A Paradox Concerning the Energy of a Dipole in a Uniform External Field

F. Javier Castro Paredes
Universidade Santiago de Compostela, Spain
Kirk T. McDonald
Joseph Henry Laboratories, Princeton University, Princeton, NJ 08544

(May 3, 2004; updated January 16, 2017)

1 Problem

A well-known result of electrostatics is that the energy of an electric dipole of moment $\mathbf{m}$ of fixed magnitude $|\mathbf{m}|$ in an external electric $\mathbf{E}_0$ is given by,

$$U_{\rm int} = -\mathbf{m} \cdot \mathbf{E}_0. \tag{1}$$

This expression can, for example, be deduced from the general prescription for the energy of interaction of a charge density ρ in and external electric potential ϕ_0, namely that,

$$U_{\rm int} = \int \rho \phi_0 \, d\mathrm{Vol}. \tag{2}$$

A useful abstraction from a charge distribution with a net dipole moment is the concept of a point dipole consisting of a pair of charges $\pm q$ separated by a small distance $\mathbf{d}$ such that we may take the limit as $d \to 0$ and $q \to \infty$ while the product $qd = m$ remains constant. Then, the electric field of the point dipole can be written (in Gaussian units) as,

$$\mathbf{E} = \frac{3(\mathbf{m} \cdot \hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{m}}{r^3} - \frac{4\pi}{3} \mathbf{m} \, \delta^3(\mathbf{r}), \tag{3}$$

where the position vector $\mathbf{r}$ is measured with respect to the center of the point dipole. The meaning of the delta function in eq. (3) is discussed, for example, in sec. 4.1 of [1].

An important contribution of Faraday and Maxwell to electrodynamics was their emphasis on the electromagnetic fields as primary physical concepts. A consequence is that the electrical energy of a system can be calculated as an integral of the electric field rather than of charges and potentials as in eq. (2). In particular, the energy of interaction of charges that create a field $\mathbf{E}$ with an external electric field $\mathbf{E_0}$ can be written as,

$$U_{\rm int} = \int \frac{\mathbf{E} \cdot \mathbf{E}_0}{4\pi} \, d\mathrm{Vol}. \tag{4}$$

The paradox concerns the use of eq. (4) for the interaction energy of a point dipole (3) in a *uniform* external electric field. In this case, we find,

$$U_{\rm int} = \frac{\mathbf{E}_0}{4\pi} \cdot \int \mathbf{E} \, d\mathrm{Vol} = -\frac{1}{3} \mathbf{m} \cdot \mathbf{E}_0, \tag{5}$$

in that the angular integral of $3(\mathbf{m} \cdot \hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{m}$ at a fixed radius r vanishes.

2 Solution

2.1 Electric Dipole

The paradox may have something to do with the use of an idealized point dipole.[1] So, we first consider the case of an electric dipole consisting of a pair of charges $\pm q$ separated by a finite distance $\mathbf{d}$, where $\mathbf{m} = q\mathbf{d}$ is the dipole moment. The electric field of the dipole can be written as,

$$\mathbf{E} = \mathbf{E}_q + \mathbf{E}_{-q}. \tag{6}$$

Using this in eq. (4), we calculate the interaction energy of this dipole in a uniform external field $\mathbf{E}_0$ to be,

$$U_{\rm int} = \frac{\mathbf{E}_0}{4\pi} \cdot \int \mathbf{E}\ d\mathrm{Vol} = \frac{\mathbf{E}_0}{4\pi} \cdot \left(\int \mathbf{E}_q\ d\mathrm{Vol} + \int \mathbf{E}_{-q}\ d\mathrm{Vol} \right) = 0, \tag{7}$$

since the volume integral of field of charge q is surely equal and opposite to that of charge $-q$.

We now appear to be in even greater difficulty than before, with Maxwell's expression for field energy predicting that there is no interaction energy for a real dipole in a uniform field!

However, on reflection, we realize that the vanishing of the interaction energy in eq. (7) depends on the cancelation of large quantities that are far separated in space, and in reality the cancelation may not be exact. For example, the volume integral of E_z due to charge q can be calculated in a spherical coordinate system (r, θ, φ) centered on the charge as,

$$\int E_z\ d\mathrm{Vol} = 2\pi \int_0^\infty r^2\ dr \int_{-1}^1 d\cos\theta\ \frac{q\cos\theta}{r^2} = 2\pi q \int_0^\infty dr \int_{-1}^1 \cos\theta\ d\cos\theta. \tag{8}$$

The integrand has the same value in any element $dr\ d\cos\theta$ at a fixed angle θ independent of r. The contributions to the integral from such elements is independent of their distance r from charge. While the integral (7) is zero mathematically, we cannot say this is true in the physical universe with confidence. In particular, we see that the interaction energy vanishes only if the external field is uniform throughout the entire universe.

That is, the implausible result (7) may be due to the unrealistic assumption of a perfectly uniform electric field $\mathbf{E}_0$ throughout all space.[2] Perhaps the form (1) can be recovered by considering electric fields from bounded distributions of electric charges, which fields are sufficiently uniform in the vicinity of the electric dipole $\mathbf{m}$.

---

[1] Other paradoxes in the discussion of point dipoles are discussed in [2].

