Notes on the Dirac Equation

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This note gathers together material from my Ph529 Lecture 6,[1] and my Ph406 Problem Sets 4, 9 and 12.[2]

1 The 4-Spinor Dirac Equation

In 1928, Dirac [1] sought a relativistic wave equation for spin-1/2 particles that would be a first-order differential equation, in contrast to the Klein-Gordon equation [2, 3] for spin-0 particles with is second order. He found this could not be done with ordinary wave functions, but rather 4-component (spinor) wave functions were required. These are a generalization of the 2-component spinors introduced by Pauli (1927) [4] in the non-relativistic quantum mechanics of spin-1/2 particles.[3]

Relativity + 1$^{\rm st}$-order differential wave equation $\Rightarrow$ spin!

Dirac 4-spinors ψ for a spin-1/2 particle (or antiparticle) of mass m in free space obey Dirac's equation,

$$i\gamma^\mu \partial_\mu \psi = m\psi, \tag{1}$$

in units with $c = 1 = \hbar$. The derivative operator is,

$$\partial_\mu = \left(\frac{\partial}{\partial t}, -\boldsymbol{\nabla}\right), \qquad \text{and} \qquad \gamma^\mu \partial_\mu = \gamma^0 \partial_0 - \sum_{i=1}^{3} \gamma^i \partial_i = \gamma_0 \frac{\partial}{\partial t} + \boldsymbol{\gamma} \cdot \boldsymbol{\nabla}, \tag{2}$$

and ψ is an object consisting of four complex numbers,

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix}, \tag{3}$$

and the 4×4 matrices γ^μ, $\mu = 0, 1, 2, 3$, *i.e.*, $\gamma^\mu = (\gamma^0, \boldsymbol{\gamma})$ with $\boldsymbol{\gamma} = (\gamma^1, \gamma^2, \gamma^3)$, have the anticommutation relations,

$$\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2\eta^{\mu\nu} I_4 \qquad \left[\Rightarrow \quad (\gamma^0)^2 = I_4, \quad (\gamma^i)^2 = -I_4\right], \tag{4}$$

[1] http://kirkmcd.princeton.edu/examples/ph529/ph529l6.pdf

[2] http://kirkmcd.princeton.edu/examples/ph406/ph406_set4_2014_sol.pdf
http://kirkmcd.princeton.edu/examples/ph406/ph406_set9_2014_sol.pdf
http://kirkmcd.princeton.edu/examples/ph406/ph406_set12_2014_sol.pdf

[3] The term spinor was invented by Ehrenfest (∼1929), as quoted in [5].

where the diagonal matrix $\eta^{\mu\nu}$ has diagonal elements $1, -1, -1, -1$, and I_4 is the unit 4×4 matrix. The Einstein summation convention is used for repeated indices in equations, with the sense that $a\,b = a_\mu b^\mu = a_0 b^0 - \mathbf{a}\cdot\mathbf{b}$ and $a_\mu = a^\mu = (a_0, \mathbf{a})$.

A particular representation of the γ-matrices was given by Dirac [1],[4,5]

$$\gamma^0 = \begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix}, \qquad \gamma^i = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix}, \qquad \gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} 0 & I_2 \\ I_2 & 0 \end{pmatrix}, \tag{5}$$

where I_2 is the 2×2 unit matrix, and $\sigma_j,\ j = 1, 2, 3$ are the Pauli spin matrices [4],[6]

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad (\sigma_j)^2 = I_2, \quad \sigma_j\sigma_k = i\epsilon_{jkl}\,\sigma_l. \tag{6}$$

As usual, we define the Pauli "spin vector" $\boldsymbol{\sigma}$ as the triplet of matrices $\boldsymbol{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$.

The Dirac representation (5) will be used throughout most of this note, with occasional use of a Weyl (chiral) representation [10]. Some alternatives are listed in the Appendix.

1.1 Feynman Slash Notation

The product of a 4-vector a_μ with the matrix 4-vector γ^μ occurs so often that Feynman introduced a special notation (easily implemented on manual typewriters via the backspace key),

$$\not{a} \equiv a_\mu \gamma^\mu. \tag{7}$$

With this notation, the Dirac equation (1) can be written as,

$$\not{P}\,\psi = m\,\psi, \tag{8}$$

if we also define the operator $P_\mu \equiv i\partial_\mu$.

Then,

$$\not{P}\,\not{P}\,\psi = \not{P}\,(m\,\psi) = m^2\,\psi = p^2\,\psi, \tag{9}$$

where $p_\mu = (E, \mathbf{p})$ is the energy momentum 4-vector of the particle described by ψ.

We also note that,

$$\not{a}\,\not{b} = a_\mu\gamma^\mu\,b_\nu\gamma^\nu = -b_\nu\gamma^\nu\,a_\mu\gamma^\mu + 2\eta^{\mu\nu}a_\mu b_\nu = \not{b}\,\not{a} + 2ab, \tag{10}$$

using eq. (4). In particular,

$$\not{P}\,\not{P} = P^2, \tag{11}$$

which is formally consistent with eq. (9) on taking $P^2\psi = p^2\psi$.

[4] We follow the notation of sec. 2.1 of [6], which is also used in [7].

[5] The concept of a fifth γ-matrix, now called γ^5, is due to Eddington [8, 9].

[6] The σ_j are both hermitian, $\sigma_j^\dagger = (\sigma_j^*)^T = \sigma_j$, and unitary, $(\sigma_j^*)^T\sigma_j = (\sigma_j)^2 = I_2$, where T means transpose.

2 Plane-Wave States

We restrict our discussion of Dirac's equation to the plane-wave solutions needed for scattering problems. We seek solutions of the form,

$$\psi = u\, e^{-i\,p\,x} = u\, e^{-i\,p_\mu x^\mu}, \tag{12}$$

with rest mass m, 3-momentum $\mathbf{p}$, energy $E = \sqrt{p^2 + m^2}$, and 4-momentum $p = p_\mu = (E, \mathbf{p})$. The 4-spinor u does not depend on $x = x^\mu = (t, \mathbf{x})$.

First, we consider the case of a particle at rest, $p_\mu = (E, \mathbf{0})$. Then, $\not{P}\psi = m\psi$ reduces to $E\gamma_0\, u = m\, u$. This simplest 4-spinors are,

$$u_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \qquad u_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \qquad u_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \qquad \text{and} \qquad u_4 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}. \tag{13}$$

Recall from eq. (5) that,

$$\gamma^0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \tag{14}$$

Hence, for spinors u_1 and u_2 we have $E = m$, but for spinors u_3 and u_4 we find $E = -m$ which are examples of the famous negative-energy solutions to the Dirac equation. The negative-energy states are troublesome in that they could emit light, dropping to a more negative energy, and then emit more light...

We readily interpret u_1 and u_2 as the spinors for the spin up and spin down states, with respect to the z axis, of a spin-1/2 particle at rest.

Dirac gave a prophetic interpretation of the "negative-energy" states u_3 and u_4 by means of a "hole theory".[7] He argued that the "vacuum" consists of a "sea" of negative-energy particles (dark energy?), occupying all possible negative-energy states. This conveniently prohibits any of the paradoxical radiative transitions. Of course, we have to ignore the infinite (negative) energy and charge of this "sea" (in a type of "renormalization" that Dirac later came to abhor).

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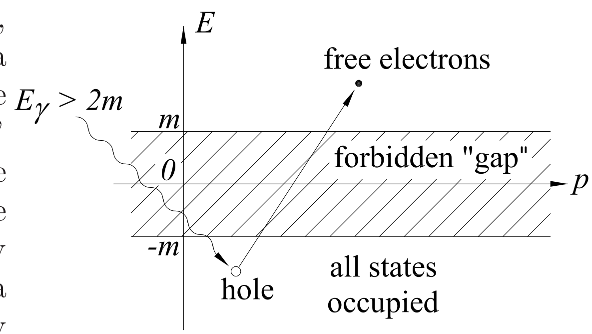
[7] Dirac interpreted electron "hole" states as protons in [11], and as what are now called positrons in [12]. The later paper also introduced the Dirac magnetic monopole. Positrons were later interpreted by Wheeler (1941), Stueckelberg [13, 14] and Feynman [15, 16] as electrons moving backwards in time. See also [18].

Transitions from the negative-energy sea are possible if enough energy is added. For example, a photon of energy $E_\gamma > 2m$ could promote a negative-energy electron across the "gap" into the realm of free electrons. This would leave a "hole" in the negative energy sea. Dirac interpreted the hole as a positively charged particle. The hole could then move around in the negative-energy sea, obeying laws for a positive energy state. To a laboratory observer, the "positron" would be very real.



If an electron collided with a hole/positron, both could disappear in a flash of light, of energy equal to that of the electron + positron.

Spinor u_3 corresponds to a spin-up, negative-energy electron at rest. Hence, in the hole theory, u_3 is interpreted as a spin-down, positive-energy, positive-charge state (at rest), *i.e.*, a positron.

Generalizing to spin-1/2 particles other than electrons, it is conventional to label particle 4-spinors as u and antiparticle 4-spinors as v. We would like to use positive energies in the antiparticle wavefunction, so it is customary to write it as,

$$\psi = v\, e^{i\,px} \qquad \text{(antiparticle)}. \tag{15}$$

This corresponds to a hole with 4-momentum $-p = (-E, -\mathbf{p})$, so the Dirac equation (1) for an antiparticle state leads to,

$$P_\mu \gamma^\mu\, v = -p_\mu \gamma^\mu v = m\, v, \qquad \not{p}\, v = -m\, v \qquad \text{(antiparticle)}, \tag{16}$$

while for a particle state, $\not{p}\, u = m\, u$.

We now consider the 4-spinors for plane-wave states of a particle of 3-momentum $\mathbf{p}$ and energy $E = \sqrt{m^2 + \mathbf{p}^2} > 0$. We write the particle 4-spinor u in terms of two, 2-components spinors χ and ζ,

$$u \propto \begin{pmatrix} \chi \\ \zeta \end{pmatrix}. \tag{17}$$

Then, eq. (8), $\not{P}\, u = \not{p}\, u = m\, u$, implies, $E\gamma^0 u - \mathbf{p} \cdot \boldsymbol{\gamma} u = m\, u$,

$$E \begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix} \begin{pmatrix} \chi \\ \zeta \end{pmatrix} - \mathbf{p} \cdot \begin{pmatrix} 0 & \boldsymbol{\sigma} \\ -\boldsymbol{\sigma} & 0 \end{pmatrix} \begin{pmatrix} \chi \\ \zeta \end{pmatrix} \propto m \begin{pmatrix} \chi \\ \zeta \end{pmatrix}, \tag{18}$$

$$E\chi - \mathbf{p} \cdot \boldsymbol{\sigma}\, \zeta \propto m\chi, \qquad \Rightarrow \qquad \chi = \frac{\mathbf{p} \cdot \boldsymbol{\sigma}}{E - m} \zeta, \tag{19}$$

$$-E\zeta + \mathbf{p} \cdot \boldsymbol{\sigma}\, \chi \propto m\zeta, \qquad \Rightarrow \qquad \zeta = \frac{\mathbf{p} \cdot \boldsymbol{\sigma}}{E + m} \chi. \tag{20}$$

Hence,

$$u \propto \begin{pmatrix} \chi \\ \frac{\mathbf{p} \cdot \boldsymbol{\sigma}}{E+m} \chi \end{pmatrix}, \tag{21}$$

is a general particle (plane-wave) 4-spinor. For antiparticles, $\not{p}\, v = -m\, v$, and we similarly find,

$$v \propto \begin{pmatrix} \frac{\mathbf{p}\cdot\boldsymbol{\sigma}}{E+m}\zeta \\ \zeta \end{pmatrix}. \tag{22}$$

For nonzero momentum $\mathbf{p}$, all four components are nonzero in both particle and antiparticle 4-spinors. Thus, the Dirac equation does not, in general, split into two independent 2-component equations.

2.1 Normalization

To use the plane-wave 4-spinors in cross-section calculations, following the relativistic version of Fermi's "golden rule",[8] they should be normalized to $2E$ particles per unit volume.

To clarify this, we develop the relativistic probability current density J^μ which obeys the conservation law,

$$\partial_\mu J^\mu = 0. \tag{23}$$

We define $\quad \psi^\dagger = (\psi_1^*, \psi_2^*, \psi_3^*, \psi_4^*) \quad$ as the conjugate of $\quad \psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix}$. (24)

We multiply the Dirac equation (1) by the $\psi^\dagger$ to obtain,

$$i\psi^\dagger\gamma^0 \frac{\partial\psi}{\partial t} + i\psi^\dagger\boldsymbol{\gamma}\cdot\boldsymbol{\nabla}\psi = m\psi^\dagger\psi. \tag{25}$$

The usual trick is to conjugate this equation and subtract. Now, $\gamma^{0\dagger} = \gamma^0$, but,

$$\boldsymbol{\gamma}^\dagger = \begin{pmatrix} 0 & \boldsymbol{\sigma} \\ -\boldsymbol{\sigma} & 0 \end{pmatrix}^\dagger = \begin{pmatrix} 0 & -\boldsymbol{\sigma} \\ \boldsymbol{\sigma} & 0 \end{pmatrix} = -\boldsymbol{\gamma}, \tag{26}$$

as $\sigma_j^\dagger = \sigma_j$, which leads to trouble. A better behaved combination is $\gamma^0\boldsymbol{\gamma}$,

$$(\gamma^0\boldsymbol{\gamma})^\dagger = \begin{pmatrix} 0 & \boldsymbol{\sigma} \\ \boldsymbol{\sigma} & 0 \end{pmatrix}^\dagger = \begin{pmatrix} 0 & \boldsymbol{\sigma} \\ \boldsymbol{\sigma} & 0 \end{pmatrix} = \gamma^0\boldsymbol{\gamma}. \tag{27}$$

So, instead we multiply the Dirac equation by $\psi^\dagger\gamma^0$,

$$i\psi^\dagger\gamma^0\gamma^0 \frac{\partial\psi}{\partial t} + i\psi^\dagger\gamma^0\boldsymbol{\gamma}\cdot\boldsymbol{\nabla}\psi = m\psi^\dagger\gamma^0\psi, \tag{28}$$

$$i\psi^\dagger \frac{\partial\psi}{\partial t} + i\psi^\dagger\gamma^0\boldsymbol{\gamma}\cdot\boldsymbol{\nabla}\psi = m\psi^\dagger\gamma^0\psi, \qquad \text{using} \qquad (\gamma^0)^2 = I_4. \tag{29}$$

[8] See p. 74 and 79 of http://kirkmcd.princeton.edu/examples/ph529/ph529l5.pdf

The conjugate equation is, recalling that $(O\psi)^\dagger = \psi^\dagger O^\dagger$,

$$-i\psi\frac{\partial\psi^\dagger}{\partial t} + i\boldsymbol{\nabla}\psi^\dagger\cdot(\gamma^0\boldsymbol{\gamma}\psi) = m\psi^\dagger\gamma^0\psi, \tag{30}$$

Subtracting eq. (30) from (29),

$$i\frac{\partial}{\partial t}(\psi^\dagger\psi) + i\boldsymbol{\nabla}\cdot(\psi^\dagger\gamma^0\boldsymbol{\gamma}\psi) = 0, \tag{31}$$

such that we can identify the probability current density as,

$$J^\mu = \psi^\dagger\gamma^0\gamma^\mu\psi, \tag{32}$$

and the probability density is,

$$\rho = J^0 = \psi^\dagger\psi. \tag{33}$$

In general, the combination $\psi^\dagger\gamma^0$ leads to clearer physical interpretations than does $\psi^\dagger$ alone, so we define,

$$\bar{\psi} = \psi^\dagger\gamma_0 \tag{34}$$

as the adjoint of ψ (and not the antiparticle of ψ). Matrix elements of a Dirac operator $f(\gamma^\mu)$ will be taken as $\bar{\psi}_2\, f\, \psi_1$ rather than $\psi_2^\dagger\, f\, \psi_1$, as the former leads to quantities with well-defined Lorentz transformations (*i.e.*, scalar, vector, tensor, ...).[9]

The adjoint of an operator f is then,

$$\bar{f} = \gamma_0 f^\dagger\gamma^0. \tag{35}$$

That is,

$$(\bar{\psi}_2\, f\, \psi_1)^\dagger = \psi_1^\dagger\, f^\dagger\gamma^{0\dagger}\,\psi_2 = \psi_1^\dagger\,\gamma^0\,\gamma^0\, f^\dagger\,\gamma^0\,\psi_2 = \bar{\psi}_1\,\bar{f}\,\psi_2. \tag{36}$$

Returning to the probability density,

$$\rho = \psi^\dagger\psi = \bar{\psi}\gamma^0\psi, \qquad \text{for the spinor} \qquad u \propto \begin{pmatrix} \chi \\ \frac{\mathbf{p}\cdot\boldsymbol{\sigma}}{E+m}\chi \end{pmatrix}, \tag{37}$$

we have, with $\boldsymbol{\sigma}^\dagger = \boldsymbol{\sigma}$,

$$\bar{u} = u^\dagger\gamma^0 \propto \left(\chi^\dagger, \frac{\mathbf{p}\cdot\chi^\dagger\boldsymbol{\sigma}}{E+m}\right)\gamma^0 = \left(\chi^\dagger, -\chi^\dagger\frac{\mathbf{p}\cdot\boldsymbol{\sigma}}{E+m}\right). \tag{38}$$

so that,

$$\rho = \bar{u}\gamma^0 u \propto \chi^\dagger\left(I_2 + \frac{(\mathbf{p}\cdot\boldsymbol{\sigma})(\mathbf{p}\cdot\boldsymbol{\sigma})}{(E+m)^2}\right)\chi. \tag{39}$$

[9]See, for example, sec. 2.4 of [6].

