# Time Dilation of Moving Clocks as a Consequence of Length Contraction

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# 1 Problem

In an article by Rindler [1], claiming that Einstein (1905) was the first to realize (in sec. 4 of [2]) that observers of moving clocks consider them to run slow,[1] the last footnote reads: *Professor Gerald Holton (private communication) suggests that the " explanation" of length contraction in terms of elementary forces no doubt helped its acceptance, whereas no single such explanation could or can be found for the many and various possible clock mechanisms.*

In what sense is Holton's remark valid?

# 2 Solution

## 2.1 Length Contraction

First, we consider in what sense is the length contraction due to "forces" as many people believe according to Holton.[2]

The length of a rod at rest in an inertial frame is somehow determined by internal, intermolecular, electromagnetic forces, although the total force on each molecule is zero. Equipotential surfaces of a charged particle at rest are spheres, but for a charged particle with uniform velocity $\mathbf{v}$ with respect to an inertial rest frame they oblate spheroids whose axial length is less than its transverse diameter by the factor $1/\gamma = \sqrt{1 - v^2/c^2}$, as first noted by Heaviside [9] (1888). This was first interpreted as leading to a length contraction (by the factor $1/\gamma$) by FitzGerald [10] (1889).[3]

However, it was already understood by Earnshaw in 1839 [13] that electric charges cannot be held in equilibrium by purely electrostatic forces, so the "explanation" of the length contraction as due to the contraction of the equipotential surfaces is only suggestive.[4] To

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[1] In 1897, p. 229 of [3], Larmor argued that "individual electrons [in a molecule with velocity $v$] describe corresponding parts of their orbits in times shorter for the [rest] system [than for the moving system] in the ratio $1 - v^2/2c^2$", and also that "one consequence of the motion thus being that the body is shrunk in the direction of its velocity $v$ in the ratio $1 - v^2/2c^2$". In 1900, sec. 112, p. 176, of [4], he inferred the length contraction of moving, physical objects, but seemed to imply that space itself is not contracted. He also noted that the analysis of moving objects involves a "time variable" whose scale is "enlarged" by the reciprocal of the length-contraction factor. But, he did not clearly associate this "time variable" with the readings of moving clocks.

In honor of these early insights, Bell [5] called the time dilation the "Larmor dilation".

[2] A somewhat skeptical commentary on the "explanation" of length contraction in terms of elementary forces is given in [6]. Recent advocates of this view include [7, 8].

[3] Lorentz (1895) gave a somewhat more explicit version of the "force explanation" of the length contraction in sec. 92 of [11].

[4] An elaboration of this suggestion is given in [14]. See also [15].

go further, one must take a quantum view (as discussed in [12]), which implies that an "explanation" of the length contraction via "forces" in the classical theory of special relativity actually requires quantum theory.

We also digress to note that the universality of the length contraction was not obvious to everyone. For example in 1900, Lord Kelvin commented [16] that the success of the length-contraction hypothesis in explaining the null result of the Michelson-Morley experiment [17] suggests that the length contraction of a moving material is inversely proportional to its stiffness.

This led Morley to repeat his experiment [18] with the rotating platform being made of wood instead of sandstone as in the original. Again, a null result was obtained, which tended to confirm the universality of the length contraction.

## 2.2 Time Dilation

Holton's remark hinges on the difficulty in providing a single, simple, physical model that explains the time dilation of all possible clock mechanisms.

However, already in sec. 2 of [2], Einstein discussed a simple model of a clock as two parallel mirrors with a beam of light that reflects back and forth between them.[5,6] In the rest frame of this clock its period $T$ is,

$$T = \frac{2L}{c}\,, \tag{1}$$

where $L$ is the separation of the mirrors in their rest frame, and $c$ is the speed of light in vacuum.

