ from the (small) source, and,

$$E_{\rm other}^2 = E^2 - E_{1/r}^2, \qquad B_{\rm other}^2 = B^2 - B_{1/r}^2. \tag{79}$$

This is shown, for example, in [23] for the case of a small electric dipole, whose fields are dual to those of a small magnetic dipole.

However, when the source region is not small compared to a wavelength, the equations (77)-(78) are not separately satisfied, although outside the source the total Poynting vector obeys,

$$\boldsymbol{\nabla}\cdot\mathbf{S} = -\frac{\partial u}{\partial t} = -\frac{\partial}{\partial t}\left(\frac{E^2 + B^2}{8\pi}\right). \tag{80}$$

This means that outside the physical current source, "other" terms in the Poynting vector and in the energy density acts as sources (and sinks) of the partial Poynting vector $\mathbf{S}_{1/r^2}$, as was also found in the examples in secs. 2.4-7. See sec. 5 of [2] for a calculation which shows that upon taking the time average, such that $\partial\langle u\rangle/\partial t = \partial\left\langle u_{1/r^2}\right\rangle/\partial t = \partial\langle u_{\rm other}\rangle/\partial t = 0$, then $\boldsymbol{\nabla}\cdot\langle\mathbf{S}\rangle = 0$ but $\boldsymbol{\nabla}\cdot\left\langle\mathbf{S}_{1/r^2}\right\rangle = -\boldsymbol{\nabla}\cdot\langle\mathbf{S}_{\rm other}\rangle \neq 0$.

Thus, the famous example of a small (Hertzian) oscillating dipole fails to illustrate the general result that "radiation to infinity" does not all proceed directly from the source currents at the speed of light, but includes contributions from "other" energy previously stored in the fields which becomes "swept up" by the "radiation to infinity". The "other" energy density $u_{\rm other}$ was emitted by the source currents as described by the "other" Poynting vector $\mathbf{S}_{\rm other}$, and the latter should be termed "radiation" in the view of this author.

2.8.1 The Current Rises as $I_0 \sin\omega t$ for 1/4 Cycle

It may be of interest to reconsider the example of sec. 2.4 now supposing that the current rises sinusoidally from zero at $t = 0$ for 1/4 cycle such that,

$$\begin{pmatrix} I(t) \\ \mathbf{m}(t) \end{pmatrix} = \begin{pmatrix} I_0 \\ m_0\,\hat{\mathbf{z}} \end{pmatrix} \begin{cases} 0 & (t < 0), \\ \sin\omega t & (0 < t < \pi/2\omega), \\ 1 & (t > \pi/2\omega). \end{cases} \tag{81}$$

Similarly to the discussion at the beginning of sec. 2.4, after time $t = r/c$ the field and the stored energy at radius r rise from zero, and at large times they are both constant. Some of the energy emitted by the currents becomes stored in the magnetic field, and some is "radiated to infinity". For times $t \gg r/c$ the electric field is zero, the magnetic field is,

$$\mathbf{B}_0(\mathbf{r}) = \frac{3(\mathbf{m}_0 \cdot \hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{m}_0}{r^3} = \frac{m_0(2\cos\theta\,\hat{\mathbf{r}} + \sin\theta\,\hat{\boldsymbol{\theta}})}{r^3} \qquad (r \gg a,\ t > r/c + \pi/2\omega), \tag{82}$$

and the magnetic energy stored outside a sphere of radius $r_0 \gg a$ at times $t \gg r_0/c$ is,

$$U_0 = \int_{r>r_0} \frac{B_0^2}{8\pi}\,d^3\mathbf{x} = 2\pi \int_{r_0}^{\infty} r^2\,dr \int_{-1}^{1} d\cos\theta\ \frac{m_0^2(3\cos^2\theta + 1)}{8\pi r^6} = \frac{m_0^2}{3r_0^3}\,. \tag{83}$$

