Second-Order Paraxial Gaussian Beam

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Many discussions of Gaussian beams emphasize a single electric field component, such as $E_y = f(r,z)\, e^{i(kz-\omega t)}$, of a cylindrically symmetric beam of angular frequency ω and wave number $k = n\omega/c$ propagating along the z axis in a medium with index of refraction n. Here, we generalize to the case of a beam with an elliptical cross section. Of course, the electric field must satisfy the free-space Maxwell equation $\boldsymbol{\nabla} \cdot \mathbf{E} = 0$. If $f(r,z)$ is not constant and $E_x = 0$, then we must have nonzero E_z. That is, the desired electric field has more than one vector component.

To deduce all components of the electric and magnetic fields of a Gaussian beam from a single scalar wave function, we follow the suggestion of Davis [2] and seek solutions for a vector potential $\mathbf{A}$ that has only a single Cartesian component (such that $(\nabla^2 \mathbf{A})_j = \nabla^2 A_j$ [4]). We work in the Lorenz gauge (and SI units), so that the electric scalar potential Φ is related to the vector potential $\mathbf{A}$ by,

$$\boldsymbol{\nabla} \cdot \mathbf{A} = -\frac{n^2}{c^2}\frac{\partial \Phi}{\partial t} = i\frac{n^2\omega}{c^2}\Phi = i\frac{k^2}{\omega}\Phi. \tag{1}$$

The vector potential can therefore have a nonzero divergence, which permits solutions having only a single component.

Of course, the electric and magnetic fields can be deduced from the potentials via,

$$\mathbf{E} = -\boldsymbol{\nabla}\Phi - \frac{\partial \mathbf{A}}{\partial t} = i\frac{\omega}{k^2}\boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \mathbf{A}) + i\omega \mathbf{A}, \tag{2}$$

using the Lorenz condition (1), and,

$$\mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A}. \tag{3}$$

The vector potential satisfies the free-space (Helmholtz) wave equation,

$$\nabla^2 \mathbf{A} - \frac{n^2}{c^2}\frac{\partial^2 \mathbf{A}}{\partial t^2} = (\nabla^2 + k^2)\mathbf{A} = 0. \tag{4}$$

We seek a solution in which the vector potential is described by a single Cartesian component A_j that propagates in the $+z$ direction with the form,

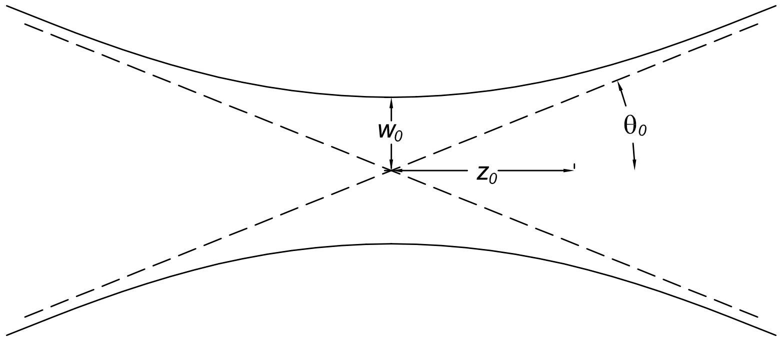
$$A_j(\mathbf{r}) = \psi(\mathbf{r})\, e^{i(kz-\omega t)}. \tag{5}$$

Inserting trial solution (5) into the wave equation (4) we find that,

$$\nabla^2\psi + 2ik\frac{\partial \psi}{\partial z} = 0. \tag{6}$$

In the usual analysis, one now assumes that the beam is cylindrically symmetric about the z axis and can be described in terms of three geometric parameters the diffraction angle θ_0, the waist w_0, and the depth of focus (Rayleigh range) z_0, which are related by,

$$\theta_0 = \frac{w_0}{z_0} = \frac{2}{kw_0}, \qquad \text{and} \qquad z_0 = \frac{kw_0^2}{2} = \frac{2}{k\theta_0^2}. \tag{7}$$



Changing variables and noting relations (7), eq. (6) takes the form,

$$\nabla_\perp^2 \psi + 4i \frac{\partial \psi}{\partial \varsigma} + \theta_0^2 \frac{\partial^2 \psi}{\partial \varsigma^2} = 0, \tag{8}$$

where,

$$\nabla_\perp^2 \psi = \frac{\partial^2 \psi}{\partial \xi^2} + \frac{\partial^2 \psi}{\partial \upsilon^2} = \frac{1}{\rho} \frac{\partial}{\partial \rho} \left(\rho \frac{\partial \psi}{\partial \rho} \right), \tag{9}$$

since ψ is assumed to be independent of the azimuth ϕ.