Recalling the 2-spinor facts,

$$(\mathbf{p}\cdot\boldsymbol{\sigma})(\mathbf{q}\cdot\boldsymbol{\sigma}) = (\mathbf{p}\cdot\mathbf{q})\, I_2 + i\; \boldsymbol{\sigma}\cdot\mathbf{p}\times\mathbf{q}, \qquad (\mathbf{p}\cdot\boldsymbol{\sigma})^2 = p^2 I_2, \tag{40}$$

we have,

$$\rho \propto \left(I_2 + \frac{p^2}{(E+m)^2}\right)\chi^\dagger\chi = \frac{2E}{E+m}\,, \tag{41}$$

supposing that $\chi^\dagger\chi = 1$.

As mentioned above, we want $\rho = 2E$, so the normalized particle 4-spinor is,

$$u = \sqrt{E+m}\begin{pmatrix} \chi \\ \frac{\mathbf{p}\cdot\boldsymbol{\sigma}}{E+m}\chi \end{pmatrix} = \begin{pmatrix} \sqrt{E+m}\,\chi \\ \frac{p}{\sqrt{E+m}}\,\hat{\mathbf{p}}\cdot\boldsymbol{\sigma}\,\chi \end{pmatrix} = \begin{pmatrix} \sqrt{E+m}\,\chi \\ \sqrt{E+m}\,\hat{\mathbf{p}}\cdot\boldsymbol{\sigma}\,\chi \end{pmatrix}, \tag{42}$$

where $\hat{\mathbf{p}} = \mathbf{p}/p$ is a unit 3-vector, such that,[10]

$$\bar{u}u = (E+m)\left(1 - \frac{p^2}{(E+m)^2}\right) = 2m. \tag{43}$$

Similarly, normalized antiparticle 4-spinors v, associated with plane-wave states $\psi = v\, e^{ipx}$, can be written as,

$$v = \sqrt{E+m}\begin{pmatrix} \frac{\mathbf{p}\cdot\boldsymbol{\sigma}}{E+m}\zeta \\ \zeta \end{pmatrix} = \begin{pmatrix} \frac{p}{\sqrt{E+m}}\,\hat{\mathbf{p}}\cdot\boldsymbol{\sigma}\,\zeta \\ \sqrt{E+m}\,\zeta \end{pmatrix} = \begin{pmatrix} \sqrt{E-m}\,\hat{\mathbf{p}}\cdot\boldsymbol{\sigma}\,\zeta \\ \sqrt{E+m}\,\zeta \end{pmatrix}, \tag{44}$$

where the 2-spinor ζ obeys $\zeta^\dagger\zeta = 1$.

In the high-energy limit ($E \gg m$, $E \approx p$), these 4-spinors simplify to,

$$u \to \sqrt{E}\begin{pmatrix} \chi \\ \hat{\mathbf{p}}\cdot\boldsymbol{\sigma}\,\chi \end{pmatrix}, \qquad v \to \sqrt{E}\begin{pmatrix} \hat{\mathbf{p}}\cdot\boldsymbol{\sigma}\,\zeta \\ \zeta \end{pmatrix}. \tag{45}$$

2.2 Plane-Wave States in the Weyl Representation

A Weyl (chiral) representation is,

$$\breve{\gamma}^0 = \begin{pmatrix} 0 & I_2 \\ I_2 & 0 \end{pmatrix}, \quad \breve{\gamma}^i = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix}, \tag{46}$$

Following [17], we designate quantities in the Weyl representation with an accent brève, *i.e.*, $\breve{\psi}$. The Weyl basis can be obtained from the Dirac basis as,

$$\breve{\gamma}^\mu = \breve{U}\gamma^\mu\breve{U}^\dagger, \qquad \breve{\psi} = \breve{U}\psi, \qquad \breve{U} = \frac{1}{\sqrt{2}}\begin{pmatrix} I_2 & -I_2 \\ I_2 & I_2 \end{pmatrix}, \qquad \breve{U}^\dagger = \frac{1}{\sqrt{2}}\begin{pmatrix} I_2 & I_2 \\ -I_2 & I_2 \end{pmatrix}. \tag{47}$$

[10] Warning: Some people prefer $\bar{u}u = 1$.

We seek plane-wave solutions of the form,

$$\breve{\psi} = \breve{u}\, e^{-i\,p\,x} = \breve{u}\, e^{-i\,p_\mu x^\mu}. \tag{48}$$

First, we consider the case of a particle at rest, $p_\mu = (E, \mathbf{0})$. Then, $\not{P}\breve{\psi} = m\breve{\psi}$ reduces to $E\breve{\gamma}_0\, \breve{u} = m\, \breve{u}$. Since,

$$\breve{\gamma}^0 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \tag{49}$$

none of the simplest spinors (13) are eigenstates.

Rather, some simple eigenspinors in the Weyl representation are,

$$\breve{u}_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \qquad \breve{u}_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}, \qquad \breve{u}_3 = \frac{1}{\sqrt{2}}\begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \qquad \breve{u}_4 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ -1 \\ 0 \\ 1 \end{pmatrix}. \tag{50}$$

For spinors $\breve{u}_1$ and $\breve{u}_2$ we have $E = m$, but for spinors $\breve{u}_3$ and $\breve{u}_4$ we find $E = -m$ which are examples of the famous negative-energy solutions to the Dirac equation.

spinors $\breve{u}_1$ and $\breve{u}_2$ are for the spin up and spin down states, with respect to the z axis, of a spin-1/2 particle at rest.

Dirac gave an interpretation of the "negative-energy" states $\breve{u}_3$ and $\breve{u}_4$ by means of a "hole theory" [11, 12]. He argued that the "vacuum" consists of a "sea" of negative-energy particles (dark energy?), occupying all possible negative-energy states, and that physical states related to negative energy correspond to their absence from (holes in) this sea.

Spinor $\breve{u}_3$ corresponds to a spin-up, negative-energy electron at rest. Hence, in the hole theory, $\breve{u}_3$ is interpreted as a spin-down, positive-energy, positive-charge state (at rest), *i.e.*, a positron.

Generalizing to spin-1/2 particles other than electrons, it is conventional to label particle 4-spinors as $\breve{u}$ and antiparticle 4-spinors as $\breve{v}$. We would like to use positive energies in the antiparticle wavefunction, so it is customary to write it as,

$$\breve{\psi} = \breve{v}\, e^{i\,px} \qquad \text{(antiparticle)}. \tag{51}$$

This corresponds to a hole with 4-momentum $-p = (-E, -\mathbf{p})$, so the Dirac equation, $i\partial_\mu\breve{\gamma}^\mu\breve{\psi} = m\breve{\psi}$, for an antiparticle state leads to,

$$P_\mu\breve{\gamma}^\mu\, \breve{v} = -p_\mu\breve{\gamma}^\mu\breve{v} = m\, \breve{v}, \qquad \not{p}\, \breve{v} = -m\, \breve{v} \qquad \text{(antiparticle)}, \tag{52}$$

while for a particle state, $\not{p}\, \breve{u} = m\, \breve{u}$.

We now consider the 4-spinors for plane-wave states of a particle of 3-momentum $\mathbf{p}$ and energy $E = \sqrt{m^2 + \mathbf{p}^2} > 0$. We write the particle 4-spinor $\breve{u}$ in terms of two, 2-components spinors χ and ζ,

$$\breve{u} \propto \begin{pmatrix} \chi \\ \zeta \end{pmatrix}. \tag{53}$$

Then, $\breve{P\!\!\!/}\, \breve{u} = \breve{p\!\!\!/}\, \breve{u} = m\, \breve{u}$ implies, $E\breve{\gamma}^0 \breve{u} - \mathbf{p} \cdot \breve{\boldsymbol{\gamma}} \breve{u} = m\breve{u}$,

$$E \begin{pmatrix} 0 & I_2 \\ I_2 & 0 \end{pmatrix} \begin{pmatrix} \chi \\ \zeta \end{pmatrix} - \mathbf{p} \cdot \begin{pmatrix} 0 & \boldsymbol{\sigma} \\ -\boldsymbol{\sigma} & 0 \end{pmatrix} \begin{pmatrix} \chi \\ \zeta \end{pmatrix} \propto m \begin{pmatrix} \chi \\ \zeta \end{pmatrix}, \tag{54}$$

$$E\zeta - \mathbf{p} \cdot \boldsymbol{\sigma}\, \zeta \propto m\chi, \qquad \Rightarrow \qquad \chi = \frac{E - \mathbf{p} \cdot \boldsymbol{\sigma}}{m} \zeta, \tag{55}$$

$$E\chi + \mathbf{p} \cdot \boldsymbol{\sigma}\, \chi \propto m\zeta, \qquad \Rightarrow \qquad \zeta = \frac{E + \mathbf{p} \cdot \boldsymbol{\sigma}}{m} \chi. \tag{56}$$

Hence,

$$\breve{u} \propto \begin{pmatrix} \chi \\ \frac{E + \mathbf{p} \cdot \boldsymbol{\sigma}}{m} \chi \end{pmatrix}, \tag{57}$$

is a general particle (plane-wave) 4-spinor. For antiparticles, $\breve{p\!\!\!/}\, v = -m\, v$, and we similarly find,

$$v \propto \begin{pmatrix} \frac{\mathbf{p} \cdot \boldsymbol{\sigma} - E}{m} \zeta \\ \zeta \end{pmatrix}. \tag{58}$$

I find the forms (57)-(58) to be awkward. Alternative forms can be obtained from the (normalized) forms (42) and (44) in the Dirac representation via the unitary transformation (47). Thus,

$$\breve{u} = \sqrt{\frac{E+m}{2}} \begin{pmatrix} I_2 & -I_2 \\ I_2 & I_2 \end{pmatrix} \begin{pmatrix} \chi \\ \frac{\mathbf{p} \cdot \boldsymbol{\sigma}}{E+m} \chi \end{pmatrix} = \sqrt{\frac{E+m}{2}} \begin{pmatrix} \left(I_2 - \frac{\mathbf{p} \cdot \boldsymbol{\sigma}}{E+m}\right) \chi \\ \left(I_2 + \frac{\mathbf{p} \cdot \boldsymbol{\sigma}}{E+m}\right) \chi \end{pmatrix}, \tag{59}$$

$$\breve{v} = \sqrt{\frac{E+m}{2}} \begin{pmatrix} I_2 & -I_2 \\ I_2 & I_2 \end{pmatrix} \begin{pmatrix} \frac{\mathbf{p} \cdot \boldsymbol{\sigma}}{E+m} \zeta \\ \zeta \end{pmatrix} = \sqrt{\frac{E+m}{2}} \begin{pmatrix} \left(\frac{\mathbf{p} \cdot \boldsymbol{\sigma}}{E+m} - I_2\right) \zeta \\ \left(\frac{\mathbf{p} \cdot \boldsymbol{\sigma}}{E+m} + I_2\right) \zeta \end{pmatrix}, \tag{60}$$

where $\chi^\dagger \chi = 1 = \zeta^\dagger \zeta$ and $\hat{\mathbf{p}} = \mathbf{p}/p$ is a unit 3-vector, such that the normalizations are,

$$\bar{\breve{u}} \breve{u} = \breve{u}^\dagger \breve{\gamma}^0 \breve{u} = \bar{\breve{v}} \breve{v} = (E+m) \left(1 - \frac{p^2}{(E+m)^2}\right) = 2m. \tag{61}$$

3 Lorentz Boosts of Spinors

In 1928, both Darwin [19] and Weyl [20] discussed this topic, which marked a new application of group theory to physics.[11]

[11] For a more recent review, see [21], which uses a different version of the Weyl representation than do we.

We first digress slightly to record some lore about rotations of spinors.

3.1 Spatial Rotations of 2-Spinors

A spatial rotation of a 2-spinor should leave its length/normalization unchanged, so the 2×2 matrix operator U_2 that describes this rotation should be unitary: $\mathsf{U}_2^\dagger \mathsf{U}_2 = I_2$. This matrix has four elements, and so can be related to a set of four independent, unitary, 2×2 matrices such as $\sigma_\mu \equiv (I_2, \boldsymbol{\sigma})$: $\mathsf{U}_2 = a I_2 + b\,\sigma_1 + c\,\sigma_2 + d\,\sigma_3$.

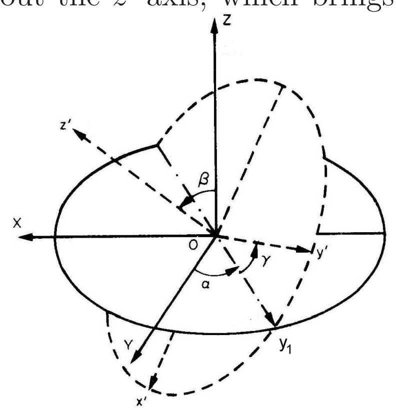
As shown in Prob. 3(c) of http://kirkmcd.princeton.edu/examples/ph410problems.pdf, a general 2×2 unitary matrix can be written as,

$$\mathsf{U}_2 = e^{i\delta}\left(\cos\frac{\theta}{2}\ I_2 + i\sin\frac{\theta}{2}\ \hat{\mathbf{u}}\cdot\boldsymbol{\sigma}\right) = e^{i\delta} e^{i\frac{\theta}{2}\hat{\mathbf{u}}\cdot\boldsymbol{\sigma}}, \tag{62}$$

where δ and θ are real numbers and $\hat{\mathbf{u}}$ is a real unit vector.[12]

A general rotation in 3-space is characterized by 3 angles. We follow Euler in naming these angles as in the figure below.[13] The rotation takes the axis (x, y, z) into the axes (x', y', z') in 3 steps:

1. A rotation by angle α about the z-axis, which brings the y-axis to the y_1 axis.
2. A rotation by angle β about the y_1-axis, which brings the z-axis to the z'-axis.
3. A rotation by angle γ about the z'-axis, which brings the y_1-axis to the y'-axis (and the x-axis to the x'-axis).



