Energy Flow in a Waveguide below Cutoff

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1 Problem

Electromagnetic waves of frequency below a cutoff cannot propagate down a hollow waveguide. One view of why this is so has been given in sec. 24-8 of [1], where it is argued that constructive interference between the physical source of the waves and the infinite set of related image sources can only occur if the free-space wavelength is short enough. In this problem, discuss the flow of energy (as described by the Poynting vector [2]) supposing that source currents in the plane $z = 0$ launch fields according to the TE_{10} mode pattern of a rectangular, vacuum waveguide of inner dimensions a in x and b in y.

2 Solution

The electromagnetic fields of the TE_{10} mode with angular frequency ω in a guide whose interior is $0 < x < a$ and $0 < y < b$ can be deduced from a sinusoidal form for $\mathbf{E} = E_y(x, z, t)\,\hat{\mathbf{y}}$ that obeys the perfect-conductor boundary condition $E_y(x = 0) = E_y(x = a) = 0$. Then, the fields are given by the real parts of (see, for example, [3]),[1]

$$E_y = E_0 \sin\frac{\pi x}{a}\, e^{i(k_g z - \omega t)}, \tag{3}$$

$$B_x = \frac{i}{\omega}\frac{\partial E_y}{\partial z} = -\frac{k_g}{\omega} E_0 \sin\frac{\pi x}{a}\, e^{i(k_g z - \omega t)}, \tag{4}$$

$$B_z = -\frac{i}{\omega}\frac{\partial E_y}{\partial x} = -\frac{i\pi}{\omega a} E_0 \cos\frac{\pi x}{a}\, e^{i(k_g z - \omega t)}, \tag{5}$$

in SI units, and the guide wave number k_g is given by,

$$k_g = \sqrt{\frac{\omega^2}{c^2} - \frac{\pi^2}{a^2}} = i\sqrt{\frac{\pi^2}{a^2} - \frac{\omega^2}{c^2}} \equiv i\alpha, \tag{6}$$

where c is the speed of light in vacuum, such that eqs. (3)-(5) satisfy the wave equation $\nabla^2\psi = \partial^2\psi/\partial(ct)^2$.

[1] The TE_{10} fields (3)-(5) can also be thought of as the sum of a pair of free-space waves that zig-zag down the guide at angle θ (see, for example, sec. 9.9 of [3]),

$$\mathbf{E} = \mathbf{E}_+ + \mathbf{E}_- = \frac{E_0}{2i}\left(e^{i(\mathbf{k}_+\cdot\mathbf{r} - \omega t)} - e^{i(\mathbf{k}_-\cdot\mathbf{r} - \omega t)}\right)\hat{\mathbf{y}} = E_0 \sin\left(\frac{\omega}{c}\cos\theta\, x\right) e^{i(k_g z - \omega t)}\,\hat{\mathbf{y}}, \tag{1}$$

$$\mathbf{B} = \frac{\mathbf{k}_+ \times \mathbf{E}_+}{\omega} + \frac{\mathbf{k}_- \times \mathbf{E}_-}{\omega}, \qquad \mathbf{k}_\pm = \pm\frac{\omega}{c}\cos\theta\,\hat{\mathbf{x}} + \frac{\omega}{c}\sin\theta\,\hat{\mathbf{z}} = \pm\frac{\omega}{c}\sin\theta\,\hat{\mathbf{x}} + k_g\,\hat{\mathbf{z}}. \tag{2}$$

The requirement that $E_y(x = a) = 0$ implies that $(\omega/c)\cos\theta = \pi/a$, so that $k_g = (\omega/c)\sin\theta$ is given by eq. (6).

2.1 Above Cutoff

For frequencies $\omega > \pi c/a$ the guide wave number k_g is real, and the fields (3)-(5) can be written as,

$$E_y = E_0 \sin\frac{\pi x}{a}\cos(k_g z - \omega t), \tag{7}$$

$$B_x = -\frac{k_g}{\omega}E_0 \sin\frac{\pi x}{a}\cos(k_g z - \omega t), \tag{8}$$

$$B_z = \frac{\pi}{\omega a}E_0 \cos\frac{\pi x}{a}\sin(k_g z - \omega t). \tag{9}$$

The (instantaneous) Poynting vector in this case is,

$$\begin{aligned}\mathbf{S} &= \frac{\mathbf{E}\times\mathbf{B}}{\mu_0} = \frac{c}{Z_0}(E_y B_z\,\hat{\mathbf{x}} - E_y B_x\,\hat{\mathbf{z}}) \\ &= \frac{cE_0^2}{Z_0}\left(\frac{\pi}{4\omega a}\sin\frac{2\pi x}{a}\sin[2(k_g z - \omega t)]\,\hat{\mathbf{x}} + \frac{k_g}{\omega}\sin^2\frac{\pi x}{a}\cos^2(k_g z - \omega t)\,\hat{\mathbf{z}}\right),\end{aligned} \tag{10}$$

where $Z_0 = \sqrt{\mu_0/\epsilon_0} = 377\ \Omega$, and ϵ_0 and μ_0 are the permittivity and the permeability of the vacuum, respectively. The time-average Poynting vector is,

$$\langle\mathbf{S}\rangle = \frac{k_g}{\omega}\frac{cE_0^2}{2Z_0}\sin^2\frac{\pi x}{a}\,\hat{\mathbf{z}} = \frac{c}{v_p}\frac{E_0^2}{2Z_0}\sin^2\frac{\pi x}{a}\,\hat{\mathbf{z}}, \tag{11}$$

corresponding to flow of energy down the guide with phase velocity,

$$v_p = \frac{\omega}{k_g} = \frac{c}{\sqrt{1-\left(\frac{\pi c}{\omega a}\right)^2}} > c, \tag{12}$$

and with group velocity,

$$v_g = \frac{d\omega}{d\,k_g} = c\sqrt{1-\left(\frac{\pi c}{\omega a}\right)^2} = \frac{c^2}{v_p} < c. \tag{13}$$

The Poynting vector (10) also describes a flow of energy in the x-direction that oscillates in time, indicating a transverse rearrangement (separately for x greater and less than $a/2$) of energy stored within the guide as this energy propagates down the guide. No energy flows into or out of the guide walls in the limit that they are perfect conductors.