In more detail, we note that according to an observer at radius r at time $t > r/c$ the retarded magnetic moment are its derivatives are,

$$[\mathbf{m}] = -m_0 \sin(kr - \omega t)\,\hat{\mathbf{z}}, \qquad [\dot{\mathbf{m}}] = -\omega m_0 \cos(kr - \omega t)\,\hat{\mathbf{z}}, \qquad [\ddot{\mathbf{m}}] = \omega^2 m_0 \sin(kr - \omega t)\,\hat{\mathbf{z}}, \tag{84}$$

where $k = \omega/c$, so the fields for $r/c < t < r/c + \pi/2\omega$ follow from eqs. (15)-(16) as,

$$\mathbf{E}(\mathbf{r}, t) = -m_0 \left(\frac{k^2 \sin(kr - \omega t)}{r} - \frac{k\cos(kr - \omega t)}{r^2} \right) \sin\theta\,\hat{\boldsymbol{\phi}}, \tag{85}$$

$$\begin{aligned} \mathbf{B}(\mathbf{r}, t) &= m_0 \frac{k^2 \sin(kr - \omega t)\sin\theta\,\hat{\boldsymbol{\theta}}}{r} \\ &\quad - m_0(2\cos\theta\,\hat{\mathbf{r}} + \sin\theta\,\hat{\boldsymbol{\theta}}) \left(\frac{k\cos(kr - \omega t)}{r^2} + \frac{\sin(kr - \omega t)}{r^3} \right). \end{aligned} \tag{86}$$

The radial component of the Poynting vector for $r/c < t < r/c + \pi/2\omega$ is,

$$S_r(\mathbf{r}, t) = \frac{c m_0^2 \sin^2\theta}{4\pi r^2} \left(k^4 \sin^2(kr - \omega t) - \left(\frac{k^3}{r} - \frac{k}{2r^3} \right) \sin 2(kr - \omega t) + \frac{k^2 \cos 2(kr - \omega t)}{r^2} \right), \tag{87}$$

which is positive for all r. The total energy that crosses a sphere of radius r is,

$$U_r = 2\pi \int_{-1}^{1} d\cos\theta \int_{r/c} r/c + \pi/2\omega r^2 S_r(\mathbf{r}, t)\,dt = \frac{m_0^2}{3} \left(\frac{1}{r^3} + \frac{\pi k^3}{2} - \frac{2k^2}{r} \right). \tag{88}$$

Of this, the first term is the energy U_0 of eq. (69) the remains stored in the fields at finite r and large times. Hence, we infer that the "radiation to infinity" at radius r is,

$$U_{\infty,r} = \frac{m_0^2}{3}\left(\frac{\pi k^3}{2} - \frac{2k^2}{r}\right). \tag{89}$$

However, according to eq. (14) the total energy "radiated to infinity" is,

$$U_\infty = \int \frac{2\,|[\ddot{\mathbf{m}}]|^2}{3c^3}\,dt = \int_0^{\pi/2\omega} \frac{2\omega^4 m_0^2 \sin^2 \omega t_{\rm source}}{3c^3}\,dt_{\rm source} = \frac{\pi m_0^2 k^3}{6}\,. \tag{90}$$

Thus, the amount of "radiation to infinity" that crosses a sphere of radius r is less than the total "radiation to infinity". As in the examples of secs. 2.4-6, part of the "radiation to infinity" comes from energy stored in the fields at large r, and which energy did not arrive there via the Poynting vector $\mathbf{S}_{1/r^2}$ associated with "radiation to infinity" at earlier times.

What is different between this case of 1/4 cycle step in the current and the steady state?

The calculations (84)-(88) and (90) hold for the steady state as well. However, the energy stored in the fields is different in the steady state than after a 1/4-cycle current step, because of the energy associated with the steady-state "radiation". In the steady state both the first and third terms in the righthand side of eq. (88) describe the flow of energy that becomes stored in the fields at finite r (or returned to the source for small r), while the middle term corresponds to the "radiation to infinity". In the sinusoidal steady state for a small oscillating dipole the "radiation fields" do not "sweep up" any of the energy of the fields they pass through, although this action occurs in general (including the steady-state case of a triangular current waveform as seen in sec. 2.5, and the case of a large array of sinusoidally oscillating dipoles as discussed in sec. 5 of [2]).

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