The 2×2 unitary matrix that corresponds to this rotation is,

$$\begin{aligned}
\mathsf{R}_2(\alpha,\beta,\gamma) &= \begin{pmatrix} \cos\frac{\beta}{2} e^{i(\alpha+\gamma)/2} & \\ -\sin\frac{\beta}{2} e^{i(\alpha-\gamma)/2} & \cos\frac{\beta}{2} e^{-i(\alpha+\gamma)/2} \end{pmatrix} \\
&= \begin{pmatrix} e^{i\gamma/2} & 0 \\ 0 & e^{-i\gamma/2} \end{pmatrix} \begin{pmatrix} \cos\frac{\beta}{2} & \sin\frac{\beta}{2} \\ -\sin\frac{\beta}{2} & \cos\frac{\beta}{2} \end{pmatrix} \begin{pmatrix} e^{i\alpha/2} & 0 \\ 0 & e^{-i\alpha/2} \end{pmatrix} \\
&= \mathsf{R}_{2,z'}(\gamma)\mathsf{R}_{2,y_1}(\beta)\mathsf{R}_{2,z}(\alpha),
\end{aligned} \tag{63}$$

[12]Note that if make the replacements $\theta \to -\theta$ and $\hat{\mathbf{u}} \to -\hat{\mathbf{u}}$ we obtain another valid representation of U_2, since the physical operation of a rotation by angle θ about an axis $\hat{\mathbf{u}}$ is identical to a rotation by $-\theta$ about the axis $-\hat{\mathbf{u}}$.

[13]From sec. 58 of Landau and Lifshitz, *Quantum Mechanics*.

where the decomposition into the product of 3 rotation matrices[14] follows from the particular rules,

$$\mathsf{R}_{2,x}(\phi) = \begin{pmatrix} \cos\frac{\phi}{2} & i\sin\frac{\phi}{2} \\ i\sin\frac{\phi}{2} & \cos\frac{\phi}{2} \end{pmatrix}, \tag{64}$$

$$\mathsf{R}_{2,y}(\phi) = \begin{pmatrix} \cos\frac{\phi}{2} & \sin\frac{\phi}{2} \\ -\sin\frac{\phi}{2} & \cos\frac{\phi}{2} \end{pmatrix}, \tag{65}$$

$$\mathsf{R}_{2,z}(\phi) = \begin{pmatrix} e^{i\phi/2} & 0 \\ 0 & e^{-i\phi/2} \end{pmatrix}. \tag{66}$$

and for a rotation of the coordinates axes by angle ϕ about direction $\hat{\mathbf{u}}$,

$$\mathsf{R}_{2,u}(\phi) = e^{i\frac{\phi}{2}\hat{\mathbf{u}}\cdot\boldsymbol{\sigma}} = I_2\cos\frac{\phi}{2} + i\,\hat{\mathbf{u}}\cdot\boldsymbol{\sigma}\sin\frac{\phi}{2}\,. \tag{67}$$

Rather than rotating the coordinate axes, we may wish to rotate vectors in Bloch space by an angle ϕ about a given axis $\hat{\mathbf{u}}$, while leaving the coordinate axes fixed. The operator,

$$\mathsf{R}_{2,u}(-\phi) = e^{-i\frac{\phi}{2}\hat{\mathbf{u}}\cdot\boldsymbol{\sigma}}, \tag{68}$$

performs this type of rotation.

3.2 Lorentz Boosts of 2-Spinors

Lorentz boosts (transformations) are more familiar for ordinary 4 vectors, such as $x^\mu = (t, x, y, z)$. Here, we seek a 2×2 matrix L_2 such that $\mathsf{L}_2\chi$ is a Lorentz transformation of the 2-spinor χ.[15]

To understand the effect of a Lorentz boost on a 2-spinor, it is helpful to consider the 2×2 matrix,

$$\mathsf{X}_2 = \begin{pmatrix} t+z & x-iy \\ x+iy & t-z \end{pmatrix} = tI_2 + x\sigma_x + y\sigma_y + z\sigma_z = x^\mu \underline{\sigma}_\mu, \tag{69}$$

where,[16]

$$\sigma_\mu = (I_2, \boldsymbol{\sigma}), \qquad \underline{\sigma}_\mu = (I_2, -\boldsymbol{\sigma}), \tag{70}$$

Then, the determinant of X_2 is,

$$|\mathsf{X}_2| = t^2 - x^2 - y^2 - z^2 = x^\mu x_\mu, \tag{71}$$

[14]The order of operations is that the rightmost rotation in eq. (63) is to be performed first.
[15]We follow sec. III of [21].
[16]Our $\underline{\sigma}_\mu$ is often written as $\bar{\sigma}_\mu$.

which is a Lorentz invariant.

The Lorentz transformation of X_2 takes the form $\mathsf{X}_2' = \mathsf{L}_2 \mathsf{X}_2 \mathsf{L}_2^\dagger$. Lorentz invariance implies that,

$$\det \mathsf{X}_2 = |\mathsf{X}_2'| = |\mathsf{L}_2|\,|\mathsf{X}_2|\,\left|\mathsf{L}_2^\dagger\right| = |\mathsf{L}_2|^2\,|\mathsf{X}_2|\,, \qquad |\mathsf{L}_2|^2 = 1. \tag{72}$$

It suffices to consider that $|\mathsf{L}_2| = 1$, for which a simple example is,

$$\mathsf{L}_{2,z} = \mathsf{L}_{2,z}^\dagger = \begin{pmatrix} e^{\frac{w}{2}} & 0 \\ 0 & e^{-\frac{w}{2}} \end{pmatrix} = e^{\frac{w}{2}\sigma_3} = \cosh(w/2)\,I_2 + \sinh(w/2)\,\sigma_3. \tag{73}$$

With this,

$$\mathsf{X}_2' = \begin{pmatrix} t' + z' & x' - iy' \\ x' + iy' & t' - z' \end{pmatrix} = \mathsf{L}_{2,z} \mathsf{X}_2 \mathsf{L}_{2,z}^\dagger = \begin{pmatrix} e^{w}(t+z) & x - iy \\ x + iy & e^{-w}(t-z) \end{pmatrix}, \tag{74}$$

and hence,

$$t' = t\cosh w + z\sinh w, \qquad x' = x, \qquad y' = y, \qquad z' = z\cosh w + t\sinh w. \tag{75}$$

A particle at rest at the origin in the unprimed frame has velocity $\mathbf{v}' = \tanh w\,\hat{\mathbf{z}}'$ in the $'$ frame. That is, $\mathsf{L}_{2,z}$ corresponds to a Lorentz boost of the particle along the z-axis. The boost parameter $w = \tanh^{-1}(v')$ is called the rapidity.[17]

Similarly, $\mathsf{L}_{2,x} = e^{\frac{w}{2}\sigma_1}$ and $\mathsf{L}_{2,y} = e^{\frac{w}{2}\sigma_2}$ correspond to Lorentz boosts along the x- and y-axes. Thus, a boost of a 2-spinor along direction $\hat{\mathbf{w}}$ is described by the 2×2 matrix,

$$\begin{aligned} \mathsf{L}_{2,\mathbf{w}} = e^{\frac{\mathbf{w}}{2}\cdot\boldsymbol{\sigma}} &= I_2 + \frac{\mathbf{w}\cdot\boldsymbol{\sigma}}{2} + \frac{1}{2!}\left(\frac{\mathbf{w}\cdot\boldsymbol{\sigma}}{2}\right)^2 + \frac{1}{3!}\left(\frac{\mathbf{w}\cdot\boldsymbol{\sigma}}{2}\right)^3 + \cdots \\ &= I_2\left(1 + \frac{1}{2!}\left(\frac{w}{2}\right)^2 + \cdots\right) + \hat{\mathbf{w}}\cdot\boldsymbol{\sigma}\left(\frac{w}{2} + \frac{1}{3!}\left(\frac{w}{2}\right)^3 + \cdots\right) \\ &= I_2\cosh\frac{w}{2} + \hat{\mathbf{w}}\cdot\boldsymbol{\sigma}\sinh\frac{w}{2}, \end{aligned} \tag{76}$$

recalling eq. (40) that $(\mathbf{w}\cdot\boldsymbol{\sigma})^2 = w^2 I_2$.

If the boost is applied to a particle of mass m at rest, it takes on momentum $\mathbf{p}$ and velocity $\mathbf{v}$ related by,

$$w = \tanh^{-1} v, \qquad \tanh w = v = \frac{p}{E}, \qquad \cosh w = \frac{E}{m},\ \sinh w = \frac{p}{m}. \tag{77}$$

$$\cosh\frac{w}{2} = \sqrt{\frac{E+m}{2m}} = \frac{\sqrt{E+p}}{2} + \frac{\sqrt{E-p}}{2}, \tag{78}$$

$$\sinh\frac{w}{2} = \sqrt{\frac{E-m}{2m}} = \frac{p}{\sqrt{2m(E+m)}} = \frac{\sqrt{E+p}}{2} - \frac{\sqrt{E-p}}{2}, \tag{79}$$

[17] The concept of rapidity was introduced in 1910 by Varićak [22] and by Whittaker [23], and given that name by Robb (1911), p. 9 of [24].

which permits eq. (76) to be rewritten in various ways, such as,

$$\mathsf{L}_{2,\mathbf{w}} = \sqrt{\frac{E+m}{2m}} \left(I_2 + \frac{\mathbf{p}\cdot\boldsymbol{\sigma}}{E+m} \right). \tag{80}$$

If we desire a highly compact notation, we can write,

$$\mathsf{L}_{2,\mathbf{w}} = e^{\frac{\mathbf{w}}{2}\cdot\boldsymbol{\sigma}} = \sqrt{e^{\mathbf{w}\cdot\boldsymbol{\sigma}}} = \sqrt{I_2 \cosh w + \hat{\mathbf{w}}\cdot\boldsymbol{\sigma} \sinh w} = \sqrt{\frac{EI_2 + \mathbf{p}\cdot\boldsymbol{\sigma}}{m}} = \sqrt{\frac{p^\mu \underline{\sigma}_\mu}{m}} = \sqrt{\frac{p\,\underline{\sigma}}{m}}. \tag{81}$$

using eq. (70). Note also that,

$$\mathsf{L}_{2,-\mathbf{w}} = e^{-\frac{\mathbf{w}}{2}\cdot\boldsymbol{\sigma}} = \sqrt{\frac{p\,\sigma}{m}}. \tag{82}$$

3.3 Lorentz Boosts of 4-Spinors

Recalling the general forms (42) and (44) of Dirac 4-spinors in the Dirac representation, the Lorentz boost L (a 4×4 matrix) of a particle/antiparticle at rest to velocity $\mathbf{v} = \mathbf{p}/E$ must obey,

$$\mathsf{L}\sqrt{2m}\begin{pmatrix} \chi \\ 0 \end{pmatrix} = \sqrt{E+m}\begin{pmatrix} \chi \\ \frac{\mathbf{p}\cdot\boldsymbol{\sigma}}{E+m}\chi \end{pmatrix}, \qquad \mathsf{L}\sqrt{2m}\begin{pmatrix} 0 \\ \zeta \end{pmatrix} = \sqrt{E+m}\begin{pmatrix} \frac{\mathbf{p}\cdot\boldsymbol{\sigma}}{E+m}\zeta \\ \zeta \end{pmatrix}. \tag{83}$$

Hence, L has the form,

$$\mathsf{L} = \begin{pmatrix} AI_2 & BI_2 \\ BI_2 & AI_2 \end{pmatrix}, \tag{84}$$

where,

$$A = \sqrt{\frac{1}{2}\left(\frac{E}{m}+1\right)} = \sqrt{\frac{\cosh w + 1}{2}} = \cosh\frac{w}{2}, \tag{85}$$

$$B = \sqrt{\frac{p^2}{2m(E+m)}}\,\hat{\mathbf{p}}\cdot\boldsymbol{\sigma} = \sqrt{\frac{1}{2}\left(\frac{E}{m}-1\right)}\,\hat{\mathbf{p}}\cdot\boldsymbol{\sigma} = \sqrt{\frac{\cosh w - 1}{2}}\,\hat{\mathbf{p}}\cdot\boldsymbol{\sigma} = \sinh\frac{w}{2}\,\hat{\mathbf{p}}\cdot\boldsymbol{\sigma}, \tag{86}$$

and w is the boost rapidity, related by eqs. (77)-(79). That is, in the Dirac representation, the boost matrix is, with $\hat{\mathbf{p}} = \hat{\mathbf{v}} = \hat{\mathbf{w}}$,

$$\mathsf{L}_{\mathbf{w}} = \begin{pmatrix} I_2 \cosh(w/2) & \hat{\mathbf{w}}\cdot\boldsymbol{\sigma}\sinh(w/2) \\ \hat{\mathbf{w}}\cdot\boldsymbol{\sigma}\sinh(w/2) & I_2\cosh(w/2) \end{pmatrix} = \sqrt{\frac{E+m}{2m}}\begin{pmatrix} I_2 & \frac{\mathbf{p}\cdot\boldsymbol{\sigma}}{E+m} \\ \frac{\mathbf{p}\cdot\boldsymbol{\sigma}}{E+m} & I_2 \end{pmatrix}. \tag{87}$$

The boost matrix in the Weyl representation can be obtained via the transformation (47),

$$\breve{\mathsf{L}}_{\mathbf{w}} = \begin{pmatrix} I_2\cosh(w/2) - \hat{\mathbf{w}}\cdot\boldsymbol{\sigma}\,\sinh(w/2) & 0 \\ 0 & I_2\cosh(w/2) + \hat{\mathbf{w}}\cdot\boldsymbol{\sigma}\,\sinh(w/2) \end{pmatrix} \tag{88}$$

$$= \sqrt{\frac{E+m}{2m}} \begin{pmatrix} I_2 - \frac{\mathbf{p}\cdot\boldsymbol{\sigma}}{E+m} & 0 \\ 0 & I_2 + \frac{\mathbf{p}\cdot\boldsymbol{\sigma}}{E+m} \end{pmatrix} = \begin{pmatrix} e^{-\frac{\mathbf{w}}{2}\cdot\boldsymbol{\sigma}} & 0 \\ 0 & e^{\frac{\mathbf{w}}{2}\cdot\boldsymbol{\sigma}} \end{pmatrix} = \frac{1}{\sqrt{m}} \begin{pmatrix} \sqrt{p\,\sigma} & 0 \\ 0 & \sqrt{p\,\underline{\sigma}} \end{pmatrix}$$

using eqs. (76) and (81)-(82).

