Motion of a Point Charge near an Electric Dipole

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1 Problem

Consider a point particle of mass m and electric charge q moving in the field of a point electric dipole $\mathbf{p} = p\,\hat{\mathbf{z}}$. Find the orbits for which the motion is at a constant radius, and comment on the stability of such motion.

2 Solution

We give two approaches, $F = ma$, and Lagrange. Aspects of this problem have been discussed in [1].[1,2]

2.1 Newtonian Method

We consider a particle of mass m and electric charge q moving in the field of an electric dipole $\mathbf{p} = p\,\hat{\mathbf{z}}$. The electric field (in Gaussian units) in polar coordinates (r, θ, ϕ) is,

$$\mathbf{E} = \frac{p}{r^3}(3\cos\theta\,\hat{\mathbf{r}} - \hat{\mathbf{z}}) = \frac{p}{r^3}(2\cos\theta\,\hat{\mathbf{r}} + \sin\theta\hat{\theta}) = \frac{p}{r^3}[3\cos\theta\sin\theta\,\hat{\boldsymbol{\rho}} + (3\cos^2\theta - 1)\,\hat{\mathbf{z}}], \qquad (1)$$

where $\hat{\boldsymbol{\rho}}$ is a unit vector perpendicular to the z axis. The torque $\mathbf{N}$ on the charge is,

$$\mathbf{N} = \mathbf{r} \times q\mathbf{E} = -\frac{pq}{r^2}(\hat{\mathbf{r}} \times \hat{\mathbf{z}}). \qquad (2)$$

The torque vanishes only along the z axis, so only the z-component of angular momentum, L_z, is conserved. Hence, the present problem has some features in common with the spherical pendulum: the main class of orbits is that which lie in planes perpendicular to the z axis, but there is a special case when $L_z = 0$, for which the motion is confined to a plane containing the z-axis.

For an orbit to be in a plane perpendicular to the z axis, the field of the dipole must lie in this plane also. This is not the case in general, but we see from the about expression for $\mathbf{E}$ that at angles such that $\cos^2\theta = 1/3$ this condition is satisfied. The corresponding angles are $\theta_0 = 54.74°$ and $125.26°$. For positive p and q, only the case of $\theta_0 = 125.26°$ corresponds to an inward force, and so is the only possibility for an orbit.

To learn more, we examine $\mathbf{F} = m\mathbf{a}$ for circular motion in an orbit of radius ρ_0 at polar angle θ_0,

$$\frac{mv_0^2}{\rho_0} = qE_\rho = \frac{\sqrt{2}pq}{\rho_0^3}. \qquad (3)$$

[1] In the quantum realm, bound states of a point electric dipole plus point electric charge exist, but only for dipole moments larger than a critical value. For reviews, see [2, 3]. See also [4].

[2] The motion of an electric magnetic in the field of a magnetic dipole is discussed in [5].

Hence,

$$mv_0^2\rho_0^2 = \sqrt{2}pq. \tag{4}$$

However, the conserved angular momentum about the z axis is $L_z = mv_0\rho_0 = \sqrt{\sqrt{2}pq/m}$ which is independent of ρ_0. This peculiar result suggests that the orbits are unstable, in that no particular value of the radius is determined by the constants of the motion.

This insight can be reinforced by consideration of the total energy,

$$U = \frac{1}{2}mv^2 + \frac{pq\cos\theta}{r} = \frac{1}{2}mv^2 + \frac{pq\cos\theta\sin\theta}{\rho}. \tag{5}$$

But for the proposed orbit we find $U = 0$! That is, the motion is not bound, but would wander from one radius to another under the slightest perturbation.

Experts will note that the special case of $L_z = 0$ deserves further attention. Here, we should consider circular motion in a plane containing the z axis, for which θ can vary. For motion on a circle of radius r_0, the radial component of $\mathbf{F} = m\mathbf{a}$ tells us that,

$$\frac{mv^2}{r_0} = qE_r = \frac{2pq\cos\theta}{r_0^3}. \tag{6}$$

The force is inward only for $pq\cos\theta < 0$, so the only hope for a stable motion is for $\pi/2 \leq \theta \leq \pi$ when p and q are positive. We see that the velocity v goes to zero at $\theta = \pi/2$ (and $3\pi/2$) so the motion lies along a half circle – no matter what the maximum velocity is!

Again an evaluation of total energy U shows that it is zero, and the motion is unstable.

There is no new form of matter to be expected from charged particles bound by dipoles!

2.2 Lagrange's Method

The potential energy for the charge-plus-dipole system is,

$$V = \frac{pq\cos\theta}{r^2}, \tag{7}$$

and so the equations of motion are obtained via Lagrange's equations as,

$$mr^2\sin^2\theta\dot{\phi} = L_z = \text{ constant}, \tag{8}$$

$$\ddot{r} = r\dot{\theta}^2 + r\sin^2\theta\dot{\phi}^2 + \frac{2pq\cos\theta}{mr^3} = r\dot{\theta}^2 + \frac{L_z^2}{m^2r^3\sin^2\theta} + \frac{2pq\cos\theta}{mr^3}, \tag{9}$$

$$\frac{d}{dt}(r^2\dot{\theta}) = r^2\sin\theta\cos\theta\dot{\phi}^2 + \frac{pq\sin\theta}{mr^2} = \frac{L_z^2\cos\theta}{m^2r^2\sin^3\theta} + \frac{pq\sin\theta}{mr^2}, \tag{10}$$

using conservation of angular momentum about the z axis to eliminate $\dot{\phi}$ from the 2nd and 3rd equations of motion.