[2] Also, the transformation from eq. (2) to eq.(4) involves an assumption about the behavior at infinity,

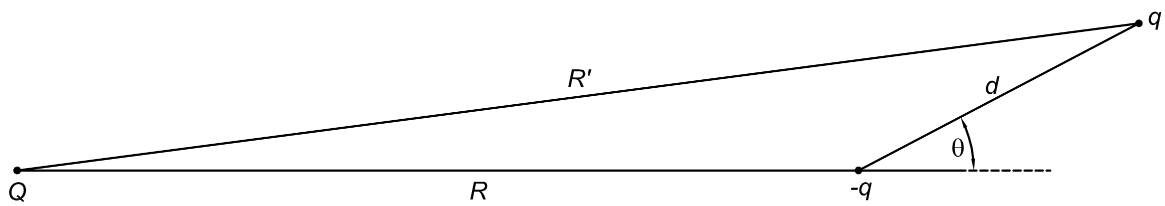
$$\begin{aligned} U_{\rm int} &= \int \rho\phi_0\, d\mathrm{Vol} = -\int \frac{\boldsymbol{\nabla}\cdot\mathbf{E}}{4\pi}\phi_0\ d\mathrm{Vol} = -\int \frac{\mathbf{E}\cdot\boldsymbol{\nabla}\phi_0}{4\pi}\, d\mathrm{Vol} - \int \frac{\boldsymbol{\nabla}\cdot\mathbf{E}\phi_0}{4\pi}\, d\mathrm{Vol} \\ &= \int \frac{\mathbf{E}\cdot\mathbf{E}_0}{4\pi}\, d\mathrm{Vol} - \oint \frac{\mathbf{E}\phi_0 \cdot d\mathbf{Area}}{4\pi}. \end{aligned} \tag{9}$$

The potential ϕ_0 at infinity is large for a mathematically uniform field, such that the surface integral cannot be neglected, and eq. (4) does not hold. Whereas, the potential ϕ_0 from any bounded charge distribution vanishes at infinity, and eq. (4) is valid in this case.

To explore this, we suppose that the external field $\mathbf{E}_0$ is created by some set of charges suitably remote from the dipole. Since the form (4) is linear in $\mathbf{E}_0$, it suffices to imagine this field as being created by a single charge Q at large distance R from the dipole such that $Q = E_0 R^2$. While charge Q creates a field $\mathbf{E}_Q$ that is essentially uniform near the dipole, this field is by no means uniform throughout the entire universe, and eq. (7) will no longer yield zero for the interaction energy.

We verify this using the geometry shown in the figure below, where,

$$\frac{1}{R'} = \frac{1}{\sqrt{R^2 + d^2 + 2Rd\cos\theta}} \approx \frac{1}{R}\left(1 - \frac{d}{R}\cos\theta + \frac{d^2}{2R^2}(3\cos^2\theta - 1) + \cdots\right). \tag{10}$$



In Appendix A we confirm that Maxwell's expression (4) for field energy can be used to calculate the interaction energy $U_{12} = qq'/D$ between a pair of point charges separated by distance D. Therefore, we calculate the interaction energy of the "external" field due to charge Q with the dipole shown in the figure above to be,

$$\begin{aligned} U_{\rm int} &= \int \frac{\mathbf{E}_0 \cdot \mathbf{E}}{4\pi}\, d\mathrm{Vol} = \int \frac{\mathbf{E}_0 \cdot \mathbf{E}_q}{4\pi}\, d\mathrm{Vol} + \int \frac{\mathbf{E}_0 \cdot \mathbf{E}_{-q}}{4\pi}\, d\mathrm{Vol} = \frac{qQ}{R'} - \frac{qQ}{R} \\ &\approx \frac{qQ}{R}\left(1 - \frac{d}{R}\cos\theta + \frac{d^2}{2R^2}(3\cos^2\theta - 1) + \cdots - 1\right) \\ &= -\frac{qdQ\cos\theta}{R^2} + \frac{qd^2Q(3\cos^2\theta - 1)}{2R^3} + \cdots = -\mathbf{m}\cdot\mathbf{E}_0 + \frac{d}{R} m E_0 \frac{3\cos^2\theta - 1}{2} + \cdots \end{aligned} \tag{11}$$

Thus, the departure of the interaction energy from $-\mathbf{m}\cdot\mathbf{E}_0$, where $\mathbf{E}_0$ is the external electric field at the center of the electric dipole $\mathbf{m}$, is of order d/R, the ratio of the size of the dipole to it distance from the sources of the external electric field. We can say that the form (1) follows from the field energy (4) for external electric fields that are not mathematically uniform, but which are "sufficiently uniform" at the location of the electric dipole.