2.2 Below Cutoff

For frequencies $\omega < \pi c/a$ the guide wave number k_g is imaginary, and the fields (3)-(5) can be written in terms of the real parameter α (defined in eq. (6)) as,

$$E_y = E_0 \sin\frac{\pi x}{a}e^{-\alpha z}\cos\omega t, \tag{14}$$

$$B_x = -\frac{\alpha}{\omega}E_0 \sin\frac{\pi x}{a}e^{-\alpha z}\sin\omega t, \tag{15}$$

$$B_z = -\frac{\pi}{\omega a}E_0 \cos\frac{\pi x}{a}e^{-\alpha z}\sin\omega t. \tag{16}$$

The Poynting vector in this case is,

$$\mathbf{S} = \frac{cE_0^2}{2Z_0}\left(-\frac{\pi}{2\omega a}\sin\frac{2\pi x}{a}\,\hat{\mathbf{x}} + \frac{\alpha}{\omega}\sin^2\frac{\pi x}{a}\,\hat{\mathbf{z}}\right)e^{-2\alpha z}\sin 2\omega t. \tag{17}$$

The time-average Poynting vector is zero, and there is no net flow of energy away from the sources in the plane $z = 0$.

The Poynting vector (17) does describe an energy flow in the z-direction, but this energy propagates only characteristic distance $z \approx 2/\alpha$ ($\approx a$ for low ω) before returning back to the sources. Again, there is also a flow of energy in the x-direction that oscillates in time, indicating an oscillatory rearrangement of energy stored within the guide and close to the sources.

In more detail, the instantaneous density u of electromagnetic energy in the guide is,

$$\begin{aligned}
u &= \frac{\epsilon_0 E^2}{2} + \frac{B^2}{2\mu_0} \\
&= \frac{E_0^2}{2}\left(\epsilon_0\sin^2\frac{\pi x}{a}e^{-2\alpha z}\cos^2\omega t + \frac{\alpha^2}{\mu_0\omega^2}\sin^2\frac{\pi x}{a}e^{-2\alpha z}\sin^2\omega t + \frac{\pi^2}{\mu_0\omega^2 a^2}\cos^2\frac{\pi x}{a}e^{-2\alpha z}\sin^2\omega t\right) \\
&= \frac{E_0^2}{2}\left(\epsilon_0\sin^2\frac{\pi x}{a}e^{-2\alpha z}\cos 2\omega t + \frac{\pi^2}{\mu_0\omega^2 a^2}e^{-2\alpha z}\sin^2\omega t\right) \\
&= \frac{cE_0^2}{2Z_0}\left(\frac{1}{c^2}\sin^2\frac{\pi x}{a}e^{-2\alpha z}\cos 2\omega t + \frac{\pi^2}{\omega^2 a^2}e^{-2\alpha z}\sin^2\omega t\right),
\end{aligned} \tag{18}$$

recalling eq. (6). Then,

$$\begin{aligned}
\frac{\partial u}{\partial t} &= \frac{cE_0^2}{2Z_0}\left(\frac{\pi^2}{\omega a^2} - \frac{2\omega}{c^2}\sin^2\frac{\pi x}{a}\right)e^{-2\alpha z}\sin 2\omega t \\
&= \frac{cE_0^2}{2Z_0}\left(\frac{\pi^2}{\omega a^2}\sin\frac{2\pi x}{a} + \frac{2\alpha^2}{\omega}\sin^2\frac{\pi x}{a}\right)e^{-2\alpha z}\sin 2\omega t = -\boldsymbol{\nabla}\cdot\mathbf{S},
\end{aligned} \tag{19}$$

again using eq. (6) for α, which verifies that electromagnetic energy is conserved under the flow (17).

To use the language of antennas, we can say that the electromagnetic fields in a waveguide operated below its cutoff frequency include only near field terms; there are no far fields in this case.[2]

References

[1] R.P. Feynman, R.B. Leighton and M. Sands, *The Feynman Lectures on Physics*, Vol. II (Addison-Wesley, 1964), http://www.feynmanlectures.caltech.edu/II_24.html#Ch24-S8

[2] Another term associated with waves is "evanescent," meaning waves that are "tied" to some matter and do not transport energy to "infinity." All waves in waveguides are "evanescent." See, for example, [4] for a general discussion.

[2] J.H. Poynting, *On the Transfer of Energy in the Electromagnetic Field*, Phil. Trans. Roy. Soc. London **175**, 343 (1884),
http://kirkmcd.princeton.edu/examples/EM/poynting_ptrsl_175_343_84.pdf

[3] S.J. Orfanidis, *Waveguides*, http://www.ece.rutgers.edu/~orfanidi/ewa/ch09.pdf

[4] K.T. McDonald, *Decomposition of Electromagnetic Fields into Electromagnetic Plane Waves* (July 11, 2010), http://kirkmcd.princeton.edu/examples/virtual.pdf