4 Electric Charge Conjugation

Pauli introduced the concept of electric charge conjugation of Dirac spinors and the operator C (but not the term charge conjugation) in 1936, p. 130 of [25] (see also [26]), such that,[18]

$$\psi^{(C)} = C\psi^{\star}, \tag{89}$$

is the charge-conjugate state of spinor ψ, with the same energy, momentum and spin components as ψ, but if ψ is a particle plane-wave state with spacetime wavefunction e^{-ipx}, then $\psi^{(C)}$ is an antiparticle state with spacetime wavefunction e^{ipx}, and *vice versa*. If state ψ has electric charge q, then $\psi^{(C)}$ has electric charge $-q$.[19] Charge conjugation leaves mass unchanged, such that a particle and its antiparticle have the same rest mass m.[20]

To deduce the charge-conjugation matrix C, we start with the Dirac equation (1) for a particle state ψ.[21] We expect that the antiparticle state $\psi^{(C)} = C\psi^*$ also satisfies the Dirac equation,

$$i\gamma^\mu\partial_\mu C\psi^* = mC\psi^*. \tag{90}$$

A clever step is to take the complex conjugate of eq. (1),

$$-i\gamma^{\mu *}\partial_\mu\psi^* = m\psi^*. \tag{91}$$

Applying the desired charge-conjugation operator C to this, we have,

$$-iC\gamma^{\mu *}\partial_\mu\psi^* = mC\psi^* = m\psi^{(C)}. \tag{92}$$

For this to be the Dirac equation (90), we require that,

$$-C\gamma^{\mu *} = \gamma^\mu C. \tag{93}$$

[18] We will also use the notation $\tilde{\psi}$ for the antiparticle $\psi^{(C)}$ of a state ψ.

[19] The term "charge conjugation" (but with the symbol L) may have been first by Kramers (1937) [27]. The term antimatter was introduced by Schuster in 1898 [28], but in his vision antimatter had negative mass. The present vision of antiparticles via electric charge conjugation of particles is perhaps closer to Kelvin's image method for a planar conductor, p. 288 of [29].

[20] This was not initially understood by Dirac, who first speculated that the antiparticle of an electron is a proton [11].

[21] This argument follows sec. 5.4, p. 107 of [30].

You can verify that this implies the electric-charge-conjugation matrix operator to be,[22]

$$C = i\gamma^2 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & i\sigma_2 \\ -i\sigma_2 & 0 \end{pmatrix}. \tag{94}$$

Then, applying the electric-charge-conjugation transformation to the particle 4-spinor u of eq. (42), we obtain (on suppression of the overall factor $\sqrt{E+m}$) the antiparticle spinor,

$$\tilde{u} = i\gamma^2 \begin{pmatrix} \chi^\star \\ \frac{\mathbf{p}\cdot\boldsymbol{\sigma}^\star}{E+m}\chi^\star \end{pmatrix} = \begin{pmatrix} i\sigma_2 \frac{\mathbf{p}\cdot\boldsymbol{\sigma}^\star}{E+m}\chi^\star \\ -i\sigma_2\chi^\star \end{pmatrix} = \begin{pmatrix} \frac{\mathbf{p}\cdot\boldsymbol{\sigma}}{E+m}(-i\sigma_2\chi^\star) \\ -i\sigma_2\chi^\star \end{pmatrix} = \begin{pmatrix} \frac{\mathbf{p}\cdot\boldsymbol{\sigma}}{E+m}\tilde{\chi} \\ \tilde{\chi} \end{pmatrix} = v, \tag{95}$$

using that fact that $\sigma_2\,\boldsymbol{\sigma}^* = -\boldsymbol{\sigma}\,\sigma_2$. Hence, the antiparticle 2-spinor $\tilde{\chi}$ is related to its corresponding particle 2-spinor χ by,

$$\tilde{\chi} = -i\sigma_2\chi^\star, \qquad \chi = i\sigma_2\tilde{\chi}^*. \tag{96}$$

4.1 Electric-Charge Conjugation in the Weyl Representation

The discussion of eqs. (90)-(92) holds in the Weyl representation as well, so we infer that the electric-charge-conjugation operator $\breve{C}$ in that representation must obey,

$$-\breve{C}\breve{\gamma}^{\mu *} = \breve{\gamma}^\mu\breve{C}. \tag{97}$$

You can verify that this implies the electric-charge-conjugation matrix operator to be,

$$\breve{C} = -i\gamma^2 = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -i\sigma_2 \\ i\sigma_2 & 0 \end{pmatrix} \qquad [= -C]. \tag{98}$$

Then, applying the electric-charge-conjugation transformation to the particle 4-spinor u of eq. (59), we obtain (on suppression of the overall factor $\sqrt{(E+m)/2}$) the antiparticle spinor, recalling that $\sigma_2\,\boldsymbol{\sigma}^* = -\boldsymbol{\sigma}\,\sigma_2$ and $\tilde{\chi} = -i\sigma_2\chi^*$,

$$\begin{aligned} \tilde{\breve{u}} = -i\breve{\gamma}^2 \begin{pmatrix} \left(1 - \frac{\mathbf{p}\cdot\boldsymbol{\sigma}^\star}{E+m}\right)\chi^\star \\ \left(1 + \frac{\mathbf{p}\cdot\boldsymbol{\sigma}^\star}{E+m}\right)\chi^\star \end{pmatrix} &= \begin{pmatrix} -i\sigma_2\left(1 + \frac{\mathbf{p}\cdot\boldsymbol{\sigma}^\star}{E+m}\right)\chi^\star \\ i\sigma_2\left(1 - \frac{\mathbf{p}\cdot\boldsymbol{\sigma}^\star}{E+m}\right)\chi^\star \end{pmatrix} = \begin{pmatrix} \left(1 - \frac{\mathbf{p}\cdot\boldsymbol{\sigma}}{E+m}\right)(-i\sigma_2\chi^\star) \\ \left(1 + \frac{\mathbf{p}\cdot\boldsymbol{\sigma}}{E+m}\right)(i\sigma_2\chi^\star) \end{pmatrix} \\ &= -\begin{pmatrix} \left(\frac{\mathbf{p}\cdot\boldsymbol{\sigma}}{E+m} - 1\right)\tilde{\chi} \\ \left(\frac{\mathbf{p}\cdot\boldsymbol{\sigma}}{E+m} + 1\right)\tilde{\chi} \end{pmatrix} = \breve{v}, \end{aligned} \tag{99}$$

as in eq. (59), taking $\zeta = -\tilde{\chi}$.

[22]Warning: Many people write $C\gamma_0$ for the matrix C of eq. (94). See, for example, https://en.wikipedia.org/wiki/Gamma_matrices

5 Helicity States

The concept of right- and lefthanded circularly polarized waves arose in optics,[23] and their equivalents for particles with spin are called helicity states,[24] which are to be distinguished from the chirality states discussed in sec. 6 below.

The helicity states of a spin-1/2 particle with nonzero velocity describe its spin component along the direction of its momentum $\mathbf{p}$.

5.1 2-Component Helicity Spinors

A general 2-component (spinor) state $\xi = a\,\xi_+ + b\,\xi_-$ where $|a|^2 + |b|^2 = 1$, can also be written as, to within an overall phase factor,

$$\xi = \cos\theta\,\xi_+ + e^{i\phi}\sin\theta\,\xi_-. \tag{100}$$

So, it is tempting to interpret parameters θ and ϕ as angles describing the orientation of a unit 3-vector that is associated with the state χ in a spherical coordinate system (r, θ, ϕ). The state ξ_+ might then correspond to the unit 3-vector $\hat{\mathbf{z}}$ that points up along the z-axis, while $\xi_- \leftrightarrow -\hat{\mathbf{z}}$.

However, this doesn't work! The suggestion is that the state ξ_+ corresponds to angles $\theta = 0$, $\phi = 0$ and state ξ_- to angles $\theta = \pi$, $\phi = 0$. With this hypothesis, eq. (100) gives a satisfactory representation of a spin-up state as ξ_+, but it implies that the spin-down state would be $e^{i\pi}\xi_+ = -\xi_+$, which is not really distinct from the spin-up state ξ_+.

We fix up things be writing,

$$\xi = e^{i\delta}\left(\cos\frac{\theta}{2}\,\xi_+ + e^{i\phi}\sin\frac{\theta}{2}\,\xi_-\right), \tag{101}$$

and identifying angles θ and ϕ with the polar and azimuthal angles of a unit 3-vector in an abstract 3-space (sometimes called the Bloch sphere). That is, we associate the state ξ with the unit 3-vector whose components are $(x, y, z) = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$. Now, the associations,

$$\text{spin up} \leftrightarrow (\theta = 0, \phi = 0) \leftrightarrow \chi_+, \qquad \text{spin down} \leftrightarrow (\theta = \pi, \phi = 0) \leftrightarrow \chi_-, \tag{102}$$

given by eq. (101) are satisfactory.

We then infer from eq. (101) that the spin-up and spin-down states in the direction (θ, ϕ) are, to within an overall phase factor,

$$\xi_+(\theta,\phi) \propto \begin{pmatrix} \cos\frac{\theta}{2} \\ \sin\frac{\theta}{2}\,e^{i\phi} \end{pmatrix}, \qquad \xi_-(\theta,\phi) \propto \chi_+(\pi-\theta,\phi+\pi) = \begin{pmatrix} \sin\frac{\theta}{2} \\ -\cos\frac{\theta}{2}\,e^{i\phi} \end{pmatrix}. \tag{103}$$

[23] Phenomena related to circular polarization were investigated by rivals Arago and Biot in 1811, whose thinking was based on a corpuscular theory of light. In a wave theory, Fresnel first distinguished between (transverse) linear and circular polarization in 1817. See, for example, [31].

[24] The term helicity may have first been used by Watanabe [33], whereas the term spirality had been proposed by Lee and Yang shortly before, on p. 1673 of [35]. The latter defined positive helicity/spirality as spin in the direction of momentum, and called this righthanded. However, the historical convention of optics was that light waves with positive helicity were called lefthanded circularly polarized.

The standard form of the spin-up/down states is,

$$\xi_+(\theta,\phi) = \begin{pmatrix} \cos\frac{\theta}{2}\,e^{-i\phi/2} \\ \sin\frac{\theta}{2}\,e^{i\phi/2} \end{pmatrix}, \qquad \xi_-(\theta,\phi) = \begin{pmatrix} \sin\frac{\theta}{2}\,e^{-i\phi/2} \\ -\cos\frac{\theta}{2}\,e^{i\phi/2} \end{pmatrix}, \tag{104}$$

which is consistent with eq. (103), but perhaps does not obviously follow from it.[25]

We can now define the positive and negative helicity 2-spinor states for a particle with 3-momentum $\mathbf{p}$ in direction (θ,ϕ) as $\chi_+ = \xi_+(\theta,\phi)$ and $\chi_- = \xi_-(\theta,\phi)$, respectively, recalling eq. (104), while the helicity states of an antiparticle are $\tilde{\chi}_+ = \xi_-(\theta,\phi) = \chi_-$ and $\tilde{\chi}_- = -\xi_+(\theta,\phi) = -\chi_+$. In all cases, positive helicity means spin in the direction of momentum $\mathbf{p}$.

The helicity 2-spinors transform under electric-charge conjugation (96) as,

$$\chi_+ = \begin{pmatrix} \cos\frac{\theta}{2}\,e^{-i\phi/2} \\ \sin\frac{\theta}{2}\,e^{i\phi/2} \end{pmatrix} \quad\to\quad \tilde{\chi}_+ = -i\sigma_2\chi_+^* = \begin{pmatrix} -\sin\frac{\theta}{2}\,e^{-i\phi/2} \\ \cos\frac{\theta}{2}\,e^{i\phi/2} \end{pmatrix} = \chi_-, \tag{105}$$

$$\chi_- = \begin{pmatrix} -\sin\frac{\theta}{2}\,e^{-i\phi/2} \\ \cos\frac{\theta}{2}\,e^{i\phi/2} \end{pmatrix} \quad\to\quad \tilde{\chi}_- = -i\sigma_2\chi_-^* = \begin{pmatrix} -\cos\frac{\theta}{2}\,e^{-i\phi/2} \\ -\sin\frac{\theta}{2}\,e^{i\phi/2} \end{pmatrix} = -\chi_+, \tag{106}$$

as claimed above.

5.1.1 Helicity Projection Operator for 2-Spinors

The helicity operator $h_2 = \hat{\mathbf{p}}(\theta,\phi)\cdot\boldsymbol{\sigma}$ has the form,

$$h_2 = \hat{\mathbf{p}}(\theta,\phi)\cdot\boldsymbol{\sigma} = \begin{pmatrix} \cos\theta & \sin\theta\,e^{-i\phi} \\ \sin\theta\,e^{i\phi} & -\cos\theta \end{pmatrix}, \tag{107}$$

such that $h_2\chi_\pm = \hat{\mathbf{p}}(\theta,\phi)\cdot\boldsymbol{\sigma}\,\chi_\pm = \pm\chi_\pm$.

If a general 2-spinor is written as $\chi = a_+\chi_+ + a_-\chi_-$, in terms of the helicity 2-spinors for the (θ,ϕ) direction, then $(I_2 \pm h_2)\chi = 2a_\pm\chi_\pm$. Hence,

$$P_{2,\pm}(\theta,\phi) = \frac{I_2 \pm h_2}{2} = \frac{I_2 \pm \hat{\mathbf{p}}(\theta,\phi)\cdot\boldsymbol{\sigma}}{2} \tag{108}$$

are the 2-spinor helicity projection operators for direction (θ,ϕ): $P_{2,\pm}\chi_\pm = \chi_\pm$, $P_{2,\pm}\chi_\mp = 0$.

[25] The standard form (104) can also be deduced via spin-1/2 rotation matrices, as discussed, for example, in Prob. 1 of http://kirkmcd.princeton.edu/examples/ph406/ph406_set4_2014_sol.pdf

5.2 4-Component Helicity Spinors

Recalling eqs. (42) and (44), the particle and antiparticle helicity 4-spinors $u_\pm$ and $v_\pm$ are, with $\tilde{\chi}_\pm = \pm\chi_\mp$, and defining $\sqrt{E_+} = \sqrt{E+m}$ and $\sqrt{E_-} = \sqrt{E-m} = p/\sqrt{E+m}$,

$$u_+(\theta,\phi) = \begin{pmatrix} \sqrt{E+m}\,\chi_+ \\ \frac{p}{\sqrt{E+m}}\,\hat{\mathbf{p}}\cdot\boldsymbol{\sigma}\,\chi_+ \end{pmatrix} = \begin{pmatrix} \sqrt{E_+}\,\chi_+ \\ \sqrt{E_-}\,\chi_+ \end{pmatrix} = \begin{pmatrix} \sqrt{E_+}\,\cos\frac{\theta}{2}\,e^{-i\phi/2} \\ \sqrt{E_+}\,\sin\frac{\theta}{2}\,e^{i\phi/2} \\ \sqrt{E_-}\,\cos\frac{\theta}{2}\,e^{-i\phi/2} \\ \sqrt{E_-}\,\sin\frac{\theta}{2}\,e^{i\phi/2} \end{pmatrix}, \quad (109)$$

$$u_-(\theta,\phi) = \begin{pmatrix} \sqrt{E+m}\,\chi_- \\ \frac{p}{\sqrt{E+m}}\,\hat{\mathbf{p}}\cdot\boldsymbol{\sigma}\,\chi_- \end{pmatrix} = \begin{pmatrix} \sqrt{E_+}\,\chi_- \\ -\sqrt{E_-}\,\chi_- \end{pmatrix} = \begin{pmatrix} -\sqrt{E_+}\,\sin\frac{\theta}{2}\,e^{-i\phi/2} \\ \sqrt{E_+}\,\cos\frac{\theta}{2}\,e^{i\phi/2} \\ \sqrt{E_-}\,\sin\frac{\theta}{2}\,e^{-i\phi/2} \\ -\sqrt{E_-}\,\cos\frac{\theta}{2}\,e^{i\phi/2} \end{pmatrix}, \quad (110)$$

$$v_+(\theta,\phi) = \begin{pmatrix} \frac{p}{\sqrt{E+m}}\,\hat{\mathbf{p}}\cdot\boldsymbol{\sigma}\,\tilde{\chi}_+ \\ \sqrt{E+m}\,\tilde{\chi}_+ \end{pmatrix} = \begin{pmatrix} -\sqrt{E_-}\,\chi_- \\ \sqrt{E_+}\,\chi_- \end{pmatrix} = \begin{pmatrix} \sqrt{E_-}\,\sin\frac{\theta}{2}\,e^{-i\phi/2} \\ -\sqrt{E_-}\,\cos\frac{\theta}{2}\,e^{i\phi/2} \\ -\sqrt{E_+}\,\sin\frac{\theta}{2}\,e^{-i\phi/2} \\ \sqrt{E_+}\,\cos\frac{\theta}{2}\,e^{i\phi/2} \end{pmatrix}, \quad (111)$$

$$v_-(\theta,\phi) = \begin{pmatrix} \frac{p}{\sqrt{E+m}}\,\hat{\mathbf{p}}\cdot\boldsymbol{\sigma}\,\tilde{\chi}_- \\ \sqrt{E+m}\,\tilde{\chi}_- \end{pmatrix} = \begin{pmatrix} -\sqrt{E_-}\,\chi_+ \\ -\sqrt{E_+}\,\chi_+ \end{pmatrix} = \begin{pmatrix} -\sqrt{E_-}\,\cos\frac{\theta}{2}\,e^{-i\phi/2} \\ -\sqrt{E_-}\,\sin\frac{\theta}{2}\,e^{i\phi/2} \\ -\sqrt{E_+}\,\cos\frac{\theta}{2}\,e^{-i\phi/2} \\ -\sqrt{E_+}\,\sin\frac{\theta}{2}\,e^{i\phi/2} \end{pmatrix}. \quad (112)$$

Note that $v_\pm = Cu_\pm^*$, using the electric-charge-conjugation operator $C = i\gamma_2$ found in eq. (94).