However, we have not completely resolved the paradox. The question remains as to whether the delta-function term in eq. (3) for the dipole field $\mathbf{E}$ should be included when using eq. (4) for a "sufficiently, but not perfectly" uniform field $\mathbf{E}_0$

2.1.1 Example: Sphere with a $\cos\theta$ Surface Charge Density

A sphere of radius a with a surface charge density,

$$\sigma_0 = -\frac{E_0}{4\pi}\cos\theta, \tag{12}$$

has interior electric field $\mathbf{E}_0 = E_0\,\hat{\mathbf{z}}$, and exterior field the same as for an electric dipole moment $\mathbf{m}_0 = -E_0 a^3\,\hat{\mathbf{z}}/3$.[3] The electric potential of the uniform field inside the sphere is,

$$\phi_0(r < a) = -E_0 z. \tag{13}$$

When electric dipole $\mathbf{m} = qd\,\hat{\mathbf{m}}$ is at the origin, inside the field $\mathbf{E}_0$, its interaction energy according to eq. (2) is,

$$U_{\rm int} = q\left(-E_0\frac{d}{2}\cos\theta\right) - q\left(-E_0\frac{-d}{2}\cos\theta\right) = -qdE_0\cos\theta = -\mathbf{m}\cdot\mathbf{E}_0, \tag{14}$$

where θ is the angle of dipole $\mathbf{m}$ with respect to the z-axis.

To verify that eq. (4) can also be used to compute the interaction energy, we simplify to the case that the magnetic dipole $\mathbf{m}$ is aligned along the z-axis, for which we expect that $U_{\rm int} = -mE_0$. For this case, with the field $\mathbf{E}$ of the electric dipole given be eq. (3),

$$\begin{aligned} U_{\rm int} &= \int \frac{\mathbf{E}\cdot\mathbf{E}_0}{4\pi}\,d\mathrm{Vol} = \int_{r<a} \frac{\mathbf{E}\cdot\mathbf{E}_0}{4\pi}\,d\mathrm{Vol} + \int_{r>a} \frac{\mathbf{E}\cdot\mathbf{E}_0}{4\pi}\,d\mathrm{Vol} \\ &= \frac{mE_0}{4\pi}\int_{r<a}\left(\frac{3cos^2\theta - 1}{r^3} - \frac{4\pi}{3}\,\delta^3(\mathbf{r})\right)d\mathrm{Vol} - \frac{mE_0 a^3}{4\pi}\int_{r>a}\frac{(2cos\theta\,\hat{\mathbf{r}} + \sin\theta\,\hat{\boldsymbol{\theta}})^2}{r^6}\,d\mathrm{Vol} \\ &= -\frac{mE_0}{3} - \frac{mE_0 a^3}{2}\int_a^\infty \frac{dr}{r^4}\int_{-1}^1 d\cos\theta\,(3\cos^2\theta + 1) = -mE_0, \end{aligned} \tag{15}$$

as expected.

In achieving the expected result, it was necessary to include the delta-function term in the field (3) for a "point" electric dipole. This reaffirms that eq. (4) can be used to compute the electric interaction energy between an electric dipole and an "external" field due to a bounded distribution of electric charge, using the full from eq. (3) for the field of the dipole. In contrast, the form (4) is not applicable to the idealized case that the "external" field is mathematically uniform.

2.2 Magnetic Dipole

The magnetic interaction energy of two electrical currents that occupy different volumes, in media with unit realtive permability, can be written in terms of the magnetic fields, say $\mathbf{B}$ and $\mathbf{B}_0$, as,

$$U_{\rm int} = \int \frac{\mathbf{B}\cdot\mathbf{B}_0}{4\pi}\,d\mathrm{Vol}. \tag{16}$$

If, the first of these currents is well described as a magnetic dipole with moment $\mathbf{m} = I\mathbf{Area}/c$, then the interaction energy is also written as (see, for example, sec. 5.7 of [1]),

$$U_{\rm int} = -\mathbf{m}\cdot\mathbf{B}_0, \tag{17}$$

supposing that the "external" field $\mathbf{B}_0$ is "suficiently uniform" over the magnetic dipole.

[3]These results can be inferred from eqs. (4.55)-(4.58) of [1].

We recall that in the case of a magnetic dipole $\mathbf{m}$, the magnetic field can be written (see, for example, sec. 5.6 of [1]),

$$\mathbf{B} = \frac{3(\mathbf{m}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{m}}{r^3} + \frac{8\pi}{3}\mathbf{m}\,\delta^3(\mathbf{r}). \tag{18}$$

Then, via an argument similar to that associated with eqs. (4)-(5), use of eqs. (16) and (18) for the case of a uniform field $\mathbf{B}_0$ appears to imply that,

$$U_{\rm int} = \frac{2}{3}\mathbf{m}\cdot\mathbf{B}_0, \tag{19}$$

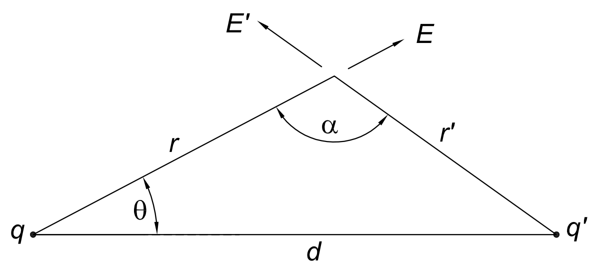
rather than eq. (17), and that if we neglect the delta-function term in eq. (18), the interaction energy (16) would be zero.

Additional comments on magnetic dipoles are given in Appendix B.

A Appendix: The Interaction Energy of Two Point Electric Charges

We verify that eq. (4) can be used to calculate the interaction energy $U_{12} = qq'/d$ of two point charges q and q' that are separated by distance d, taking $\mathbf{E}$ to be the field of charge q and $\mathbf{E}_0 = \mathbf{E}'$ to be that of charge q'.