### 2.2.1 Clock with Motion Parallel to Its Light Beams

When this clock moves with velocity $\mathbf{v}$ parallel to the path of its beam of light (as discussed in sec. 2 of [2]), the transit time (in the lab frame) from the first mirror to the second (in the direction of velocity $\mathbf{v}$) is related by,[7]

$$c\,t_1 = \frac{L}{\gamma} + vt_1, \qquad t_1 = \frac{L}{\gamma\,(c - v)}\,, \tag{2}$$

where $\gamma = 1/\sqrt{1 - v^2/c^2}$ and $L/\gamma$ is the Lorentz-contracted separation of the moving mirrors. Then, the transit time of the reflected beam from the second mirror back to the first is,

$$c\,t_2 = \frac{L}{\gamma} - vt_2, \qquad t_2 = \frac{L}{\gamma\,(c + v)}\,, \tag{3}$$

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[5] This type of clock may have been inspired by Poincaré's discussion of clock synchronization via light beams, sec. 3, point 2 of [19] (1900).

[6] While a light clock can be considered to indicate proper time in an inertial frame, this is not the case for an accelerated light clock [20].

[7] It is important to note that the speed of the light beams relative to the lab frame is $c$ for both the beam emitted by the first (moving) mirror and reflected by the second (moving) mirror.

so the period $T'$ of the moving clock according to observers in the lab frame is,

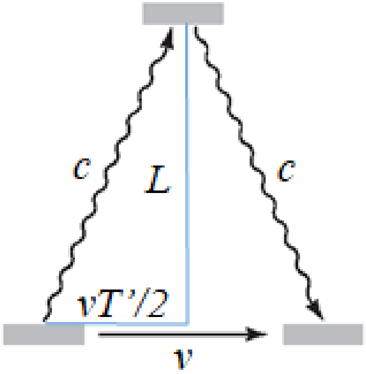
$$T' = t_1 + t_2 = \frac{L}{\gamma}\left(\frac{1}{c-v} + \frac{1}{c+v}\right) = \frac{2Lc}{\gamma\,(c^2 - v^2)} = \frac{2L\gamma}{c} = \gamma T > T. \tag{4}$$

Thus, the time-dilation factor $\gamma$ can be considered as a consequence of length contraction (and also of the universality of the speed of light), for this particular type of clock.[8]

However, those, like Lord Kelvin, who seek a "mechanical" explanation for every physical phenomenon may not immediately accept the universality of the time dilation for moving clocks.[9]

### 2.2.2 Clock with Motion Perpendicular to Its Light Beams

We can also consider the case in which the clock has velocity $\mathbf{v}$ perpendicular to the direction of its light beams when at rest, as show in the figure below (from [7]).



Here, the path of the light beam from the first mirror to the second and back has length $cT'$ in the lab frame, while the first (bottom) mirror travels distance $vT'$, Then,

$$cT' = 2\sqrt{L^2 + (vT'/2)^2}, \qquad T' = \gamma\frac{2L}{c} = \gamma T, \tag{5}$$

as in the previous case.

For this case, we did not need to consider a length contraction, but only that the speed of light is the same in the lab frame and in the rest frame of the clock.[10,11]

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[8] Einstein did not explicitly display the time-dilation factor $\gamma$ in [2], but only reported it to second order in $v/c$ as $1 + v^2/2c^2$.

[9] In contrast, an Einsteinian view is that clocks if clocks of various physical construction all run at the same rate in one inertial frame, then they will all be observed to run at the same rate in any inertial frame. Hence, an understanding of the "mechanism" of time dilation for one type of clock suffices as a kind of understanding of the time dilation for all types of clocks.

[10] An interesting variation of the light clock is discussed in [21], which has four mirrors at the corners of a square. Comparison of such a clock in motion with its behavior at rest permits one to deduce both the length contraction and the time dilation, independently of one another.

[11] It was noted in [22, 23] that if one accepts the time dilation, then the argument of sec. 2.1 above can be inverted to deduce the length contraction.

# A Appendix: Time Dilation of a Spring Clock

A clock could be based on a mass $m$ connected to a "fixed point" by a spring of constant $k$.

In the (inertial) rest frame of the "fixed point", the period of this clock is,

$$T = 2\pi\sqrt{\frac{m}{k}}\,, \tag{6}$$

ignoring the tiny correction that as the oscillating mass $m$ moves with velocity $\mathbf{u}(t)$ respect to this frame, its "relativistic mass" is $m/\sqrt{1 - u^2/c^2}$, which slightly increases its period.