5.2.1 Helicity Projection Operators for 4-Spinors.

The generalization to 4-spinors of the 2-spinor helicity projection operators (108) is,

$$h = \begin{pmatrix} h_2 & 0 \\ 0 & h_2 \end{pmatrix}, \qquad P_\pm = \frac{I_4 \pm h}{2}, \quad (113)$$

such that $h\,u_\pm = \pm u_\pm$, $P_\pm u_\pm = u_\pm$, $P_\pm u_\mp = 0$. Then, recalling eqs. (109)-(112), the operator P_+ projects positive helicity for particles, but negative helicity for antiparticles, while P_- projects negative helicity for particles, but positive helicity for antiparticles, $P_\pm v_\mp = v_\mp$, $P_\pm v_\pm = 0$.

We also note that,

$$\gamma^0 \boldsymbol{\gamma} \gamma^5 = \begin{pmatrix} \mathbf{I} & 0 \\ 0 & -\mathbf{I} \end{pmatrix} \begin{pmatrix} 0 & \boldsymbol{\sigma} \\ -\boldsymbol{\sigma} & 0 \end{pmatrix} \begin{pmatrix} 0 & \mathbf{I} \\ \mathbf{I} & 0 \end{pmatrix} = \begin{pmatrix} \boldsymbol{\sigma} & 0 \\ 0 & \boldsymbol{\sigma} \end{pmatrix}, \qquad P_\pm = \frac{I_4 \pm \gamma_0\, \hat{\mathbf{p}}(\theta,\phi) \cdot \boldsymbol{\gamma}\, \gamma_5}{2}, \tag{114}$$

and $\gamma_\mu \gamma_0 \hat{\mathbf{p}} \cdot \boldsymbol{\gamma}\, \gamma_5 = \gamma_0 \hat{\mathbf{p}} \cdot \boldsymbol{\gamma}\, \gamma_5 \gamma_\mu$. Consequently, the helicity 4-spinor states $\psi_\pm$ satisfy the Dirac equation, $i\gamma^\mu \partial_\mu \psi_\pm = m\, \psi_\pm$,

$$P_\pm(i\gamma^\mu \partial_\mu \psi) = i\gamma^\mu \partial_\mu P_\pm \psi = i\gamma^\mu \partial_\mu \psi_\pm = P_\pm(m\psi) = m\, \psi_\pm. \tag{115}$$

5.3 Helicity Conservation in the Electromagnetic Interaction of High-Energy Spin-1/2 Particles

We consider the electromagnetic interaction of a pointlike particle of charge e, in which it scatters from motion along the z axis to that at angle θ to the z-axis, and in which the particle has high energy at all times ($E \gg m,\ E \approx p$). The part of the scattering matrix element involving the 4-spinor of this particle is $e\bar{u}\gamma^\mu u$ or $e\bar{v}\gamma^\mu u$.[26] We now show that in this approximation, matrix elements $\langle \bar{u}_-(\theta)|\gamma_\mu|u_+(0)\rangle$ vanish for $\mu = 0, 1, 2, 3$, and similarly that $\langle \bar{v}_+(\theta)|\gamma_\mu|u_+(0)\rangle = 0$, *i.e.*, initial particle couples only to final states of the same helicity.

For high-speed motion ($E_+ \approx E_- \approx E$) along the $+z$-axis, the 4-spinors are,

$$u_+(0) \to \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad u_-(0) \to \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix}, \quad v_+(0) \to \begin{pmatrix} 0 \\ -1 \\ 0 \\ 1 \end{pmatrix}, \quad v_-(0) \to \begin{pmatrix} -1 \\ 0 \\ -1 \\ 0 \end{pmatrix}. \tag{116}$$

Recalling that,

$$\gamma_0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \qquad \gamma_1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix},$$

$$\gamma_2 = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix}, \qquad \gamma_3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \tag{117}$$

[26] See, for example, pp. 89-92 of http://kirkmcd.princeton.edu/examples/ph529/ph529l6.pdf

we have that,

$$\gamma_0 u_+(0) = \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix}, \quad \gamma_1 u_+(0) = \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix}, \quad \gamma_2 u_+(0) = \begin{pmatrix} 0 \\ i \\ 0 \\ -i \end{pmatrix}, \quad \gamma_3 u_+(0) = \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix} \tag{118}$$

To evaluate matrix elements such as $\bar{u}_f \gamma_\mu u_i$ we recall that this equals $u_f^\dagger \gamma_0 \gamma_\mu u_i$, so we multiply eq. (115) by γ_0 to obtain,

$$\gamma_0 \gamma_0 u_+(0) = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad \gamma_0 \gamma_1 u_+(0) = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}, \quad \gamma_0 \gamma_2 u_+(0) = \begin{pmatrix} 0 \\ i \\ 0 \\ i \end{pmatrix}, \quad \gamma_0 \gamma_3 u_+(0) = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \tag{119}$$

Then, using eqs. (110)-(111) we see that,

$$\langle \bar{u}_-(\theta, \phi) | \gamma_\mu | u_+(0) \rangle = u_-^\dagger(\theta, \phi) \gamma_0 \gamma_\mu u_+(0) = 0 = \langle \bar{v}_+(\theta, \phi) | \gamma_\mu | u_+(0) \rangle. \tag{120}$$

Hence, a high-energy pointlike spin-1/2 particle of a given helicity cannot couple a particle of the opposite helicity via the electromagnetic interaction, nor can it annhilate with an antiparticle of the same helicity. It is possible for a high-energy spin-1/2 particle of a given helicity to scatter into a particle of the same helicity, or annihilate with an antiparticle of opposite helicity, via single-photon emission.

Examples where helicity conservation in the high-energy limit is useful in providing a simplified understanding include e^+e^- annihilation into a pair of spin-0 or spin-1/2 particles, as well as elastic scattering of electrons off spin-0 and spin-1/2 particles, as discussed on p. 118 ff of http://kirkmcd.princeton.edu/examples/ph529/ph529l7.pdf.

5.4 Orthogonality Relations of the Helicity 4-Spinors

If we label the four components of a general spinor ψ as ψ_i, $i = 1, 4$, then,

$$\phi^\dagger \psi = \phi_1^* \phi_1 + \phi_2^* \phi_2 + \phi_3^* \phi_3 + \phi_4^* \phi_4, \tag{121}$$

while,

$$\bar{\phi} \psi = \phi^\dagger \gamma_0 \psi = \phi_1^* \phi_1 + \phi_2^* \phi_2 - \phi_3^* \phi_3 - \phi_4^* \phi_4. \tag{122}$$

The helicity 4-spinors (109)-(112) obey,

$$u_+^\dagger u_+ = u_-^\dagger u_- = v_+^\dagger v_+ = v_-^\dagger v_- = 2E, \tag{123}$$

$$u_+^\dagger u_- = u_+^\dagger v_+ = u_-^\dagger v_- = v_+^\dagger v_- = 0, \tag{124}$$

$$u_+^\dagger v_- = -2p = -u_-^\dagger v_+, \tag{125}$$

while,

$$\bar{u}_+ u_+ = \bar{u}_- u_- = \bar{v}_+ v_+ = \bar{v}_- v_- = 2m, \tag{126}$$

$$\bar{u}_+ u_- = \bar{u}_+ v_+ = \bar{u}_- v_- = \bar{v}_+ v_- = \bar{u}_+ v_- = \bar{u}_- v_+ = 0. \tag{127}$$

That is, the $u_\pm$ and $v_\pm$ spinors are fully orthogonal with respect to the scalar product $\bar{\phi}\psi$, but not with respect to $\phi^\dagger\psi$.

5.5 Helicity Is Not Lorentz Invariant

Helicity is only well defined for particles in motion, which indicates that it is not a Lorentz-invariant concept. A particle moving in the $+z$ direction with spin up with respect to the z-axis has positive helicity. But, that particle can be viewed in a frame where its velocity is in the $-z$ direction, while its spin remains up with respect to the z-axis, such that its velocity is negative in this frame.

It is of interest to define a Lorentz-invariant attribute of particles with spin that corresponds to their helicity in the high-energy limit. This is chirality, discussed in the next section.

5.6 Helicity in the Weyl Representation

The helicity spinors (109)-(112) can be transformed to the Weyl representation via $\breve{\psi} = \breve{U}\psi$ of eq. (47), with $A_\pm \equiv (\sqrt{E+m} \pm \sqrt{E-m})/\sqrt{2} = (\sqrt{E_+} \pm \sqrt{E_-})/\sqrt{2}$,

$$\breve{u}_+(\theta,\phi) = \begin{pmatrix} A_- \, \chi_+ \\ A_+ \, \chi_+ \end{pmatrix} = \begin{pmatrix} A_- \cos\frac{\theta}{2} \, e^{-i\phi/2} \\ A_- \sin\frac{\theta}{2} \, e^{i\phi/2} \\ A_+ \cos\frac{\theta}{2} \, e^{-i\phi/2} \\ A_+ \sin\frac{\theta}{2} \, e^{i\phi/2} \end{pmatrix}, \quad \breve{u}_+(0,0;\mathbf{p}=0) = \sqrt{m} \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \tag{128}$$

$$\breve{u}_-(\theta,\phi) = \begin{pmatrix} A_+ \, \chi_- \\ -A_- \, \chi_- \end{pmatrix} = \begin{pmatrix} -A_+ \sin\frac{\theta}{2} \, e^{-i\phi/2} \\ A_+ \cos\frac{\theta}{2} \, e^{i\phi/2} \\ A_- \sin\frac{\theta}{2} \, e^{-i\phi/2} \\ -A_- \cos\frac{\theta}{2} \, e^{i\phi/2} \end{pmatrix}, \quad \breve{u}_-(0,0;\mathbf{p}=0) = \sqrt{m} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}, \tag{129}$$

$$\breve{v}_+(\theta,\phi) = \begin{pmatrix} -A_+ \, \chi_- \\ A_- \, \chi_- \end{pmatrix} = \begin{pmatrix} A_+ \sin\frac{\theta}{2} \, e^{-i\phi/2} \\ -A_+ \cos\frac{\theta}{2} \, e^{i\phi/2} \\ -A_- \sin\frac{\theta}{2} \, e^{-i\phi/2} \\ A_- \cos\frac{\theta}{2} \, e^{i\phi/2} \end{pmatrix}, \quad \breve{v}_+(0,0;\mathbf{p}=0) = \sqrt{m} \begin{pmatrix} 0 \\ -1 \\ 0 \\ 1 \end{pmatrix}, \tag{130}$$

$$\breve{v}_-(\theta,\phi) = \begin{pmatrix} A_-\,\chi_+ \\ -A_+\,\chi_+ \end{pmatrix} = \begin{pmatrix} A_-\,\cos\frac{\theta}{2}\,e^{-i\phi/2} \\ A_-\,\sin\frac{\theta}{2}\,e^{i\phi/2} \\ -A_+\,\cos\frac{\theta}{2}\,e^{-i\phi/2} \\ -A_+\,\sin\frac{\theta}{2}\,e^{i\phi/2} \end{pmatrix}, \qquad \breve{u}(0,0;\mathbf{p}=0) = \sqrt{m}\begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix} \tag{131}$$

The four spinors $\breve{u}_\pm(\theta=0,\phi=0,\mathbf{p}=0)$ and $\breve{v}_\pm(\theta=0,\phi=0,\mathbf{p}=0)$ can be regarded as the simplest physical spinors in the Weyl representation.[27]

The helicity projection operators of eq. (113) hold in that form in the Weyl representation as well.

Note that $\breve{v}_\pm = \breve{C}\breve{u}^*_\pm$, using the electric-charge-conjugation operator $\breve{C} = -i\breve{\gamma}_2$ found in eq. (98).

In the high-energy limit, $E \approx p$, $A_+ \approx \sqrt{2E}$ and $A_- \approx 0$.

6 Chirality States

In the high-energy limit ($E \gg m$), the helicity spinors (109)-(110) simplify to,

$$u_+(\theta,\phi) \propto \begin{pmatrix} \cos\frac{\theta}{2}\,e^{-i\phi/2} \\ \sin\frac{\theta}{2}\,e^{i\phi/2} \\ \cos\frac{\theta}{2}\,e^{-i\phi/2} \\ \sin\frac{\theta}{2}\,e^{i\phi/2} \end{pmatrix}, \qquad u_-(\theta,\phi) \propto \begin{pmatrix} -\sin\frac{\theta}{2}\,e^{-i\phi/2} \\ \cos\frac{\theta}{2}\,e^{i\phi/2} \\ \sin\frac{\theta}{2}\,e^{-i\phi/2} \\ -\cos\frac{\theta}{2}\,e^{i\phi/2} \end{pmatrix}, \tag{132}$$

which are eigenstates of the matrix,

$$\Gamma = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & I_2 \\ I_2 & 0 \end{pmatrix}, \tag{133}$$

i.e., $\Gamma\,u_\pm = \pm u_\pm$. This matrix already has a name,[28]

$$\Gamma = \gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3, \qquad (\gamma^5)^2 = I_4, \qquad \bar{\gamma}^5 = -\gamma^5, \qquad \gamma^5\gamma^\mu = -\gamma^\mu\gamma^5. \tag{134}$$

[27] Compare with eq. (11.18) of [32], which calls $\breve{v}_+$ a spin-down state.

[28] The matrix γ^5 was first defined by Pauli [25] as $\gamma^5 = \gamma^1\gamma^2\gamma^3\gamma^4$ ($= \gamma^1\gamma^2\gamma^3\gamma^0$), and this notation was used in such notable papers as [35, 36]. The form (134) may have first been used by Case (1957) [37].