We calculate in a spherical coordinate system (r, θ, φ) whose origin is at charge q and whose z axis points towards charge q', as shown in the figure below.



By the law of cosines we have,

$$r' = \left(r^2 + d^2 - 2rd\cos\theta\right)^{1/2}, \tag{20}$$

and also,

$$\cos\alpha = \frac{r^2 + r'^2 - d^2}{2rr'} = \frac{r - d\cos\theta}{r'}. \tag{21}$$

The interaction energy of the two charges is,

$$\begin{aligned} U_{12} &= \int \frac{\mathbf{E}\cdot\mathbf{E}'}{4\pi} d\mathrm{Vol} = \int \frac{qq'\cos\alpha}{4\pi r^2 r'^2} d\mathrm{Vol} = \frac{qq'}{2}\int_0^\infty dr \int_{-1}^1 d\cos\theta \frac{r - d\cos\theta}{(r^2 + d^2 - 2rd\cos\theta)^{3/2}} \\ &= \frac{qq'}{2}\int_0^\infty dr\left[\left(\frac{1}{|r-d|} - \frac{1}{r+d}\right)\left(\frac{1}{2d} - \frac{d}{2r^2}\right) - \frac{1}{2r^2 d}\left(|r-d| - r - d\right)\right] \\ &= \frac{qq'}{2}\int_0^d dr\left[\left(\frac{1}{d-r} - \frac{1}{r+d}\right)\frac{r^2-d^2}{2r^2 d} + \frac{1}{rd}\right] + \frac{qq'}{2}\int_d^\infty dr\left[\left(\frac{1}{r-d} - \frac{1}{r+d}\right)\frac{r^2-d^2}{2r^2 d} + \frac{1}{r^2}\right] \\ &= qq'\int_d^\infty \frac{dr}{r^2} = \frac{qq'}{d}. \end{aligned} \tag{22}$$

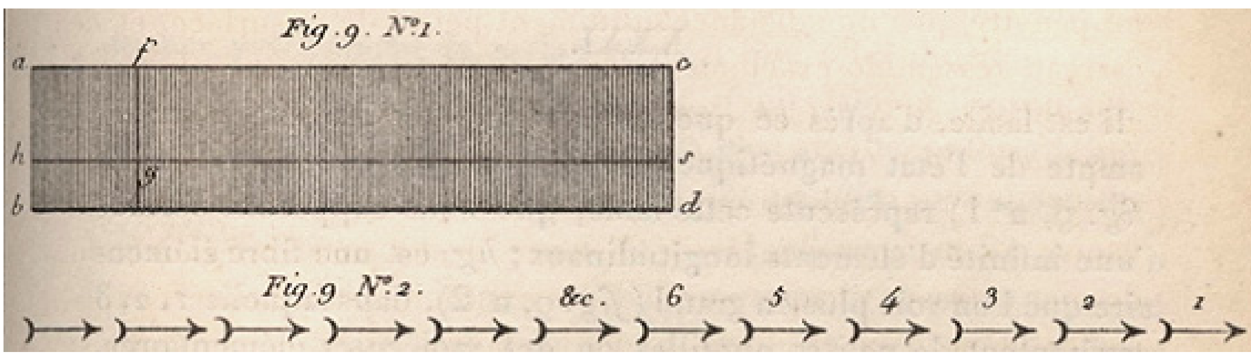
Note how the contribution to the energy U_{12} at distances $r < d$ vanishes, and the energy is accounted for in Maxwell's view entirely by the contribution for $r > d$, *i.e.*, at relatively large distances.

B Further Comments on Magnetic Dipoles

B.1 Gilbertian Magnetism

Human awareness of magnetism came earlier than that of electricity. In 1269, Peregrinus [3] conluded from experiment that a (permanent) magnet has two "poles," and that like poles of different magnets repel, while unlike poles attract. In 1600, Gilbert published a treatise [4], which includes qualitative notions of magnetic energy and lines of force.[4] In 1785, Coulomb confirmed (and made widely known) that the static force pairs of electric charges q_1 and q_2 varies as $q_1 q_2/r^2$ [5], and that the force between idealized magnetic poles p_1 and p_2 at the ends of long, thin magnets varies as $p_1 p_2/r^2$ [6].[5] The electric and magnetic forces were considered to be unrelated, except that they obeyed the same functional form.

Coulomb also noted that magnetic poles appear not to be isolatable, conjecturing (p. 306 of [7]) that the fundamental constituent of magnetism, a *molécule de fluide magnétique*, is a dipole, such that effective poles appear at the ends of a long, thin magnet.



B.2 Ampèrian Magnetism

In 1820, Ørsted [11, 12] published decisive evidence that electric currents exert forces on permanent magnets, indicating the electricy and magnetism are related.[6] Ørsted's term "electric conflict," used in his remarks on p. 276 of [12], is a precursor of the later concept of the magnetic field.

[4]Then, as now, magnetism seems to have inspired claims of questionable merit, which led Gilbert to pronounce on p. 166, "May the gods damn all such sham, pilfered, distorted works, which do but muddle the minds of students!"

[5]The $1/r^2$ law for the force between magnetic poles had been stated earlier by Michell [8, 9] and by Priestly [10].