The form of eq. (8) suggests the series expansion,

$$\psi = \psi_0 + \theta_0^2 \psi_2 + \theta_0^4 \psi_4 + ... \tag{10}$$

in terms of the small parameter θ_0^2. Inserting this into eq. (8) and collecting terms of order θ_0^0 and θ_0^2, we find,

$$\nabla_\perp^2 \psi_0 + 4i \frac{\partial \psi_0}{\partial \varsigma} = 0, \tag{11}$$

and,

$$\nabla_\perp^2 \psi_2 + 4i \frac{\partial \psi_2}{\partial \varsigma} = -\frac{\partial^2 \psi_0}{\partial \varsigma^2}, \tag{12}$$

etc.

1 Zeroth-Order Gaussian Beam

Equation (11) is called the paraxial wave equation, whose solution we obtain by an "educated guess". Namely, we expect the transverse behavior of the wave function ψ_0 to be Gaussian, but with a width that varies with z. Also, the amplitude of the wave should vary with z, asymptotically falling as $1/z$. We work in the scaled coordinates ρ and ς, and write a trial solution as,

$$\psi_0 = h(\varsigma)\, e^{-f(\varsigma)\rho^2}, \tag{13}$$

where the possibly complex functions f and h are defined to obey $f(0) = 1 = h(0)$. Since the transverse coordinate ρ is scaled by the waist w_0, we see that $Re(f) = w_0^2/w^2(\varsigma)$ where $w(\varsigma)$ is the beam width at position ς. From the geometric parameters (6) we see $w(\varsigma) \approx \theta_0 z = w_0 \varsigma$ for large ς. Hence, we expect that $Re(f) \approx 1/\varsigma^2$ for large ς. Also, we expect the amplitude h to obey $|h| \approx 1/\varsigma$ for large ς.

Plugging the trial solution (13) into the paraxial wave equation (11) we find that,

$$-fh + ih' + \rho^2 h(f^2 - if') = 0. \tag{14}$$

For this to be true at all values of ρ we must have,

$$\frac{f'}{f^2} = -i, \qquad \text{and} \qquad \frac{h'}{fh} = -i. \tag{15}$$

We see that $f = h$ is a solution – despite the different physical origin of these two functions as the transverse width and amplitude of the wave. We integrate the first of eq. (15) to obtain,

$$\frac{1}{f} = C + i\varsigma. \tag{16}$$

Our definition $f(0) = 1$ determines that $C = 1$. That is,

$$f = \frac{1}{1 + i\varsigma} = \frac{1 - i\varsigma}{1 + \varsigma^2} = \frac{e^{-i\tan^{-1}\varsigma}}{\sqrt{1 + \varsigma^2}}. \tag{17}$$

Note that $Re(f) = 1/(1 + \varsigma^2) = w_0^2/w^2(\varsigma)$, while $|f| = 1/\sqrt{1 + \varsigma^2}$, so that $f = h$ is in fact consistent with the asymptotic expectations discussed above. The longitudinal dependence of the width of the Gaussian beam is now seen to be,

$$w(\varsigma) = w_0\sqrt{1 + \varsigma^2}. \tag{18}$$

The lowest-order wave function is,

$$\psi_0 = f\, e^{-f\rho^2} = \frac{e^{-i\tan^{-1}\varsigma}}{\sqrt{1 + \varsigma^2}}\, e^{-\rho^2/(1+\varsigma^2)}\, e^{i\varsigma\rho^2/(1+\varsigma^2)}. \tag{19}$$

The factor $e^{-i\tan^{-1}\varsigma}$ in ψ_0 is the so-called Gouy phase shift, which changes from 0 to $\pi/2$ as z varies from 0 to ∞, with the most rapid change near the z_0. For large z the phase factor $e^{i\varsigma\rho^2/(1+\varsigma^2)}$ can be written as $e^{ikr_\perp^2/(2z)}$, recalling eq. (6). When this is combined with the traveling wave factor $e^{i(kz-\omega t)}$ we have,

$$e^{i[kz(1+r_\perp^2/2z^2)-\omega t]} \approx e^{i(kr-\omega t)}, \tag{20}$$

where $r = \sqrt{z^2 + r_\perp^2}$. Thus, the wave function ψ_0 is a modulated spherical wave for large z, but is a modulated plane wave near the focus.

2 Second-Order Gaussian Beam

The solution to eq. (12) for ψ_2 has been given in [2], and that for ψ_4 has been discussed in [3].

In particular,

$$\psi_2 = \left(\frac{f}{2} - \frac{f^3\rho^4}{4}\right)\psi_0 = \left(\frac{f^2}{2} - \frac{f^4\rho^4}{4}\right)e^{-f\rho^2}. \tag{21}$$