We now define the chirality projection operators,[29,30]

$$P_{R,L} \equiv \frac{I_4 \pm \gamma^5}{2} \quad \left[= \frac{1}{2} \begin{pmatrix} I_2 & \pm I_2 \\ \pm I_2 & I_2 \end{pmatrix} \right]. \tag{135}$$

We seek an eigenstate ψ_R of the chirality operator P_R, such that $P_R\psi_R = \psi_R$,

$$P_R\psi_R \propto P_R \begin{pmatrix} \chi_R \\ \zeta_R \end{pmatrix} = \frac{1}{2} \begin{pmatrix} \chi_R + \zeta_R \\ \zeta_R + \chi_R \end{pmatrix} = \psi_R = \begin{pmatrix} \chi_R \\ \zeta_R \end{pmatrix}, \tag{136}$$

which implies that $\zeta_R = \chi_R$,

$$\psi_R \propto \begin{pmatrix} \chi_R \\ \chi_R \end{pmatrix}. \tag{137}$$

When we consider eigenstates ψ_L of P_L, we cannot obtain a simple form with $P_L\psi_L = -\psi_L$, so we suppose that $P_L\psi_L = \psi_L$,

$$P_L\psi_L \propto P_L \begin{pmatrix} \chi_L \\ \zeta_L \end{pmatrix} = \frac{1}{2} \begin{pmatrix} \chi_L - \zeta_L \\ \zeta_L - \chi_L \end{pmatrix} = \psi_L = \begin{pmatrix} \chi_L \\ \zeta_L \end{pmatrix}, \tag{138}$$

which implies that $\zeta_L = -\chi_L$,

$$\psi_L \propto \begin{pmatrix} \chi_L \\ -\chi_L \end{pmatrix}. \tag{139}$$

Note that the chirality spinors $\psi_{R,L}$ are eigenstates of γ^5: $\gamma^5\psi_{R,L} = \pm\psi_{R,L}$.

Already in 1929, Weyl [10] had commented that the right- and lefthanded chirality states $\psi_{R,L}$ of Dirac 4-spinors ψ might play an important role in physics. But, because these states are not invariant under space inversion (parity), they were initially not considered to be physically relevant.[31]

We note from eqs. (42) and (44) that in the rest frame of a spin-1/2 particle/antiparticle, their 4-spinors which obey the Dirac equation (1)) have the forms,

$$u = \sqrt{2m} \begin{pmatrix} \chi \\ 0 \end{pmatrix}, \qquad v = \sqrt{2m} \begin{pmatrix} 0 \\ \zeta \end{pmatrix}. \tag{140}$$

Hence, the chirality states (137) and (139) are not ordinary solutions to the Dirac equation.

[29] R and L stand for righthanded and lefthanded, such that in the high-energy limit, $P_R u_+ = u_+$, where the positive-helicity state u_+ was defined to be righthanded in [35].

[30] The term chirality as applied to spinors was first used by Watanabe (1957) [33], although Eddington (1949) had used this term, p. 111 of [38], as distinguishing left- and righthanded frames in his (bizarre) theory of elementary particles.

[31] See, for example, p. 226 of [34].

6.1 Chirality States and the Dirac Equation

The right- and lefthanded-chirality states $u_{R,L}$ (and $v_{R,L}$) do not strictly satisfy the Dirac equation $i\gamma^\mu\partial_\mu u = m\,u$, but rather,

$$i\gamma^\mu\partial_\mu u_{R,L} = m\,u_{L,R}, \tag{141}$$

(and similarly $i\gamma^\mu\partial_\mu v_{R,L} = m\,v_{R,L}$), since $\gamma^5\gamma^\mu = -\gamma^\mu\gamma^5$:

$$\frac{I_4 \pm \gamma^5}{2} i\gamma^\mu\partial_\mu u = i\gamma^\mu\partial_\mu \frac{I_4 \mp \gamma^5}{2} u = i\gamma^\mu\partial_\mu u_{L,R} = \frac{I_4 \pm \gamma^5}{2} m u = m\,u_{R,L}. \tag{142}$$

Expanding eq. (141) (in the Dirac representation),

$$E \begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix} \begin{pmatrix} \chi_{R,L} \\ \pm\chi_{R,L} \end{pmatrix} - \mathbf{p}\cdot \begin{pmatrix} 0 & \boldsymbol{\sigma} \\ -\boldsymbol{\sigma} & 0 \end{pmatrix} \begin{pmatrix} \chi_{R,L} \\ \pm\chi_{R,L} \end{pmatrix} \propto m \begin{pmatrix} \chi_{L,R} \\ \mp\chi_{L,R} \end{pmatrix}, \tag{143}$$

$$E\chi_{R,L} \mp \mathbf{p}\cdot\boldsymbol{\sigma}\,\chi_{R,L} \propto m\chi_{L,R} \qquad \Rightarrow \qquad E\chi_{R,L} - m\chi_{L,R} = \pm\mathbf{p}\cdot\boldsymbol{\sigma}\,\chi_{R,L}. \tag{144}$$

In the high-energy limit, $E \approx |\mathbf{p}| \gg m$, eq. (144) simplifies to $\hat{\mathbf{p}}\cdot\boldsymbol{\sigma}\,\chi_{R,L} \approx \pm\chi_{R,L}$, such that (in this limit) $\chi_{R,L} \approx \chi_\pm$, the helicity 2-spinors of sec. 5.1. That is, chirality (particle) states are the same as helicity states in the high-energy limit, as anticipated in eq. (132).

$$u_{R,L}(E \gg m) = \sqrt{E} \begin{pmatrix} \chi_\pm \\ \pm\chi_\pm \end{pmatrix} \approx u_\pm(E \gg m), \tag{145}$$

At the other extreme, a particle a rest, $\mathbf{p} = 0$, $E = m$ and $\chi_{R,L} = \chi_{L,R}$

The 2-spinors in the chirality states, eqs. (137) and (139), can be considered as sums of the helicity 2 spinors χ_+ and χ_-. In the rest frame, where $\chi_L = \chi_R$, we write,

$$u_{R,L}(\mathbf{p} = 0) = \sqrt{m} \begin{pmatrix} a\chi_+ + b\chi_- \\ \pm(a\chi_+ + b\chi_-) \end{pmatrix}, \tag{146}$$

where $|a|^2 + |b|^2 = 1$. The boosted states with 3-momentum $\mathbf{p}$ follow from eq. (87) as, in the high-energy limit where $E \approx p \gg m$ and the rapidity is related by $\tanh w = p/E \approx 1$, $\cosh(w/2) \approx \sinh(w/2) \approx \sqrt{E/2m}$,

$$\begin{aligned} u_R(E \gg m) &= \sqrt{\frac{E}{2}} \begin{pmatrix} I_2(a\chi_+ + b\chi_-) + \hat{\mathbf{p}}\cdot\boldsymbol{\sigma}(a\chi_+ + b\chi_-) \\ I_2(a\chi_+ + b\chi_-) + \hat{\mathbf{p}}\cdot\boldsymbol{\sigma}(a\chi_+ + b\chi_-) \end{pmatrix} \\ &= \sqrt{\frac{E}{2}} \begin{pmatrix} a\chi_+ + b\chi_- + a\chi_+ - b\chi_- \\ a\chi_+ + b\chi_- + a\chi_+ - b\chi_- \end{pmatrix} = \sqrt{\frac{E}{2}} \begin{pmatrix} 2a\chi_+ \\ 2a\chi_+ \end{pmatrix}. \end{aligned} \tag{147}$$

Comparing with eq. (145), we see that $a = 1/\sqrt{2}$. Similarly,

$$u_L(E \gg m) = \sqrt{\frac{E}{2}} \begin{pmatrix} I_2(a\chi_+ + b\chi_-) - \hat{\mathbf{p}} \cdot \boldsymbol{\sigma}(a\chi_+ + b\chi_-) \\ -I_2(a\chi_+ + b\chi_-) + \hat{\mathbf{p}} \cdot \boldsymbol{\sigma}(a\chi_+ + b\chi_-) \end{pmatrix}$$

$$= \sqrt{\frac{E}{2}} \begin{pmatrix} a\chi_+ + b\chi_- - a\chi_+ + b\chi_- \\ -a\chi_+ - b\chi_- + a\chi_+ - b\chi_- \end{pmatrix} = \sqrt{\frac{E}{2}} \begin{pmatrix} 2b\chi_+ \\ -2b\chi_+ \end{pmatrix}, \tag{148}$$

such that $b = 1/\sqrt{2} = a$.

We can now boost the rest-frame chirality states, eq. (146) with $a = b = 1/\sqrt{2}$,

$$u_{R,L}(\mathbf{p} = 0) = \sqrt{\frac{m}{2}} \begin{pmatrix} \chi_+ + \chi_- \\ \pm(\chi_+ + \chi_-) \end{pmatrix}, \tag{149}$$

to an arbitrary momentum $\mathbf{p}$,

$$u_{R,L}(\mathbf{p}) = \sqrt{\frac{m}{2}} \begin{pmatrix} \sqrt{\frac{E+m}{2m}}\, I_2(\chi_+ + \chi_-) \pm \sqrt{\frac{E-m}{2m}}\, \hat{\mathbf{p}} \cdot \boldsymbol{\sigma}(\chi_+ + \chi_-) \\ \pm\sqrt{\frac{E+m}{2m}}\, I_2(\chi_+ + \chi_-) + \sqrt{\frac{E+m}{2m}}\, \hat{\mathbf{p}} \cdot \boldsymbol{\sigma}(\chi_+ + \chi_-) \end{pmatrix}$$

$$= \frac{1}{2} \begin{pmatrix} \sqrt{E_+}\,(\chi_+ + \chi_-) \pm \sqrt{E_-}\,(\chi_+ - \chi_-) \\ \pm\sqrt{E_+}\,(\chi_+ + \chi_-) + \sqrt{E_-}\,(\chi_+ - \chi_-) \end{pmatrix}, \tag{150}$$

which agrees with eq. (145) in the high energy limit, $E \approx E_+ \approx E_- \gg m$.

While the plane-wave particle chirality spinors $u_{R,L}$ are associated with the spacetime factor e^{-ipx}, they could be written as,

$$u_{R,L} = \frac{1}{2}(u_+ + u_- \mp v_+ \pm v_-), \tag{151}$$

recalling eqs. (109)-(112). As this may lead to confusion, we do not use it further.

6.2 Chirality and Antiparticles

Suppose that the antiparticle of state u is $v = \tilde{u} = u^{(C)}$. We can decompose u and v into chirality states, $u = u_R + u_L$, $v = v_R + v_L$. Then, recalling the electric-charge-conjugation operator (94) and that $\gamma^2\gamma^5 = -\gamma^5\gamma^2$,

$$\tilde{u}_{R,L} = i\gamma^2 u^*_{R,L} = i\gamma^2 \frac{I_4 \pm \gamma^5}{2} u^* = \frac{I_4 \mp \gamma^5}{2}(i\gamma^2 u^*) = \frac{I_4 \mp \gamma^5}{2}\tilde{u} = \frac{I_4 \mp \gamma^5}{2} v = v_{R,L}. \tag{152}$$

Similarly, $\tilde{v}_{R,L} = u_{R,L}$. Note that the antiparticle of u_R (in the sense of electric-charge conjugation) is v_R and not v_L.

The eigenstates of the chirality projection operators are called the **righthanded(lefthanded)-chirality states** for particles(antiparticles),[32,33]

$$P_{R,L}\, u_{R,L} = u_{R,L}, \qquad P_{R,L}\, v_{L,R} = v_{L,R}. \tag{153}$$

The electric-charge conjugates of the chirality states (150) are, in the Dirac representation, recalling eqs. (105)-(106),

$$\begin{aligned} v_{R,L} = u^{(C)}_{R,L} = i\gamma^2 u^*_{R,L} &= \frac{1}{2}\begin{pmatrix} i\sigma_2[\pm\sqrt{E_+}\,(\chi^*_+ + \chi^*_-) + \sqrt{E_-}\,(\chi^*_+ - \chi^*_-)] \\ -i\sigma_2[\sqrt{E_+}\,(\chi^*_+ + \chi^*_-) \pm \sqrt{E_-}\,(\chi^*_+ - \chi^*_-)] \end{pmatrix} \\ = i\gamma^2 u^*_{R,L} &= \frac{1}{2}\begin{pmatrix} \pm\sqrt{E_+}\,(\chi_+ - \chi_-) - \sqrt{E_-}\,(\chi_+ + \chi_-) \\ -\sqrt{E_+}\,(\chi_+ - \chi_-) \pm \sqrt{E_-}\,(\chi_+ + \chi_-) \end{pmatrix}. \end{aligned} \tag{154}$$

In the rest frame of a spin-1/2 antiparticle $\sqrt{E_+} = \sqrt{2m}$ while $\sqrt{E_-} = 0$, so that,

$$v_{R,L}(\mathbf{p} = 0) = \sqrt{\frac{m}{2}}\begin{pmatrix} \pm(\chi_+ - \chi_-) \\ -(\chi_+ - \chi_-) \end{pmatrix}, \tag{155}$$

and in the high-energy limit,

$$v_{R,L}(E \gg m) \approx \sqrt{E}\begin{pmatrix} -\chi_\mp \\ \pm\chi_\mp \end{pmatrix} \approx v_\pm(E \gg m), \tag{156}$$

recalling eqs. (111)-(112).

6.3 Helicity Conservation when Chirality Approximates Helicity.

For relativistic spin-1/2 particles, with $E \gg m$, their chirality and helicity states are essentially identical, as noted in eqs. (145) and (156). Then, for example, a matrix element between helicity states such as $\bar{u}_-\gamma^\mu u_+$ in eq. (120) is well approximated by the matrix element of chirality states,

$$\begin{aligned} \bar{u}_-\gamma^\mu u_+ &\approx \bar{u}_L\gamma^\mu u_R \\ &= \frac{1}{4}[(I_4 - \gamma^5)u]^\dagger\gamma^0\gamma^\mu(I^4 + \gamma^5 u = \frac{1}{4}u^\dagger(I_4 - \gamma^5)\gamma^0\gamma^\mu(I_4 + \gamma^5)u \\ &= \frac{1}{4}u^\dagger\gamma^0(I_4 + \gamma^5)\gamma^\mu(I_4 + \gamma^5)u = \frac{1}{4}\bar{u}\gamma^\mu(I_4 - \gamma^5)(I_4 + \gamma^5)u \\ &= 0, \end{aligned} \tag{157}$$

recalling that $\gamma^5\gamma^\mu = -\gamma^\mu\gamma^5$ and $(\gamma^5)^2 = I^4$.

That is, only chirality-conserving matrix elements of the operator γ^μ are nonzero for relativistic spin-1/2 states.

[32] In the hole theory, a righthanded-chirality antiparticle corresponds to the absence of a lefthanded, negative-energy particle.

[33] The definition (135) and the relations (153) are independent of the representation of the Dirac matrices.