[6]Reports have existed since at least the 1600's that lightning can affect ship's compasses (see, for example, p. 179 of [13]), and an account of magnetization of iron knives by lightning was published in Phil. Trans. in 1735 [14]. In 1797, von Humboldt conjectured that certain patterns of terrestrial magnetism were due to lighting strikes (see p. 13 of [15], a historical review of magnetism). A somewhat indecisive experiment involving a voltaic pile and a compass was performed by Romagnosi in 1802 [16].

> It is sufficiently evident from the preceding facts that the electric conflict is not confined to the conductor, but dispersed pretty widely in the circumjacent space. From the preceding facts we may likewise infer that this conflict performs circles.

Between 1820 and 1825 Ampère made extensive studies [17, 18, 19, 20, 21, 22] of the magnetic interactions of electrical currents.[7] Already in 1820 Ampère came to the vision that all magnetic effects are due to electrical currents.[8,9]

B.3 Interaction Energy of Electric Currents

The magnetic interaction energy of two electrical currents that occupy different volumes, in media with unit relative permeability, was first considered by Neumann, sec. 11 of [32],[10]

$$U_{\rm int} = \oint_1 \oint_2 \frac{I_1\, d\mathbf{l}_1 \cdot I_2\, d\mathbf{l}_2}{c^2 r} \,. \tag{23}$$

We now also write this as[11]

$$U_{\rm int} = \oint_i \frac{I_i\, d\mathbf{l}_i \cdot \mathbf{A}_j}{c} \,, \qquad \mathbf{A}_j = \oint_j \frac{I_j\, d\mathbf{l}_j}{cr} \,. \tag{24}$$

With the change of notation the (static) current I_1 is I, and that (static) current I_2 is I_0, the interaction energy of the two circuits is

$$\begin{aligned} U_{\rm int} &= \oint \frac{I\, d\mathbf{l} \cdot \mathbf{A}_0}{c} = \int \frac{\mathbf{J} \cdot \mathbf{A}_0}{c}\, d\mathrm{Vol} = \int \frac{(\boldsymbol{\nabla} \times \mathbf{B}) \cdot \mathbf{A}_0}{c}\, d\mathrm{Vol} \\ &= \int \frac{(\boldsymbol{\nabla} \times \mathbf{A}_0) \cdot \mathbf{B}}{4\pi}\, d\mathrm{Vol} + \int \frac{\boldsymbol{\nabla} \cdot (\mathbf{B} \times \mathbf{A}_0)}{4\pi}\, d\mathrm{Vol} \\ &= \int \frac{\mathbf{B} \cdot \mathbf{B}_0}{4\pi}\, d\mathrm{Vol} + \oint \frac{\mathbf{B} \times \mathbf{A}_0 \cdot d\mathbf{Area}}{4\pi} = \int \frac{\mathbf{B} \cdot \mathbf{B}_0}{4\pi}\, d\mathrm{Vol}, \end{aligned} \tag{25}$$

where $\mathbf{J}$ is the current density corresponding to the electric current I, and we suppose that the vector potential $\mathbf{A}_0$ falls off sufficiently fast at infinity. However, the if magnetic field $\mathbf{B}_0$ is mathematically uniform, and along the z-axis, then $\mathbf{A}_0 = \varrho B_0\, \hat{\boldsymbol{\phi}}/2$ in a cylindrical coordinate system (ϱ, ϕ, z), and the surface integral in eq. (25) cannot be neglected. Hence, eq. (25) does not hold in case of a mathematically uniform magnetic field $\mathbf{B}_0$, just as eq. (4) does not hold for a mathematically uniform electric field $\mathbf{E}_0$.

---

[7] An extensive discussion in English of Ampère's attitudes on the relation between magnetism and mechanics is given in [23]. A historical survey of 19th-century electrodynamics is given in [24]; another survey [25] gives little credit to Ampère. See also [26], sec. IIA regarding Ampère.

[8] See, for example, [27].

[9] The confirmation that permanent magnetism, due to the magnetic moments of electrons, is Ampèrian (rather than Gilbertian = due to pairs of opposite magnetic charges) came only after detailed studies of positronium (e^+e^- "atoms") in the 1940's [28, 31].

[10] If we write eq. (23) as $U_{\rm int} = I_1 I_2 M_{12}$, then M_{12} is the mutual inductance of circuits 1 and 2. Neumann included a factor of 1/2 in his version of eq. (23), associated with his choice of units.

[11] Neumann is often credited with inventing the vector potential $\mathbf{A}$, although he appears not to have factorized his eq. (23) into eq. (24).

B.4 Interaction Energy of an Ampèrian Magnetic Dipole, I

For current density $\mathbf{J}$ due to a small loop of electric current I with area $d\mathbf{Area}$ in an external magnetic field $\mathbf{B}_0 = \boldsymbol{\nabla} \times \mathbf{A}_0$, eq. (24) gives,

$$U_{\rm int}^{\rm (I)} = \oint \frac{I\, d\mathbf{l} \cdot \mathbf{A}_0}{cr} = \int \frac{I d\mathbf{Area}}{c} \cdot \boldsymbol{\nabla} \times \mathbf{A}_0 = \mathbf{m} \cdot \mathbf{B}_0 \tag{26}$$

noting that the magnetic dipole of the small current loop is $\mathbf{m} = I\, d\mathbf{Area}/c$.