We first seek an "orbit" for which $\dot{r} = 0 = \dot{\theta}$ but $\dot{\phi} \neq 0$ (*i.e.*, $L_z \neq 0$). Let r_0 and θ_0 be the constant values of radius and polar angle on the orbit. Then, eq. (9) tells us that,

$$\sin^2\theta_0\cos\theta_0 = -\frac{L_z^2}{2mpq}, \tag{11}$$

and eq. (10) implies,

$$\frac{\sin^4\theta_0}{\cos\theta_0} = -\frac{L_z^2}{mpq}. \tag{12}$$

These two relations combine to yield,

$$\cos\theta_0 = \pm\frac{\sqrt{3}}{3}, \qquad \text{so} \qquad \theta_0 = 54.74^\circ \text{ or } 125.26^\circ. \tag{13}$$

At angle θ_0, the dipole field lines are perpendicular to the z axis, as can be seen from the expression (1) for the field. This permits circular orbits in planes perpendicular to the z axis. However, there is no preferred radius r_0 for orbits of a given angular momentum L_z. That is, any r_0 is possible and the motion is unstable against a change in radius!

We can see this another way by considering the (conserved) total energy of the system,

$$U = \frac{m}{2}(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\dot{\phi}^2) + \frac{pq\cos\theta}{r^2}. \tag{14}$$

For the orbit with $\dot{r} = 0 = \dot{\theta}$ we then have,

$$U = \frac{L_z^2}{2mr_0^2\sin^2\theta_0} + \frac{pq\cos\theta_0}{r_0^2}, \tag{15}$$

However, on comparing with eq. (9), we recognize that the total energy U vanishes for orbits with $\dot{r} = 0 = \dot{\theta}$. That is, the system is not bound, and any slight perturbation in the energy causes arbitrarily large change in the radius.

A mathematical curiosity is that the motion is stable against perturbation in the $\hat{\theta}$ direction if L_z is not too large. To see this, consider motion in which $\dot{r} = 0$, but $\dot{\theta} \neq 0$ as well as $\dot{\phi} \neq 0$. Then, we quickly find that,

$$\dot{\theta}^2 = -\frac{L_z^2}{m^2r_0^4\sin^2\theta} - \frac{2pq\cos\theta}{mr_0^4}. \tag{16}$$

Since $\dot{\theta}^2$ must be positive definite, we see that for $pq > 0$ we can only have $\cos\theta < 0$. The motion is oscillatory in θ between angles that satisfy,

$$\sin^2\theta\cos\theta = -\frac{L_z^2}{2mpq}. \tag{17}$$

If,

$$\frac{L_z^2}{2mpq} < \frac{2\sqrt{3}}{9} = 0.3849, \tag{18}$$

then there are two roots to the transcendental equation. The "central" angle of the motion is $\theta_0 = 125.26^\circ$, as holds for the case that $\dot{\theta} = 0$ always. (Motion about $\theta_0 = 54.74^\circ$ is unstable for $pq > 0$.) As $L_z \to 0$ the turning points approach $\theta = \pi/2$ and π, and $\dot{\phi} \to 0$. The motion consists of large oscillations in θ combined with a slow precession about the z axis. This behavior is unusual in that there is no option for small oscillations in θ as $\dot{\phi} \to 0$.

Although the particle cannot reach the z axis so long as $L_z > 0$, the motion for very small L_z looks a lot like pendulum motion.

We also consider the case that $L_z = 0$ ($\Rightarrow \dot{\phi} = 0$) along with $\dot{r} = 0$. Then, one finds from the 2nd equation of motion that,

$$\dot{\theta}^2 = -\frac{2pq\cos\theta}{mr_0^4}. \tag{19}$$

For $pq > 0$ we can again only have $\cos\theta < 0$. The motion can reach the $-z$ axis, passing through it at maximum $\dot{\theta}$. The turning points of the orbit are at $\theta = \pi/2$ and $\theta = 3\pi/2$ (slightly stretching the meaning of the polar angle θ to accommodate the present case). The motion is like that of a pendulum with angular amplitude $\pi/2$, and lies along a great semicircle centered on the $-z$ axis.

This motion could be considered stable against perturbation in the $\hat{\theta}$ or $\hat{\phi}$ directions. Such perturbations would change angular momentum L_z from a zero to a nonzero value, and we found above that for small L_z the motion is like a precessing pendulum.

But like the case of $L_z > 0$, the total energy U vanishes for this orbit, so it is unstable against a radial (or energy) perturbation.

References

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