6.4 Orthogonality of Chirality States

The chirality states (150) and (154) obey,

$$u_R^\dagger u_R = u_L^\dagger u_L = v_R^\dagger v_R = v_L^\dagger v_L = 2E, \tag{158}$$
$$u_R^\dagger v_L = u_L^\dagger v_R = v_R^\dagger u_L = v_L^\dagger u_R = 2p, \tag{159}$$
$$u_R^\dagger u_L = u_L^\dagger u_R = v_R^\dagger v_L = v_L^\dagger v_R = u_R^\dagger v_R = u_L^\dagger v_L = v_R^\dagger u_R = v_L^\dagger u_L = 0. \tag{160}$$

Noting that, in the Dirac representation,

$$\gamma^0 u_{R,L} = \begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix} u_{R,L} = \frac{1}{2} \begin{pmatrix} \sqrt{E_+}\,(\chi_+ + \chi_-) \pm \sqrt{E_-}\,(\chi_+ - \chi_-) \\ \mp\sqrt{E_+}\,(\chi_+ + \chi_-) - \sqrt{E_-}\,(\chi_+ - \chi_-) \end{pmatrix}, \tag{161}$$

$$\gamma^0 v_{R,L} = \frac{1}{2} \begin{pmatrix} \pm\sqrt{E_+}\,(\chi_+ - \chi_-) - \sqrt{E_-}\,(\chi_+ + \chi_-) \\ \sqrt{E_+}\,(\chi_+ - \chi_-) \mp \sqrt{E_-}\,(\chi_+ + \chi_-) \end{pmatrix}, \tag{162}$$

and that $\bar{a}b = a^\dagger \gamma^0\, b$, we have,

$$\bar{u}_R u_L = \bar{u}_L u_R = \bar{v}_R v_L = \bar{v}_L v_R = 2m, \tag{163}$$
$$\bar{u}_R u_R = \bar{u}_L u_L = \bar{v}_R v_R = \bar{v}_L v_L = 0, \tag{164}$$
$$\bar{u}_R v_L = \bar{u}_L v_R = \bar{v}_R u_L = \bar{v}_L u_R = \bar{u}_R v_R = \bar{u}_L v_L = \bar{v}_R u_R = \bar{v}_L u_L = 0. \tag{165}$$

6.5 Sterile Neutrinos

In the so-called V-A theory [36, 39, 40], only lefthanded particle (righthanded antiparticle) states participate in the weak interaction. Since the neutrino has no strong or electromagnetic interaction (presuming that the neutrino has no magnetic moment as well as no electric charge), then a righthanded neutrino (lefthanded antineutrino) would have no interactions (except gravity) and could be called sterile.[34]

While a massless, sterile neutrino is a somewhat trivial concept, the possibility of a sterile neutrino with mass has led to considerable discussion/controversy, despite lack of clear experimental evidence for such a particle.[35]

6.6 Weak-Charge Conjugation

In 1960, Glashow [43] postulated a new symmetry, $\mathrm{SU}(2)_T \otimes \mathrm{U}(1)_Y$, based on weak isospin T, and the conserved quantum numbers/charges T_3 and weak hypercharge $Y_\mathrm{W} = 2(Q - T_3)$. This set the stage for the development of the Standard Model [44, 45].

[34] The notion of a sterile neutrino seems to have been introduced by Pontecorvo, on p. 986 of [41].
[35] A recent experimental limit on the existence of sterile neutrinos is [42].

Antiparticles have the negative quantum numbers of those in the table for particles. Recall that with respect to electric-charge conjugation,

$$e^{-(C)}_{R,L} = e^+_{R,L}, \qquad \nu^{(C)}_{R,L} = \bar{\nu}_{R,L}. \qquad (166)$$

We can introduce the weak-charge-conjugation operator $C_{\rm W}$ such that $C_{\rm W}\psi^*$ is the antiparticle of spin-1/2 particle state ψ in the sense of the table. Then,

$$e^{-(C_{\rm W})}_{R,L} = e^+_{L,R}, \qquad \nu^{(C_{\rm W})}_{R,L} = \bar{\nu}_{L,R}. \qquad (167)$$

Charge:	Isotopic	Hyper	Electric
d_L	-1/2	+1/3	-1/3
e_L	-1/2	-1	-1
$(\nu_e)_L$	+1/2	-1	0
u_L	+1/2	-1/3	+2/3
d_R	0	+2/3	-1/3
e_R	0	-2	-1
$(\nu_e)_R$	0	0	0
u_R	0	+4/3	+2/3

In the Dirac and Weyl representations,[36]

$$C_{\rm W} = \gamma^5 C = \begin{pmatrix} -i\sigma_2 & 0 \\ 0 & i\sigma_2 \end{pmatrix}, \qquad \breve{C}_{\rm W} = \breve{\gamma}^5 \breve{C} = \begin{pmatrix} 0 & i\sigma_2 \\ i\sigma_2 & 0 \end{pmatrix}. \qquad (168)$$

6.7 Chirality in the Weyl Representation

The transformation of states from the Dirac representation to the Weyl representation is given in eq. (47). Then, using eqs. (150) and (155) we find,

$$\breve{u}_L(\mathbf{p}) = \begin{pmatrix} \sqrt{E_+}\,(\chi_+ + \chi_-) - \sqrt{E_-}\,(\chi_+ - \chi_-) \\ 0 \end{pmatrix}, \ \breve{u}_L(0) = \sqrt{m}\begin{pmatrix} \chi_+ + \chi_- \\ 0 \end{pmatrix}, \qquad (169)$$

$$\breve{u}_R(\mathbf{p}) = \begin{pmatrix} 0 \\ \sqrt{E_+}\,(\chi_+ + \chi_-) + \sqrt{E_-}\,(\chi_+ - \chi_-) \end{pmatrix}. \qquad (170)$$

$$\breve{v}_L(\mathbf{p}) = \begin{pmatrix} 0 \\ -\sqrt{E_+}\,(\chi_+ - \chi_-) - \sqrt{E_-}\,(\chi_+ + \chi_-) \end{pmatrix}, \qquad (171)$$

$$\breve{v}_R(\mathbf{p}) = \begin{pmatrix} \sqrt{E_+}\,(\chi_+ - \chi_-) - \sqrt{E_-}\,(\chi_+ + \chi_-) \\ 0 \end{pmatrix}. \qquad (172)$$

In their rest frame,

$$\breve{u}_L(0) = \sqrt{m}\begin{pmatrix} \chi_+ + \chi_- \\ 0 \end{pmatrix}, \qquad \breve{u}_R(0) = \sqrt{m}\begin{pmatrix} 0 \\ \chi_+ + \chi_- \end{pmatrix}, \qquad (173)$$

[36]Here, $C_{\rm W}$ means $C_{\rm Weak}$ rather than $C_{\rm Weyl}$ as in eqs. (200) and (202).

$$\breve{v}_L(0) = \sqrt{m} \begin{pmatrix} 0 \\ -\chi_+ + \chi_- \end{pmatrix}, \qquad \breve{v}_R(0) = \sqrt{m} \begin{pmatrix} \chi_+ - \chi_- \\ 0 \end{pmatrix}, \tag{174}$$

and in the high-energy limit,

$$\breve{u}_L(E \gg m) \approx \sqrt{2E} \begin{pmatrix} \chi_- \\ 0 \end{pmatrix} \approx \breve{u}_-(E \gg m),\ \breve{u}_R(E \gg m) \approx \sqrt{2E} \begin{pmatrix} 0 \\ \chi_+ \end{pmatrix} \approx \breve{u}_+(E \gg m), \tag{175}$$

$$\breve{v}_L(E \gg m) \approx \sqrt{2E} \begin{pmatrix} 0 \\ -\chi_+ \end{pmatrix} \approx \breve{v}_-(E \gg m),\ \breve{v}_R(E \gg m) \approx \sqrt{2E} \begin{pmatrix} -\chi_- \\ 0 \end{pmatrix} \approx \breve{v}_+(E \gg m), \tag{176}$$

recalling eqs. (128)-(131).[37]

The simplicity of the chirality states in the Weyl representation leads it also to be called the chiral representation.

The antiparticle (in the sense of electric-charge conjugation) of $\breve{u}_{R,L}$ is $\breve{u}_{R,L}^{(C)} = \breve{C}\breve{u}_{R,L}^* = \breve{v}_{R,L}$. For neutrinos, $\breve{u}_R$ and $\breve{v}_L$ are sterile in the V-A theory.

The chirality projection operators in the Weyl representation are,

$$\breve{P}_R = \frac{I_4 + \breve{\gamma}^5}{2} = \begin{pmatrix} 0 & 0 \\ 0 & I_2 \end{pmatrix}, \qquad \breve{P}_L = \frac{I_4 - \breve{\gamma}^5}{2} = \begin{pmatrix} I_2 & 0 \\ 0 & 0 \end{pmatrix}, \tag{179}$$

such that $\breve{P}_{R,L}\,\breve{u}_{R,L} = \breve{u}_{R,L}$, while $\breve{P}_{L,R}\,\breve{v}_{R,L} = \breve{v}_{L,R}$.

7 Two-Component Theory of Massless Fermions

Until relatively recently, experimental evidence was consistent with neutrinos being massless. The character of Dirac 4-spinors for massless spin-1/2 states was considered by Weyl (1929) [10], who formulated a "two component" theory.

For example, the four helicity states (109)-(112) reduce to two independent states when $m = 0$, since then $v_\pm = -u_\mp$. When $m = 0$ then $E = E_+ = E_-$, so the 4-spinors of massless particles correspond to the high-energy limit of massive particles, where the helicity spinors

[37] Using the last form of the Lorentz boost (88) in the Weyl representation on the rest-frame chirality spinors (173)-(174), we can also write the chirality spinors (169)-(172) with 3-momentum $\mathbf{p}$ as,

$$\breve{u}_L(\mathbf{p}) = \begin{pmatrix} \sqrt{p\sigma}(\chi_+ + \chi_-) \\ 0 \end{pmatrix}, \qquad \breve{u}_R(\mathbf{p}) = \begin{pmatrix} 0 \\ \sqrt{p\underline{\sigma}}(\chi_+ + \chi_-) \end{pmatrix}, \tag{177}$$

$$\breve{v}_L(\mathbf{p}) = \begin{pmatrix} 0 \\ \sqrt{p\underline{\sigma}}(-\chi_+ + \chi_-) \end{pmatrix}, \qquad \breve{v}_R(\mathbf{p}) = \begin{pmatrix} \sqrt{p\sigma}(\chi_+ - \chi_-) \\ 0 \end{pmatrix}, \tag{178}$$

as favored by many theorists.

and the chirality spinors are identical, $u_R = u_+ = -v_- = -v_L$ and $u_L = u_- = -v_+ = -v_R$.

$$u_R(\theta,\phi) = u_+(\theta,\phi) = \sqrt{E}\begin{pmatrix}\chi_+\\ \chi_+\end{pmatrix}, \qquad u_L(\theta,\phi) = u_-(\theta,\phi) = \sqrt{E}\begin{pmatrix}\chi_-\\ -\chi_-\end{pmatrix}, \tag{180}$$

$$v_R(\theta,\phi) = v_+(\theta,\phi) = \sqrt{E}\begin{pmatrix}-\chi_-\\ \chi_-\end{pmatrix}, \qquad v_L(\theta,\phi) = v_-(\theta,\phi) = \sqrt{E}\begin{pmatrix}-\chi_+\\ -\chi_+\end{pmatrix}. \tag{181}$$

Note that $v_\pm = Cu^*_\pm$, using the electric-charge-conjugation operator $C = i\gamma_2$, eq. (94).

In the Weyl representation, we have,

$$\breve{u}_R(\theta,\phi) = -\breve{v}_L(\theta,\phi) = \sqrt{2E}\begin{pmatrix}0\\ \chi_+\end{pmatrix}, \qquad \breve{u}_L(\theta,\phi) = -\breve{v}_R(\theta,\phi) = \sqrt{2E}\begin{pmatrix}\chi_-\\ 0\end{pmatrix}. \tag{182}$$

8 Majorana States

In 1937, Majorana speculated [46] that spin-1/2 neutrinos might not be "Dirac" particles (as considered above in this note), but rather are their own antiparticles (with respect to electric-charge conjugation).[38,39] While this is automatic for massless neutrinos, as seen in sec. 6, it is nontrivial for massive neutrinos.

We saw in sec. ecc above that the electric-charge-conjugation transformation $\tilde\psi = i\gamma_2\psi^*$ of a 4-spinor state ψ leads to its antiparticle state $\tilde\psi$. Majorana spinors (when constructed from spinors that obey the Dirac equation) are their own antiparticles, which suggests that they are combinations of Dirac particles u and antiparticles v. So, recalling eqs. (42) and (95) we consider a general Majorana spinor of the form,

$$\frac{\psi}{\sqrt{E_+}} = au\, e^{-ipx} + bv\, e^{ipx} = a\, e^{-ipx}\begin{pmatrix}\chi\\ \frac{\mathbf{p}\cdot\boldsymbol{\sigma}}{E+m}\chi\end{pmatrix} + b\, e^{ipx}\begin{pmatrix}\frac{\mathbf{p}\cdot\boldsymbol{\sigma}}{E+m}\tilde\chi\\ \tilde\chi\end{pmatrix}, \tag{183}$$

where χ and $\tilde\chi = -i\sigma_2\chi^*$ ($\chi = i\sigma_2\tilde\chi^*$, from eq. (96)) are 2-spinors with unit normalization, $E_+ = E + m$, and $|a|^2 + |b|^2 = 1$, so that $\bar\psi\psi = 2m$. Then, the requirement that this state be its own antiparticle implies (omitting some factors $e^{\pm i\,px}$),

$$\psi = \psi^{(C)} = \tilde\psi = i\gamma_2\psi^*, \tag{184}$$

and in more detail,
$$\frac{\psi}{\sqrt{E_+}} = \frac{\psi^{(C)}}{\sqrt{E_+}} = \frac{\tilde\psi}{\sqrt{E_+}} = \widetilde{au} + \widetilde{bv} = a^*\tilde u + b^*\tilde v = a^* i\gamma^2 u^* + b i\gamma^2 v^*$$
$$= \begin{pmatrix}0 & i\sigma_2\\ -i\sigma_2 & 0\end{pmatrix}\left[a^*\, e^{ipx}\begin{pmatrix}\chi^*\\ \frac{\mathbf{p}\cdot\boldsymbol{\sigma}^*}{E+m}\chi^*\end{pmatrix} + b^*\, e^{-ipx}\begin{pmatrix}\frac{\mathbf{p}\cdot\boldsymbol{\sigma}^*}{E+m}\tilde\chi^*\\ \tilde\chi^*\end{pmatrix}\right] \tag{185}$$
$$= a^*\begin{pmatrix}\frac{\mathbf{p}\cdot\boldsymbol{\sigma}}{E+m}(-i\sigma_2\chi^*)\\ -i\sigma_2\chi^*\end{pmatrix} + b^*\begin{pmatrix}i\sigma_2\tilde\chi^*\\ \frac{\mathbf{p}\cdot\boldsymbol{\sigma}}{E+m}(i\sigma_2\tilde\chi^*)\end{pmatrix} = a^*\begin{pmatrix}\frac{\mathbf{p}\cdot\boldsymbol{\sigma}}{E+m}\tilde\chi\\ \tilde\chi\end{pmatrix} + b^*\begin{pmatrix}\chi\\ \frac{\mathbf{p}\cdot\boldsymbol{\sigma}}{E+m}\chi\end{pmatrix},$$

[38] See also [47].

[39] Much of the discussion below follows Prob. 7.51 of [7].

recalling that $\sigma_2\boldsymbol{\sigma}^* = -\boldsymbol{\sigma}\sigma_2$. Hence, $b = a^*$ $(= 1/\sqrt{2})$, and $v = \tilde{u}$ $(u = \tilde{v})$, such that Majorana 4-spinors have only 2 independent components (and always contain both a Dirac-particle and -antiparticle spinor).[40] A consequence is that the top and bottom 2-spinors, χ_t and χ_b, of a Majorana 4-spinor state ψ are related by,

$$\frac{\psi}{\sqrt{E_+}} = \begin{pmatrix} \chi_t \\ \chi_b \end{pmatrix}, \qquad \chi_b = -i\sigma_2\chi_t^*, \qquad \chi_t = i\sigma_2\chi_b^*. \tag{186}$$

8.1 Majorana and the Dirac Equation

The Dirac equation (1), $i\gamma^\mu\partial_\mu\psi = m\psi$, can also be written for Majorana states (184) as,

$$i\gamma^\mu\partial_\mu\psi = m\psi^{(C)} = m\tilde{\psi}, \tag{187}$$

which is sometimes called the Majorana equation.