If we wish to use the interaction energy $U_{\rm int}^{\rm (I)}$ to discuss the force on the magnetic dipole, we must note that the above derivation tacitly assumed that the electric currents were kept constant as the two circuits (two magnetic dipoles) were brought in from infinity. Some sort of "batteries" are required to maintain the constant currents, and these batteries do work as the circuits are moved. Indeed, this work is $-2U_{\rm int}^{\rm (I)}$, such that the force on a circuit with magnetic moment $\mathbf{m}$ is,

$$\mathbf{F} = -\boldsymbol{\nabla} U_{\rm total} = \boldsymbol{\nabla} U_{\rm int}^{\rm (I)} = \boldsymbol{\nabla}(\mathbf{m} \cdot \mathbf{B}_0). \tag{27}$$

B.5 Interaction Energy of Gilbertian Magnetic Dipoles

In 1860, Tait [33] used a model of magnetic dipoles as pairs of opposite magnetic poles to deduce that the interaction energy $U_{\rm int}$ of a pair of magnetic dipoles $\mathbf{m}_1$ and $\mathbf{m}_2$ is,

$$U_{\rm int} = \frac{\mathbf{m}_1 \cdot \mathbf{m}_2 - 3(\mathbf{m}_1 \cdot \hat{\mathbf{r}})(\mathbf{m}_1 \cdot \hat{\mathbf{r}})}{r^3} = -\mathbf{m}_1 \cdot \mathbf{B}_{21} = -\mathbf{m}_2 \cdot \mathbf{B}_{12}, \tag{28}$$

where $\mathbf{r}$ is the vector from the center of one dipole to the center of the other, and,

$$\mathbf{B}_{ij} = \frac{3(\mathbf{m}_i \cdot \hat{\mathbf{r}})(\mathbf{m}_i \cdot \hat{\mathbf{r}})}{r^3} - \frac{4\pi \mathbf{m}_i}{3}\, \delta^3(\mathbf{r}) \qquad \text{(Gilbertian)} \tag{29}$$

is the magnetic field due to (Gilbertian) dipole i at the location of dipole j.[12]

Equation (28) suggests that the interaction energy of a magnetic dipole $\mathbf{m}$ in an external magnetic field $\mathbf{B}_0$ has the form,

$$U_{\rm int}^{\rm (II)} = -\mathbf{m} \cdot \mathbf{B}_0, \tag{30}$$

B.6 Interaction Energy of an Ampèrian Magnetic Dipole, II

If the magnetic dipole is associated with Ampèrian electric currents, then the magnetic field has the form (see, for example, sec. 5.6 of [1]),

$$\mathbf{B}_{ij} = \frac{3(\mathbf{m}_i \cdot \hat{\mathbf{r}})(\mathbf{m}_i \cdot \hat{\mathbf{r}})}{r^3} + \frac{8\pi \mathbf{m}_i}{3}\, \delta^3(\mathbf{r}) \qquad \text{(Ampèrian)}, \tag{31}$$

which makes no difference to the interaction energy (28). This suggests that the interaction energy (30) also holds for an Ampèrian magnetic dipole in an external magnetic field.

[12]Of course, Tait used only the magnetic field $\mathbf{H}$ in 1860, prior to the introduction of the field $\mathbf{B}$ in 1871 by W. Thomson, pp. 398-402 of [34].

A derivation of eq. (30) based on the assumption of an Ampèrian magnetic dipole proceeds by consideration of the force $\mathbf{F} = -\boldsymbol{\nabla} U_{\rm int}$ when the magnetic moment $\mathbf{m}$ is held constant. Then, from the Biot-Savert law (and a version of Stokes' theorem) for a loop of electrical current I with magnetic momentum $\mathbf{m} = I\,\mathbf{Area}/c$,

$$\mathbf{F} = \frac{I}{c}\oint d\mathbf{l}\times\mathbf{B}_0 = \frac{I}{c}\int \boldsymbol{\nabla}(d\mathbf{Area}\cdot\mathbf{B}_0) - \frac{I}{c}\int (\boldsymbol{\nabla}\cdot\mathbf{B}_0)\, d\mathbf{Area} = \boldsymbol{\nabla}(\mathbf{m}\cdot\mathbf{B}_0) = -\boldsymbol{\nabla} U_{\rm int}^{\rm (II)}. \quad (32)$$

Comparing with sec. B.4, we see that the consideration of the Biot-Savart law in the present section leads us to an interaction energy that includes the energy associated with maintaining constant currents/dipole moments, if that is indeed the case, while that of sec. B.4 only included the energy of the magnetic field.

B.7 Interaction Energy of an Ampèrian Magnetic Dipole, III

Yet another scenario is of interest, in which a permeable medium with zero conduction current is brought into an "external" field $\mathbf{B}_0$ that is maintained by fixed currents.

