“Hidden” Momentum of an Electric Dipole in a Magnetic Field

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1 Problem

Consider an electric dipole $\mathbf{d} = q\mathbf{l}$,[1] with electric charges $\pm q$ separated by distance $\mathbf{l}$ where $l = |\mathbf{l}|$ is constant, in an external magnetic field $\mathbf{B}$ that is quasistatic. First discuss the electromagnetic-field momentum supposing the dipole is at rest, and then consider that the dipole is slowly moving.

2 Solution

This problem is closely related to the case of a capacitor in a magnetic field, which is discussed in [1]. An electric dipole in a magnetic field is the electromagnetic dual of a magnetic dipole (current loop) in an electric field, which is discussed in [2].

2.1 Electric Dipole at Rest in a Static, Uniform Magnetic Field

An electric dipole at rest (in an inertial frame) in a static, uniform magnetic field is an example of a system “at rest”, for which the total 3-momentum of the system is zero.[2] However, the electromagnetic-field momentum of this system is nonzero.

The contemporary view of (classical) electromagnetic-field momentum dates back to discussion in 1891 by J.J. Thomson,[3] that the “field-only” momentum[4] is (in Gaussian units)

$$\mathbf{P}_{\rm EM} = \int \frac{\mathbf{E} \times \mathbf{B}}{4\pi c}\, d{\rm Vol}, \tag{1}$$

where $\mathbf{E}$ is the electric field, $\mathbf{B}$ is the magnetic field and c is the speed of light in vacuum. In quasistatic examples, such as the present case, alternative expressions for the electromagnetic-field momentum can be given [7], including the version favored by Maxwell,

$$\mathbf{P}_{\rm EM}^{\rm (M)} = \int \frac{\rho \mathbf{A}^{\rm (C)}}{c}\, d{\rm Vol}, \tag{2}$$

[1]Electric dipoles are often represented by the symbol $\mathbf{p}$, but in this note there is emphasis on 3-momentum, which we will denote by the vector $\mathbf{p}$ in accordance with common usage, perhaps derived from the Greek *pteron* ($\pi\tau\epsilon\rho o\nu$, wing) or *pero* ($\pi\epsilon\rho\omega$, to fly).

[2]See, for example, Appendix A of [3], which is based on discussion in [4].

[3]See, for example, Sec. 1 of [5], and references therein.

[4]When electromagnetic media are involved, one considers the electromechanical fields $\mathbf{D}$ and $\mathbf{H}$, which leads to “perpetual” debates as to whether $\mathbf{E} \times \mathbf{B}$ should be replaced by $\mathbf{E} \times \mathbf{H}$ or by $\mathbf{D} \times \mathbf{B}$. See, for example, [6].

where ρ is the density of electric charge and $\mathbf{A}^{(\mathrm{C})}$ is the magnetic vector potential in the Coulomb gauge (where $\boldsymbol{\nabla}\cdot\mathbf{A}^{(\mathrm{C})}=0$).

For the present example, we consider a static, uniform magnetic field $\mathbf{B}=B_0\,\hat{\mathbf{z}}$ whose source is axially symmetric about the z-axis, such that the Coulomb-gauge vector potential can be written as[5]

$$\mathbf{A}^{(\mathrm{C})}=\frac{\mathbf{B}\times\mathbf{x}}{2}\,. \tag{5}$$

Then, for an electric dipole $\mathbf{d}=q\mathbf{l}$ with charges $\pm q$ at $\mathbf{x}_0\pm\mathbf{l}/2$, the electromagnetic-field momentum is[6,7]

$$\mathbf{P}_{\mathrm{EM}}^{(\mathrm{M})}=\frac{1}{2c}qB_0\,\hat{\mathbf{z}}\times\left(\mathbf{x}_0+\frac{\mathbf{l}}{2}\right)-\frac{1}{2c}qB_0\,\hat{\mathbf{z}}\times\left(\mathbf{x}_0-\frac{\mathbf{l}}{2}\right)=\frac{B_0\,\hat{\mathbf{z}}\times q\mathbf{l}}{2c}=\frac{\mathbf{B}\times\mathbf{d}}{2c}\,. \tag{6}$$