8.2 Majorana Helicity 4-Spinors

First, we consider the spin-up/down helicity 2-spinors $\chi_\pm$, eqs. (105)-(106),

$$\chi_+ = \begin{pmatrix} \cos\frac{\theta}{2}\, e^{-i\phi/2} \\ \sin\frac{\theta}{2}\, e^{i\phi/2} \end{pmatrix}, \qquad \chi_- = \begin{pmatrix} -\sin\frac{\theta}{2}\, e^{-i\phi/2} \\ \cos\frac{\theta}{2}\, e^{i\phi/2} \end{pmatrix}, \tag{188}$$

for which $\tilde{\chi}_\pm = -i\sigma_2\chi_\pm^* = \pm\chi_\mp$.Taking $a = b = 1/\sqrt{2}$, the Majorana helicity 4-spinors $\psi_\pm$ are their own antiparticles, $\psi_\pm = \tilde{\psi}_\pm$, with,

$$\frac{\psi_\pm}{\sqrt{E_+/2}} = e^{-ipx}\begin{pmatrix} \chi_\pm \\ \pm\frac{p}{E+m}\chi_\pm \end{pmatrix} + e^{ipx}\begin{pmatrix} -\frac{p}{E+m}\chi_\mp \\ \pm\chi_\mp \end{pmatrix}. \tag{189}$$

8.3 Majorana Chirality 4-spinors.

The following states, related to chirality,

$$\psi_{R,L} = \tilde{\psi}_{R,L} = \frac{u_{R,L} + v_{R,L}}{\sqrt{2}} = \frac{\nu_{R,L} + \bar{\nu}_{R,L}}{\sqrt{2}}, \tag{190}$$

are Majorana states (but note that $P_{R,L}\psi_{R,L} \neq \psi_{R,L}$). Furthermore,

$$\psi = \tilde{\psi} = a\psi_R + b\psi_L = \frac{a\nu_R + a\overline{\nu}_R + b\nu_L + b\overline{\nu}_L}{\sqrt{2}}, \tag{191}$$

[40]The Majorana two-component theory applies to particle with mass (but without electric charge), unlike the Weyl two-component theory of massless fermions (sec. 6 above).

is a Majorana state for any real numbers a and b that obey $|a|^2 + |b|^2 = 1$.[41]

An implication is that a Majorana state created with nominally one chirality can later interact via the opposite chiralty, which permits the phenomenon of neutrinoless double-beta decay (sec. 8.6.2 below).

8.4 Majorana States Based on Weak-Charge Conjugation

Another possible vision of Majorana states is that antiparticles are defined with respect to weak-charge conjugation, as mentioned in sec. 6.6 above.

Then, instead of, eq. (190), we would consider,

$$\psi'_{R,L} = \frac{\nu_{R,L} + \bar{\nu}_{R,L}}{\sqrt{2}}\,. \tag{193}$$

8.5 Present Experiments Exclude that the Observed Neutrinos are Majorana States

In my note [51] I discuss how present experimental data exclude that the observed (massive) neutrinos are Majorana states of the forms (190) and (193). The argument for the form (190) also applies to (191), since in both cases the production of such a Majorana neutrino state includes a sterile neutrino with 50% probability, which cannot be accommodated in the observed decays of the Z^0 weak boson.

This conclusion does not depend on the use of the Dirac representation, and hold for other representations, such as that of Weyl, as well.

8.6 Majorana Mass

However, the observed, massive neutrinos could be "Majorana" in a different sense, having "Majorana mass" terms in their Lagrangian, such that Majorana mass m_L is associated with a transition between ν_L and $\overline{\nu}_R$, and m_R with a transition between the sterile neutrinos ν_R and $\overline{\nu}_L$.[42]

Majorana mass terms do not necessarily imply the existence of Majorana (neutrino) states.[43]

8.6.1 The "See-Saw" Mechanism

An appealing application of Majorana mass terms is the so-called "see-saw" mechanism [53, 54], in which neutrinos are associated with a Dirac mass m_D as well as Majorana masses

[41] In the compressed notation introduced in secs. 3.2-3.3, Majorana states of the form (191) can be written in the Weyl representation as,

$$\breve{\psi} = \begin{pmatrix} \sqrt{p\,\boldsymbol{\sigma}}\,[a(\chi_+ - \chi_-) + b(\chi_+ + \chi_-)] \\ \sqrt{p\,\underline{\boldsymbol{\sigma}}}\,[a(\chi_+ + \chi_-) + b(-\chi_+ + \chi_-)] \end{pmatrix} = \begin{pmatrix} \sqrt{p\,\boldsymbol{\sigma}}\,[(a+b)\chi_+ - (a-b)\chi_-] \\ \sqrt{p\,\underline{\boldsymbol{\sigma}}}\,[(a-b)\chi_+ + (a+b)\chi_-] \end{pmatrix}. \tag{192}$$

[42] The concept of a Majorana mass may be been first discussed (briefly) by McClennan (1957) [52].

[43] These two concepts are often confused. See the warning in sec. 5 of [55].

m_L and m_R, with a mass matrix of the form,

$$\begin{pmatrix} m_L & m_D \\ m_D & m_R \end{pmatrix}. \tag{194}$$

where the "ideal" mass m_L of the ordinary neutrino could be 0, while that of the (sterile) partner is m_R, and m_D describes the coupling between the two "ideal" states. In a "grand-unified" theory, it could be that $m_R \approx 10^{15}$ GeV, the grand-unified energy scale, and $m_D \approx m_{\rm Higgs}$ as representative of the electroweak energy scale, such that $m_L \ll m_D \ll m_R$.

The eigenvalues λ of the mass matrix (194) are the roots of the determinant equation,

$$\begin{vmatrix} m_L - \lambda & m_D \\ m_D & m_R - \lambda \end{vmatrix} = \lambda^2 - (m_R + m_L)\lambda + m_R m_L - m_D^2 = 0. \tag{195}$$

Then, for $m_L \ll m_D \ll m_R$,

$$\begin{aligned} \lambda &= \frac{m_R + m_L \pm \sqrt{(m_R + m_L)^2 - 4m_R m_L + 4m_D^2}}{2} \\ &= \frac{m_R + m_L \pm \sqrt{(m_R - m_L)^2 + 4m_D^2}}{2} \approx \frac{m_R + m_L}{2} \pm \frac{m_R - m_L}{2}\left(1 + \frac{2m_D^2}{(m_R - m_L)^2}\right) \\ &\approx m_R,\ m_L - \frac{m_D^2}{m_R}. \end{aligned} \tag{196}$$

The mass of the heavier eigenstate is essentially unaffected, $m_{\nu'} \approx m_R$, while the mass of the "ordinary" neutrino, $m_\nu \approx m_L - m_D^2/m_R$, is affected by the coupling m_D only if $m_L \approx m_D^2/m_R \approx m_\nu$. We suppose the latter holds, so the "prediction" here is both a bit tentative and qualitative, with $m_\nu \approx (100\ \text{GeV})^2/10^{15}\ \text{GeV} = 10^{-11}\ \text{GeV} = 0.01\ \text{eV}$, for $m_R = 10^{15}$ GeV and $m_D = 100$ GeV.

This prescription as to how a very light mass might appear in a theory involving two larger, and disparate mass scales is generically called the "see-saw" mechanism, and many variants appear in the literature.

8.6.2 Neutrinoless Double-Beta Decay

In 1935, Goeppert-Mayer made a calculation [56] of the process $A \to A' ee\overline{\nu\nu}$, so-called **2-neutrino double-beta decay**, which was first definitively observed in 1987 [57].[44]

It was realized by Furry (1939) [58] that, if neutrinos behave according to Majorana's view, there could exist the phenomenon of **neutrinoless double-beta decay**, $A \to A' ee$.

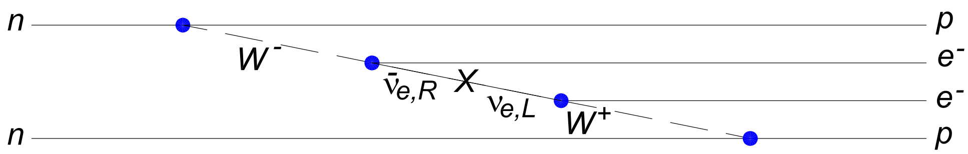
[44] For an estimate of the 2-neutrino-double-beta-decay lifetime, we note that a neutron has radius $r \approx 1$ fermi, so the time scale for collisions of quarks inside it is $r/c \approx 3 \times 10^{-23}$ s. The neutron lifetime is about 10^3 s, so the probability of a weak interaction occurring during each collision is about 3×10^{-26}.

The probability of two such weak interactions occuring in a nucleus within a single collision time is the square of this, and hence the lifetime for double-beta decay is about 3×10^{25} times the neutron lifetime, *i.e.*, $\approx 3 \times 10^{28}$ s. Recalling that a year contains about $\pi \times 10^7$ s, we estimate that the lifetime against double-beta decay is 10^{21} years (which agrees with experiment to within an order of magnitude).

Further, if the nuclear matrix elements were similar for 2-neutrino double-beta decay and neutrinoless double-beta decay, then the rate for the latter would be much larger (and the lifetime much shorter), because the 3-body phase space for the final state of the latter reaction is much larger than the 5-body phase space for the former.

To date, no neutrinoless double-beta decay has been observed, with limits on the lifetime ($> 10^{25}$ yr) being longer than that for observed 2-neutrino decays.[45]

In the neutrinoless double-beta-decay reaction $\nu\, A \to A'\, e^- e^-$ via the decay of two neutrons in the nucleons A, the virtual neutrino (ν in the figure below) needs to have amplitudes to be both a $\overline{\nu}_R$ and a ν_L.



This could be due to the neutrino being a Majorana state of the form (191) with both nonzero a and b, or it could be due to a Dirac neutrino associated with Majorana-mass terms. According to the claim in sec. 8.5 above, only the latter could be the case.

A Other Representations of the Dirac Matrices

The Dirac representation used in this note is,[46]

$$\gamma^0_{\rm D} = \begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix}, \ \gamma^i_{\rm D} = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix}, \ \gamma^5_{\rm D} = \begin{pmatrix} 0 & I_2 \\ I_2 & 0 \end{pmatrix}, \ C_{\rm D} = i\gamma^2 = \begin{pmatrix} 0 & i\sigma_2 \\ -i\sigma_2 & 0 \end{pmatrix}, \tag{197}$$

where I_2 is the 2×2 unit matrix, $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$, and $\sigma_i,\ i = 1,2,3$ are the Pauli spin matrices,

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \ \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \ \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \ (\sigma_j)^2 = I_2, \ \sigma_j\sigma_k = i\epsilon_{jkl}\,\sigma_l. \tag{198}$$

Pauli "4-vectors" can be defined as,

$$\sigma_\mu = (I_2, \boldsymbol{\sigma}), \qquad \underline{\sigma}_\mu = (I_2, -\boldsymbol{\sigma}). \tag{199}$$

The Weyl (chiral) representation used in this note is,[47]

$$\gamma^\mu_{\rm W} = \begin{pmatrix} 0 & \sigma_\mu \\ \underline{\sigma}_\mu & 0 \end{pmatrix}, \quad \gamma^5_{\rm W} = \begin{pmatrix} -I_2 & 0 \\ 0 & I_2 \end{pmatrix}, \quad C_{\rm W} = -i\gamma^2_{\rm W} = \begin{pmatrix} 0 & -i\sigma_2 \\ i\sigma_2 & 0 \end{pmatrix}. \tag{200}$$

[45] See, for example, [59].

[46] $\gamma^0 = \alpha_4 = \beta = \rho_3$ of pp. 614-615 of [1], while $\gamma^j = \alpha_j = \rho_1\sigma_j$, where the 4×4 matrices σ_j and ρ_j are given on p. 614.

[47] This form of the Weyl representation may have first appeared in eq. (3) of [60].

The Weyl basis can be obtained from the Dirac basis as,

$$\gamma^\mu_{\rm W} = \mathsf{U}_{\rm W}\gamma^\mu_{\rm D}\mathsf{U}^\dagger_{\rm W}, \qquad \psi_{\rm W} = \mathsf{U}_{\rm W}\psi_{\rm D}, \qquad \mathsf{U}_{\rm W} = \frac{1}{\sqrt{2}}\begin{pmatrix} I_2 & -I_2 \\ I_2 & I_2 \end{pmatrix}. \tag{201}$$

The Weyl (chiral) representation is sometimes written with all matrices the negative of the above.

Yet another Weyl representation is,

$$\gamma^0_{\rm W'} = \begin{pmatrix} 0 & I_2 \\ I_2 & 0 \end{pmatrix}, \ \gamma^i_{\rm W'} = \begin{pmatrix} 0 & -\sigma_i \\ \sigma_i & 0 \end{pmatrix}, \ \gamma^5_{\rm W'} = \begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix}, \ C_{\rm W'} = \begin{pmatrix} 0 & i\sigma_2 \\ -i\sigma_2 & 0 \end{pmatrix}. \tag{202}$$

This alternate Weyl basis can be obtained from the Dirac basis as,

$$\gamma^\mu_{\rm W'} = \mathsf{U}_{\rm W'}\gamma^\mu_{\rm D}\mathsf{U}^\dagger_{\rm W'}, \qquad \psi_{\rm W'} = \mathsf{U}_{\rm W'}\psi_{\rm D}, \qquad \mathsf{U}_{\rm W'} = \frac{1}{\sqrt{2}}\begin{pmatrix} I_2 & I_2 \\ I_2 & -I_2 \end{pmatrix}. \tag{203}$$

A Majorana representation, in which the γ^μ are imaginary, is,

$$\gamma^0_{\rm M} = \begin{pmatrix} 0 & \sigma_2 \\ \sigma_2 & 0 \end{pmatrix}, \ \gamma^1_{\rm M} = \begin{pmatrix} i\sigma_3 & 0 \\ 0 & i\sigma_3 \end{pmatrix}, \ \gamma^2_{\rm M} = \begin{pmatrix} 0 & -\sigma_2 \\ \sigma_2 & 0 \end{pmatrix}, \ \gamma^3_{\rm M} = \begin{pmatrix} -i\sigma_1 & 0 \\ 0 & -i\sigma_1 \end{pmatrix},$$

$$\gamma^5_{\rm M} = \begin{pmatrix} \sigma_2 & 0 \\ 0 & \sigma_2 \end{pmatrix}, \quad C_{\rm M} = \begin{pmatrix} -iI_2 & 0 \\ 0 & iI_2 \end{pmatrix}. \tag{204}$$

This Majorana basis can be obtained from the Dirac basis as,

$$\gamma^\mu_{\rm M} = \mathsf{U}_{\rm M}\gamma^\mu_{\rm D}\mathsf{U}^\dagger_{\rm M}, \qquad {}_{\rm M}\psi_{\rm M} = \mathsf{U}\psi_{\rm D}, \qquad \mathsf{U}_{\rm M} = \frac{1}{\sqrt{2}}\begin{pmatrix} I_2 & \sigma_2 \\ -\sigma_2 & -I_2 \end{pmatrix} = \frac{\gamma^0_{\rm D}+\gamma^2_{\rm D}}{\sqrt{2}}. \tag{205}$$

Another Majorana representation is,

$$\gamma^0_{\rm M'} = \begin{pmatrix} 0 & \sigma_2 \\ \sigma_2 & 0 \end{pmatrix}, \ \gamma^1_{\rm M'} = \begin{pmatrix} i\sigma_1 & 0 \\ 0 & i\sigma_1 \end{pmatrix}, \ \gamma^2_{\rm M'} = \begin{pmatrix} 0 & \sigma_2 \\ -\sigma_2 & 0 \end{pmatrix}, \ \gamma^3_{\rm M'} = \begin{pmatrix} i\sigma_3 & 0 \\ 0 & i\sigma_3 \end{pmatrix}. \tag{206}$$

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