We now consider this clock as observed in an inertial frame that has velocity $\mathbf{v}$ with respect to the rest frame of the "fixed point" of the clock. For simplicity, we suppose that $v \gg u$max, where $u_{\rm max}$ is the maximum speed of mass $m$ with respect to the "fixed point". We also suppose that $\mathbf{v}$ is perpendicular to the direction of motion of mass $m$ in the rest frame of the "fixed point"

In this frame (the $'$ frame), the relativistic mass is $m' = \gamma m$, where $\gamma = 1/\sqrt{1 - v^2/c^2}$, while the spring constant remains $k$, so the period of the clock is,

$$T' = 2\pi\sqrt{\frac{m'}{k}} = \frac{T}{\sqrt{1 - v^2/c^2}} = \gamma\, T > T, \tag{7}$$

again ignoring the small variation with time in the oscillating relativistic mass.

This result is satisfactory, but one might be uncomfortable with the neglect of the small variation with time in the oscillating relativistic mass.

*A similar analysis could be made for a clock based on a torsional spring (with neglect, for simplicity, of the small variation in time of the moment of inertia of the oscillating relativistic mass).*

# B Appendix: Pendulum Clock

The period of a pendulum clock depends on the acceleration due to gravity, which should be described by general, rather than special, relativity.[12]

As discussed in [24], the acceleration due to gravity of a moving object at the Earth's surface depends on one's choice of coordinates, and on the direction of the motion. A moving pendulum clock at the Earth's surface does run at a different rate (according to observers at rest on the Earth), but not at the rate implied by a naïve application of special relativity.

# References

[1] W. Rindler, *Einstein's Priority in Recognizing Time Dilation Physically*, Am. J. Whys. **38**, 1111 (1970), `http://kirkmcd.princeton.edu/examples/GR/rindler_ajp_38_1111_70.pdf`

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[12] The footnote at the end of sec. 4 of the English translation of [2], which excludes consideration of pendulum clocks, was not in the German original.

[2] A. Einstein, *Zur Elektrodynamik bewegter Körper*, Ann. d. Phys. **17**, 891 (1905),
http://kirkmcd.princeton.edu/examples/EM/einstein_ap_17_891_05.pdf
http://kirkmcd.princeton.edu/examples/EM/einstein_ap_17_891_05_english.pdf

[3] J. Larmor, *A Dynamical Theory of the Electric and Luminiferous Medium. — Part III. Relations with Material Media*, Phil. Trans. Roy. Soc. London A **190**, 205 (1897),
http://kirkmcd.princeton.edu/examples/EM/larmor_ptrsla_190_205_97.pdf

[4] J. Larmor, *Aether and Matter* (Campridge U. Press, 1900),
http://kirkmcd.princeton.edu/examples/EM/larmor_aether_matter_00.pdf

[5] J.S. Bell, *George Francis FitzGerald*, Phys. World **5**(9), 31 (1992),
http://kirkmcd.princeton.edu/examples/GR/bell_pw_5-9_31_92.pdf

[6] E.L. Feĭnberg, *Can the relativistic change in the scales of length and time be considered the result of the action of certain forces?* Sov. Phys. Usp. **18**, 624 (1975),
http://kirkmcd.princeton.edu/examples/GR/feinberg_spu_18_624_75.pdf

[7] W.M. Nelson, *Special relativity from the dynamical viewpoint*, Am. J. Phys. **83**, 600 (2015), http://kirkmcd.princeton.edu/examples/GR/nelson_ajp_83_600_15.pdf

[8] D.V. Peregoudov, *Relativistic length contraction and time dilation as dynamical phenomena*, Eur. J. Phys. **41**, 015602 (2020),
http://kirkmcd.princeton.edu/examples/GR/peregoudov_ejp_41_015602_20.pdf

[9] O. Heaviside, *On the Electromagnetic Effects due to the Motion of Electrification through a Dielectric*, Phil. Mag. **27**, 324 (1889),
http://kirkmcd.princeton.edu/examples/EM/heaviside_pm_27_324_89.pdf

[10] G.F. FitzGerald, *The Ether and the Earth's Atmosphere*, Science **13**, 390 (1889),
http://kirkmcd.princeton.edu/examples/EM/fitzgerald_science_13_390_89.pdf