We consider a small element in the permeable medium, with magnetic moment $\mathbf{m} = \chi\mathbf{B}_0$, where $\chi < 0$ corresponds to a diamagnetic moment and $\chi > 0$ is a paramagnetic moment. From the use of Stokes' theorem in eq. (32), we see that the operator $\boldsymbol{\nabla}$ is not meant to act on the dipole $\mathbf{m}$ in $\boldsymbol{\nabla}(\mathbf{m}\cdot\mathbf{B}_0) = (\mathbf{m}\cdot\boldsymbol{\nabla})\mathbf{B}_0 + \mathbf{m}\times(\boldsymbol{\nabla}\times\mathbf{B}_0)$. Rather, for $\mathbf{m} = \chi\mathbf{B}_0$ we have,

$$\begin{aligned}\boldsymbol{\nabla}(\mathbf{m}\cdot\mathbf{B}_0) &= \chi\boldsymbol{\nabla}(\mathbf{B}_0^2) = 2\chi[(\mathbf{B}_0\cdot\boldsymbol{\nabla})\mathbf{B}_0 + \mathbf{B}_0\times(\boldsymbol{\nabla}\times\mathbf{B}_0)] = 2[(\mathbf{m}\cdot\boldsymbol{\nabla})\mathbf{B}_0 + \mathbf{m}\times(\boldsymbol{\nabla}\times\mathbf{B}_0)] \\ &= 2\mathbf{F}. \end{aligned} \quad (33)$$

Hence, the force on the magnetic dipole element of the permeable medium can be written as,

$$\mathbf{F} = \frac{\chi}{2}\boldsymbol{\nabla}(B_0^2) = -\boldsymbol{\nabla} U_{\rm int}^{\rm (III)}, \qquad \text{with} \qquad U_{\rm int}^{\rm (III)} = -\frac{\mathbf{m}\cdot\mathbf{B}_0}{2} = -\frac{\chi B_0^2}{2}. \quad (34)$$

in terms of a third form of an interaction energy, $U_{\rm int}^{\rm (III)}$. This last result is deduced by rather different arguments in eq. (32.8) of [29], where $\mathbf{B}_0$ is written as $\mathbf{H}$.

The first form of the force in eq. (33) implies that paramagnetic objects are pulled into regions of stronger magnetic field, while diamagnetic objects are driven to regions of weaker field, as noted by Faraday in Art. 2269 of [30], in which he first identified diamagnetism.

B.8 Force on a Moving Magnetic Dipole

The results above strictly apply only to a magnetic dipole at rest. For discussion of the force on a moving magnetic dipole, see sec. 2.6 of [31].

References

[1] J.D. Jackson, *Classical Electrodynamics*, 2nd ed. (Wiley, 1975),
http://kirkmcd.princeton.edu/examples/EM/jackson_ce2_75.pdf

[2] E. Parker, *An apparent paradox concerning the field of an ideal dipole*, Eur. J. Phys. **38**, 025205 (2017), http://kirkmcd.princeton.edu/examples/EM/parker_ejp_38_025205_17.pdf

[3] P. de Maricourt (Peregrinus), *Epistola ad Sigerum* (1269), http://kirkmcd.princeton.edu/examples/EM/peregrinus_magnet.pdf

[4] W. Gilbert, *De Magnete* (1600), English translation (1893), http://kirkmcd.princeton.edu/examples/EM/gilbert_de_magnete.pdf

[5] C.-A. de Coulomb, *Premier Mémoire sur l'Électricité et le Magnétisme*, Mém. Acad. Roy. Sci., 578 (1785), http://kirkmcd.princeton.edu/examples/EM/coulomb_mars_88_569_85.pdf

[6] C.-A. de Coulomb, *Second Mémoire sur l'Électricité et le Magnétisme*, Mém. Acad. Roy. Sci., 578 (1785), http://kirkmcd.princeton.edu/examples/EM/coulomb_mars_88_578_85.pdf

[7] C.-A. de Coulomb, *Collection de Mémoires relatifs a la Physique* (Gauthier-Villars, 1883), http://kirkmcd.princeton.edu/examples/EM/coulomb_memoires.pdf

[8] J. Michell, *A Treatise on Artificial Magnets* (Cambridge U. Press, 1750), http://kirkmcd.princeton.edu/examples/EM/michell_magnets.pdf

[9] J. Michell, *On the Means of discovering the Distance, Magnitude, &c. of the Fixed Stars, in consequence of the Diminution of the Velocity of their Light, in case such a Distinction Should be found to take place in any of them, and such other Data Should be procured from Observations, as would be farther necessary for that Purpose*, Phil. Trans. Roy. Soc. London **74** , 35 (1783), http://kirkmcd.princeton.edu/examples/GR/michell_ptrsl_74_35_84.pdf

[10] J. Priestly, *The History and Present State of Electricity, with Original Experiments* (1767), http://kirkmcd.princeton.edu/examples/EM/priestly_electricity_67_p731-732.pdf

[11] J.C. Örsted, *Experimenta circa Effectum Conflictus Electrici in Acum Magneticam* (pamphlet, Berlin, 1829), http://kirkmcd.princeton.edu/examples/EM/oersted_experimenta.pdf

[12] J.C. Oersted, *Experiments on the Effect of a Current of Electricity on the Magnetic Needle*, Ann. Phil. **16**, 273 (1820), http://kirkmcd.princeton.edu/examples/EM/oersted_ap_16_273_20.pdf

[13] W. de Fonveille, *Thunder and Lightning* (Scribner, 1869), http://kirkmcd.princeton.edu/examples/EM/fonvielle_lightning.pdf

[14] P. Dod and Dr. Cookson, *An Account of an extraordinary Effect of Lightning in communicating Magnetism*, Phil. Trans. Roy. Soc. London **39**, 74 (1735), http://kirkmcd.princeton.edu/examples/EM/dod_ptrsl_39_74_35.pdf