As the total momentum of a system at rest is zero, the present system must contain a mechanical momentum equal and opposite to that of eq. (6), which has come to be called a "hidden" momentum (following [8]),

$$\mathbf{P}_{\mathrm{hidden}}=-\frac{\mathbf{B}\times\mathbf{d}}{2c}=\frac{\mathbf{d}\times\mathbf{B}}{2c}\,. \tag{7}$$

The "hidden" mechanical momentum[8] does not reside in the electric dipole, but in the conduction current that is the source of the magnetic field $\mathbf{B}$. This is a "relativistic" effect associated with the variation with position of the relativistic momentum $\gamma m\mathbf{v}$ of the charge carriers as affected by the electric field of the electric dipole. See, for example, [2].

Such "hidden" mechanical momentum can also exist in systems with nonzero total momentum, where it is typically negligible compared to the total momentum.

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[5] We confirm that for a uniform magnetic field $\mathbf{B}$,

$$\boldsymbol{\nabla}\cdot\mathbf{A}^{(\mathrm{C})}=\boldsymbol{\nabla}\cdot\frac{\mathbf{B}\times\mathbf{x}}{2}=\frac{\mathbf{B}}{2}\cdot\boldsymbol{\nabla}\times\mathbf{x}-\frac{\mathbf{x}}{2}\cdot\boldsymbol{\nabla}\times\mathbf{B}=0, \tag{3}$$

and

$$\mathbf{B}=\boldsymbol{\nabla}\times\mathbf{A}^{(\mathrm{C})}=\boldsymbol{\nabla}\times\frac{\mathbf{B}\times\mathbf{x}}{2}=\frac{\mathbf{B}}{2}\boldsymbol{\nabla}\cdot\mathbf{x}-\frac{\mathbf{x}}{2}\boldsymbol{\nabla}\cdot\mathbf{B}+(\mathbf{x}\cdot\boldsymbol{\nabla})\mathbf{B}-(\mathbf{B}\cdot\boldsymbol{\nabla})\mathbf{x}=\frac{3B_0\,\hat{\mathbf{z}}}{2}-0+0-\frac{B_0\,\hat{\mathbf{z}}}{2}$$
$$=B_0\,\hat{\mathbf{z}}=\mathbf{B}. \tag{4}$$

[6] The result (6) was confirmed in Sec. 2.4 of [1] using eq. (1) for the special case of a spherical capacitor inside a spherical magnet, for which the computation of $\int\mathbf{E}\times\mathbf{B}\,d\mathrm{Vol}$ is tractable. See also Sec. IV of [9].

[7] The electromagnetic-field momentum of a magnetic dipole $\mathbf{m}$ in an external electric field $\mathbf{E}$ was computed using both our eqs. (1) and (2) by J.J. Thomson [10, 5] to be $\mathbf{P}_{\mathrm{EM}}=\mathbf{E}\times\mathbf{m}/c$. This case might seem to be the electromagnetic dual of an electric dipole in an external magnetic field, which casts doubt on the factor of 2 in the denominator of eq. (6). However, the duality transformation takes electric charges and currents into magnetic charges and currents, which latter do not exist so far as we know. Hence, we cannot use the duality transform of Thomson's result to infer that the field momentum of an electric dipole (based on electric charges $\pm q$) in a magnetic field (based on electric currents as per Ampère, and not on magnetic charges) would be $\mathbf{B}\times\mathbf{d}/c$.

[8] As discussed in [3], the electromagnetic field momentum (1) can also be called a "hidden" momentum.

2.2 The System is in Quasistatic Motion

We now suppose the system of electric dipole $\mathbf{d} = q\mathbf{l}$ plus external magnetic field $\mathbf{B}$ includes motion of the dipole, space and time dependence of the magnetic field, as well as a possible external electric field $\mathbf{E}$, where the motion is quasistatic such that radiation can be ignored.