We now verify that the form (21) satisfies eq. (12). First,

$$
\begin{aligned}
-\frac{\partial^2 \psi_0}{\partial \varsigma^2} &= -\frac{\partial^2}{\partial \varsigma^2} f\, e^{-f\rho^2} = -\frac{\partial}{\partial \varsigma} f'(1 - f\rho^2)\, e^{-f\rho^2} = i\frac{\partial}{\partial \varsigma}(f^2 - f^3\rho^2)\, e^{-f\rho^2} \\
&= if'[2f - 3f^2\rho^2 - \rho^2(f^2 - f^3\rho^2)]\, e^{-f\rho^2} \\
&= \left(2f^3 - 4f^4\rho^2 + f^5\rho^4\right)\, e^{-f\rho^2},
\end{aligned} \tag{22}
$$

recalling eq. (15). Next,

$$
\begin{aligned}
\frac{\partial^2 \psi_2}{\partial \xi^2} &= \frac{\partial^2}{\partial \xi^2}\left(\frac{f^2}{2} - \frac{f^4\rho^4}{4}\right)\, e^{-f\rho^2} = \frac{\partial}{\partial \xi}\xi\left(-f^3 + \frac{f^5\rho^4}{2} - f^4\rho^2\right)\, e^{-f\rho^2} \\
&= \left[-f^3 + \frac{f^5\rho^4}{2} - f^4\rho^2 + \xi^2\left(4f^5\rho^2 - f^6\rho^4\right)\right]\, e^{-f\rho^2}
\end{aligned} \tag{23}
$$

Hence,

$$
\nabla_\perp^2 \psi_2 = \left(-2f^3 + 5f^5\rho^4 - 2f^4\rho^2 - f^6\rho^4\right)\, e^{-f\rho^2} \tag{24}
$$

Finally,

$$
\begin{aligned}
4i\frac{\partial \psi_2}{\partial \varsigma} &= 4i\frac{\partial}{\partial \varsigma}\left(\frac{f^2}{2} - \frac{f^4\rho^4}{4}\right)\, e^{-f\rho^2} = 4if'\left(f - f^3\rho^4 - \frac{f^2\rho^2}{2} + \frac{f^4\rho^4}{4}\right)\, e^{-f\rho^2} \\
&= \left(4f^3 - 4f^5\rho^4 - 2f^4\rho^2 + f^6\rho^4\right)\, e^{-f\rho^2}
\end{aligned} \tag{25}
$$

Thus,

$$
\nabla_\perp^2 \psi_2 + 4i\frac{\partial \psi_2}{\partial \varsigma} = \left(2f^3 + f^5\rho^4 - 4f^4\rho^2\right)\, e^{-f\rho^2} = -\frac{\partial^2 \psi_0}{\partial \varsigma^2}. \tag{26}
$$

3 Third-Order Electric and Magnetic Fields

To obtain the electric and magnetic fields of a second-order Gaussian beam that is polarized in the y direction,[1] we take the vector potential to be,

$$
A_x = 0, \quad A_y = \frac{E_0}{i\omega}(\psi_0 + \theta_0^2\psi_2)\, e^{i(kz-\omega t)} = \frac{E_0}{i\omega}\left[f + \theta_0^2\left(\frac{f^2}{2} - \frac{f^4\rho^4}{4}\right)\right]\, e^{-f\rho^2}\, e^{i(kz-\omega t)}, \quad A_z = 0. \tag{27}
$$

Then,

$$
i\frac{\omega}{k^2}\boldsymbol{\nabla}\cdot\mathbf{A} = -\frac{E_0\theta_0^2 y}{4}\left[2f^2 + \theta_0^2\left(f^3 - \frac{f^5\rho^4}{4} - f^4\rho^2\right)\right]\, e^{-f\rho^2} e^{i(kz-\omega t)} \approx -\frac{E_0\theta_0^2 y}{2} f^2\, e^{-f\rho^2}\, e^{i(kz-\omega t)}. \tag{28}
$$

[1]Other polarizations are, of course, possible. A vector potential with only an x-component leads to x-polarization, while one with only a z-component leads to radial polarization, as discussed, for example, in secs. 2.4 and 2.5 of [1], respectively.

and the electric field follows from eq. (2) as,

$$
\begin{aligned}
E_x &\approx \theta_0^2 \frac{xy}{w_0^2} f^2 E_{0y}, \\
E_y &\approx \left[1 + \theta_0^2 \left(\frac{f^2 y^2}{w_0^2} - \frac{f^3 \rho^4}{4}\right)\right] E_{0y} \qquad \text{where} \qquad E_{0y} = E_0 f\, e^{-f\rho^2}\, e^{i(kz-\omega t)}, \\
E_z &\approx -i\theta_0 \frac{y}{w_0} \left[f + \theta_0^2 \left(1 - \frac{f\rho^2}{2}\right)\right] E_{0y},
\end{aligned}
\tag{29}
$$

where we neglect terms of order θ_0^4, and note that $f' = -if^2/z_0 = -if^2 k\theta_0^2/2 = -if^2\theta_0/w_0$. Similarly, the magnetic field follows from eq. (3) as,

$$
\begin{aligned}
B_x &= -\left(1 - \theta_0^2 \frac{f^3\rho^4}{4}\right) \frac{n}{c} E_{0y}, \\
B_y &= 0, \\
B_z &= i\theta_0 \frac{x}{w_0} f \left[1 + \theta_0^2 \left(\frac{f}{2} - \frac{f^3\rho^4}{4} + \frac{f^2\rho^2}{2}\right)\right] \frac{n}{c} E_{0y}.
\end{aligned}
\tag{30}
$$

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