[11] H.A. Lorentz, *Versuch einser Theorie der Electrischen und Optischen Erscheinungen in Bewegten Körpern* (Brill, 1985),
http://kirkmcd.princeton.edu/examples/EM/lorentz_versuch_95.pdf
http://kirkmcd.princeton.edu/examples/EM/lorentz_versuch_95_english.pdf

[12] W.F.G. Swann, *Relativity, the Fitzgerald-Lorentz Contraction, and Quantum Theory*, Rev. Mod. Phys. **13**, 197 (1941), http://kirkmcd.princeton.edu/examples/GR/swann_rmp_13_197_41.pdf

[13] S. Earnshaw, *On the Nature of the Molecular Forces which regulate the Constitution of the Luminiferous Ether*, Trans. Camb. Phil. Soc. **7**, 97 (1839),
http://kirkmcd.princeton.edu/examples/EM/earnshaw_tcps_7_97_39.pdf

[14] R.A. Sorenson, *Lorentz contraction: A real change of shape*, Am. J. Phys. **63**, 413 (1995), http://kirkmcd.princeton.edu/examples/GR/sorenson_ajp_63_413_95.pdf

[15] D.V. Redžić, *Direct Calculation of Length Contraction and Clock Retardation*, Serb. Astron. J. **190**, 49 (2015), http://kirkmcd.princeton.edu/examples/GR/redzic_saj_190_49_15.pdf

[16] Lord Kelvin, *On the Motion produced in an Infinite Elastic Solid by the Motion through the Space occupied by it of a body acting on it only by Attraction or Repulsion*, Phil. Mag. **50**, 181 (1900), `http://kirkmcd.princeton.edu/examples/GR/kelvin_pm_50_181_00.pdf`

[17] A.A. Michelson and E.W. Morley, *Influence of the Motion of the Medium on the Velocity of Light*, Am. J. Sci. **31**, 377 (1886),
`http://kirkmcd.princeton.edu/examples/GR/michelson_ajs_31_377_86.pdf`
*On the Relative Motion of the Earth and the Luminiferous Æther*, Phil. Mag. **24**, 449 (1887), `http://kirkmcd.princeton.edu/examples/GR/michelson_pm_24_449_87.pdf`

[18] E.W. Morley and D.C. Miller, *Extract from a Letter dated Cleveland, Ohio, August 5th, 1904, to Lord Kelvin*, Phil. Mag. **8**, 753 (1904),
`http://kirkmcd.princeton.edu/examples/GR/morley_pm_8_753_04.pdf`
*Report of an Experiment to detect the FitzGerald-Lorentz Effect*, Phil. Mag. **9**, 680 (1905), `http://kirkmcd.princeton.edu/examples/GR/morley_pm_9_680_05.pdf`

[19] H. Poincaré, *La Théorie de Lorentz et la Principe de Réaction*, Arch. Neer. **5**, 252 (1900), `http://kirkmcd.princeton.edu/examples/EM/poincare_an_5_252_00.pdf`
Translation: *The Theory of Lorentz and the Principle of Reaction*,
`http://kirkmcd.princeton.edu/examples/EM/poincare_an_5_252_00_english.pdf`

[20] F.V. Kowalski, *Accelerating light clocks*, Phys. Rev. A **53**, 3761 (1996),
`http://kirkmcd.princeton.edu/examples/GR/kowalski_pra_53_3761_96.pdf`

[21] J.R. Galli and F. Amiri, *The Square Light Clock and Special Relativity*, Phys. Teach. **50**, 212 (2012), `http://kirkmcd.princeton.edu/examples/GR/galli_pt_50_212_12.pdf`

[22] J.M. Lévy, *A simple derivation of the Lorentz transformation and of the accompanying velocity and acceleration changes*, Am. J. Phys. **75**, 615 (2007),
`http://kirkmcd.princeton.edu/examples/GR/levy_ajp_75_615_07.pdf`

[23] F. Behroozi, *A Simple Derivation of Time Dilation and Length Contraction in Special Relativity*, Phys. Teach. **52**, 410 (2014),
`http://kirkmcd.princeton.edu/examples/GR/behroozi_pt_52_410_14.pdf`

[24] K.T. McDonald, *Gravitational Acceleration of a Moving Object at the Earth's Surface* (Apr. 4, 2016), `http://kirkmcd.princeton.edu/examples/gravity_moving.pdf`