We begin by consideration of the (Lorentz) force on the dipole (using an argument due to Bruno Klajn, private commnication),

$$\begin{aligned}\mathbf{F}_{\rm Lorentz} = q\left(\mathbf{E}_+ + \frac{\mathbf{v}_+}{c}\times\mathbf{B}_+\right) - q\left(\mathbf{E}_- + \frac{\mathbf{v}_-}{c}\times\mathbf{B}_-\right) &= q(\mathbf{l}\cdot\boldsymbol{\nabla})\left(\mathbf{E} + \frac{\mathbf{v}}{c}\times\mathbf{B}\right) + \frac{q}{c}\frac{d\mathbf{l}}{dt}\times\mathbf{B} \\ &= (\mathbf{d}\cdot\boldsymbol{\nabla})\left[\mathbf{E} + \left(\frac{\mathbf{v}}{c}\times\mathbf{B}\right)\right] + \frac{1}{c}\frac{d\mathbf{d}}{dt}\times\mathbf{B} \\ &= \boldsymbol{\nabla}\left[\mathbf{d}\cdot\mathbf{E} + \mathbf{d}\cdot\left(\frac{\mathbf{v}}{c}\times\mathbf{B}\right)\right] - \mathbf{d}\times\left[\boldsymbol{\nabla}\times\left(\mathbf{E} + \frac{\mathbf{v}}{c}\times\mathbf{B}\right)\right] + \frac{1}{c}\frac{d\mathbf{d}}{dt}\times\mathbf{B} \\ &= \boldsymbol{\nabla}\left[\mathbf{d}\cdot\mathbf{E} + \left(\mathbf{d}\times\frac{\mathbf{v}}{c}\right)\cdot\mathbf{B}\right] - \mathbf{d}\times\left[-\frac{1}{c}\frac{\partial\mathbf{B}}{\partial t} + \frac{\mathbf{v}}{c}(\boldsymbol{\nabla}\cdot\mathbf{B}) - \left(\frac{\mathbf{v}}{c}\cdot\boldsymbol{\nabla}\right)\mathbf{B}\right] + \frac{1}{c}\frac{d\mathbf{d}}{dt}\times\mathbf{B} \\ &= \boldsymbol{\nabla}(\mathbf{d}\cdot\mathbf{E} + \mathbf{m}\cdot\mathbf{B}) + \frac{\mathbf{d}}{c}\times\frac{d\mathbf{B}}{dt} + \frac{1}{c}\frac{d\mathbf{d}}{dt}\times\mathbf{B} \quad = \quad -\boldsymbol{\nabla}U_{\rm int} + \frac{d}{dt}\left(\frac{\mathbf{d}}{c}\times\mathbf{B}\right), \end{aligned} \tag{8}$$

where $\mathbf{E}$, $\mathbf{B}$ and $\mathbf{v}$ refer to quantities at the center of the electric dipole $\mathbf{d}$, the apparent magnetic moment of a moving electric dipole is $\mathbf{m} = \mathbf{d}\times\mathbf{v}/c$ is to lowest order in v/c,[9] the convective derivative is $d\mathbf{B}/dt = \partial\mathbf{B}/\partial t + (\mathbf{v}\cdot\boldsymbol{\nabla})\mathbf{B}$, the interaction energy of an electric dipole $\mathbf{d}$ in an external electric field $\mathbf{E}$ and of a magnetic dipole $\mathbf{m}$ in an external magnetic field $\mathbf{B}$ is $U_{\rm int} = -\mathbf{d}\cdot\mathbf{E} - \mathbf{m}\cdot\mathbf{B}$, and we presume that $\boldsymbol{\nabla}$ does not act on the length $\mathbf{l}$ of the electric dipole.