[15] V. Courtillot and J.-L. Le Mouël, *The Study of Earth's Magnetism (1269-1950): A Foundation by Peregrinus and Subsequent Development of Geomagnetism and Paleomagnetism*, Rev. Geophys. **45**, RG3008 (2007), http://kirkmcd.princeton.edu/examples/EM/courtillot_rg_45_RG3008_07.pdf

[16] S. Stringari and R.R. Wilson, *Romagnosi and the discovery of electromagnetism*, Rend. Fis. Acc. Lincei **11**, 115 (2000), http://kirkmcd.princeton.edu/examples/EM/stringari_rfal_11_115_00.pdf

[17] A.M. Ampère, *Mémoire sur les effets courans électriques*, Ann. Chem. Phys. **15**, 59 (1820), http://kirkmcd.princeton.edu/examples/EM/ampere_acp_15_59_20.pdf

[18] A.M. Ampère, *Expériences relatives à de nouveaux phénomènes électro-dynamiques*, Ann. Chem. Phys. **20**, 60 (1822), http://kirkmcd.princeton.edu/examples/EM/ampere_acp_20_60_22.pdf

[19] A.M. Ampère, *Mémoire sur la détermination de la formule qui représente l'action mutuelle de deux portions infiniment petites de conducteurs voltaïques*, Ann. Chem. Phys. **20**, 398, 422 (1822), http://kirkmcd.princeton.edu/examples/EM/ampere_acp_20_398_22.pdf

[20] A.M. Ampère, *Extrait d'un Mémoire sur les Phénomènes électro-dynamiques*, Ann. Chem. Phys. **26**, 134, 246 (1824), http://kirkmcd.princeton.edu/examples/EM/ampere_acp_26_134_24.pdf

[21] A.M. Ampère, *Description d'un appareil électro-dynamique*, Ann. Chem. Phys. **26**, 390 (1824), http://kirkmcd.princeton.edu/examples/EM/ampere_acp_26_390_24.pdf

[22] A.M. Ampère, *Mémoire sur une nouvelle Éxperience électro-dynamique, sur son application à la formule qui représente l'action mutuelle de deux élemens de conducteurs voltaïques, et sur de nouvelle conséquences déduites de cette formule*, Ann. Chem. Phys. **29**, 381 (1825), http://kirkmcd.princeton.edu/examples/EM/ampere_acp_29_381_25.pdf

[23] R.A.R. Tricker, *Early Electrodynamics, the First Law of Circulation* (Pergamon, 1965), http://kirkmcd.princeton.edu/examples/EM/tricker_early_em.pdf

[24] O. Darrigol, *Electrodynamics from Ampère to Einstein* (Oxford U. Press, 2000), http://kirkmcd.princeton.edu/examples/EM/darrigol_em_00.pdf

[25] E.T. Whittaker, *A History of the Theories of Aether and Electricity from the Age of Descartes to the Close of the Ninenteenth Century* (Longmans, Green, London, 1910), http://kirkmcd.princeton.edu/examples/EM/whittaker_history.pdf

[26] J.D. Jackson and L.B. Okun, *Historical roots of gauge invariance*, Rev. Mod. Phys. **73**, 663 (2001), http://kirkmcd.princeton.edu/examples/EM/jackson_rmp_73_663_01.pdf

[27] L.P. Williams, *What were Ampère's Earliest Discoveries in Electrodynamics?* Isis **74**, 492 (1983), http://kirkmcd.princeton.edu/examples/EM/williams_isis_74_492_83.pdf

[28] J.D. Jackson, *The Nature of Intrinsic Magnetic Dipole Moments*, CERN-77-17 (1977), http://kirkmcd.princeton.edu/examples/EM/jackson_CERN-77-17.pdf

[29] L.D. Landau and E.M. Lifshitz and L.P. Pitaevskii, *Electrodynamics of Continuous Media*, 2$^{\text{nd}}$ ed. (Butterworth-Heinemann, 1984), http://kirkmcd.princeton.edu/examples/EM/landau_ecm2.pdf

[30] M. Faraday, *Experimental Researches in Electricity.—Twentieth Series*, Phil. Trans. Roy. Soc. London **136**, 21 (1846),
http://kirkmcd.princeton.edu/examples/EM/faraday_ptrsl_136_21_46.pdf

[31] K.T. McDonald, *Forces on Magnetic Dipoles* (Oct. 26, 2014),
http://kirkmcd.princeton.edu/examples/neutron.pdf

[32] F.E. Neumann, *Allgemeine Gesetze der inducirten elektrischen Ströme*, Abh. König. Akad. Wiss. Berlin **1**, 45 (1845),
http://kirkmcd.princeton.edu/examples/EM/neumann_akawb_1_45.pdf

[33] P.G. Tait, *Quaternion Investigations Connected with Electrodynamics and Magnetism*, Quart. J. Pure Appl. Math. **3**, 331 (1860),
http://kirkmcd.princeton.edu/examples/EM/tait_qjapm_3_331_60.pdf

[34] W. Thomson, *Reprints of Papers on Electrostatics and Magnetism*, 2nd ed. (Macmillan, 1884), http://kirkmcd.princeton.edu/examples/EM/thomson_electrostatics_magnetism.pdf