We also have that the force on the electric dipole is the time rate of change of the total (mechanical) momentum, $\mathbf{P}_{\mathbf{d}} = m\mathbf{v} + \mathbf{P}_{\rm hidden}$, of the dipole,

$$\frac{d\mathbf{P}_{\mathbf{d}}}{dt} = \mathbf{F}_{\rm Lorentz} = -\boldsymbol{\nabla}U_{\rm int} + \frac{d}{dt}\left(\frac{\mathbf{d}}{c}\times\mathbf{B}\right), \tag{9}$$

$$\frac{d}{dt}\left(\mathbf{P}_{\mathbf{d}} + \frac{\mathbf{B}\times\mathbf{d}}{c}\right) = -\boldsymbol{\nabla}U_{\rm int} \approx \frac{d}{dt}\left(\mathbf{P}_{\mathbf{d}} + \mathbf{P}_{\rm EM} - \mathbf{P}_{\rm hidden}\right) = \frac{d}{dt}\left(m\mathbf{v} + \mathbf{P}_{\rm EM}\right), \tag{10}$$

in the approximation that the electromagnetic-field momentum and "hidden" mechanical momentum are the same as for the system at rest and in quasistatic motion.

The interpretation of eq. (10) is awkward. Neither the electromagnetic field momentum $\mathbf{P}_{\rm EM}$ nor the "hidden" mechanical momentum $\mathbf{P}_{\rm hidden}$ are solely properties of the electric dipole $\mathbf{d}$; rather, they are properties of the system as a whole. Klajn proposes that the term $\mathbf{B}\times\mathbf{d}/c$ be called a "potential momentum", but that name is more commonly associated with the electromagnetic-field momentum $\mathbf{P}_{\rm EM}$,[10] which is $\approx \mathbf{B}\times\mathbf{d}/2c$ in the present example.[11] Note that the last form of eq. (10) is consistent with calling $\mathbf{P}_{\rm EM}$ a "potential momentum".

[9] See, for example, [11].

[10] For a review of "potential momentum", see, for example, Sec. III of [12].

[11] That the field momentum in the quasistatic case is not necessarily $\mathbf{B}\times\mathbf{d}/2c$ is discussed in [13, 14].

For what it's worth, we can write a Lagrangian for the "particle" with mass m and velocity $\mathbf{v}$ that interacts with external electric field $\mathbf{E}$ and magnetic field $\mathbf{B}$ when $v \ll c$ as[12]

$$\mathcal{L} = \frac{mv^2}{2} - U_{\rm int} = \frac{mv^2}{2} + \mathbf{d} \cdot \mathbf{E} + \mathbf{m} \cdot \mathbf{B} = \frac{mv^2}{2} + \mathbf{d} \cdot \left(\mathbf{E} + \frac{\mathbf{v}}{c} \times \mathbf{B} \right). \tag{11}$$

Then,

$$\frac{\partial \mathcal{L}}{\partial \mathbf{v}} = m\mathbf{v} + \frac{\mathbf{B} \times \mathbf{d}}{c} = \mathbf{P_d} + \frac{\mathbf{B} \times \mathbf{d}}{c}, \qquad \frac{\partial \mathcal{L}}{\partial \mathbf{x}} = -\boldsymbol{\nabla} U_{\rm int}, \tag{12}$$

such that $(d/dt)(\partial \mathcal{L}/\partial \mathbf{v}) = \partial \mathcal{L}/\partial \mathbf{x}$ leads to eq. (10). We recognize $\partial \mathcal{L}/\partial \mathbf{v} = \mathbf{P_d} + \mathbf{B} \times \mathbf{d}/c$ as the canonical momentum of this example, which differs from the mechanical momentum $\mathbf{P_d}$ of the electric dipole by the term $\mathbf{B} \times \mathbf{d}/c$, which happens to be twice the (approximate) electromagnetic-field momentum (6) of the system. There is no general name for such differences between a canonical momentum and a mechanical momentum.

Some controversy about the quasistatic case of the present example arose in 2014 via an eprint by Hu [14]. The author made some remarks on this in footnotes 6, 8, 10 and Appendix B.2 of [1].

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