Distortionless Transmission Line

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1 Problem

Deduce the differential equation for current (or voltage) in a two-conductor transmission line that is characterized by resistance R (summed over both conductors), inductance L, capacitance C and leakage conductance K, all defined per unit length. The leakage conductance describes the undesirable current I_{leakage} (per unit length) that flows directly from one conductor to the other across the dielectric that separate them according to,

$$I_{\text{leakage}} = KV, \tag{1}$$

where $V(x, t)$ is the voltage between the two conductors, taken to be along the x axis.[1]
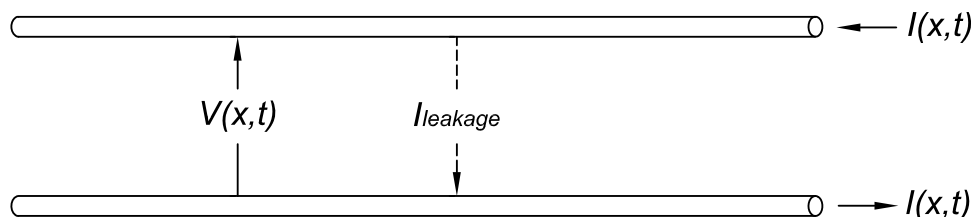
Deduce a relation among R, L, C and K that permits "distortionless" waves of the form,

$$e^{-\gamma x} f(x - vt) \tag{4}$$

to propagate along the transmission line. Give expressions for v and γ in terms of R, L and C, and discuss the line impedance $Z = V/I$ for distortionless transmission.

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[1]The leakage current $I_{\text{leakage}} = KV$ per unit length can be related to the conductivity σ of the medium surrounding the conductors by considering length dx along one conductor,

$$I_{\text{leakage}}\, dz = \sigma \int E_\perp\, d\text{Area} = \frac{\sigma}{\epsilon} \int D_\perp\, d\text{Area} = \frac{\sigma}{\epsilon} Q = \frac{\sigma}{\epsilon} VC\, dx, \tag{2}$$

where ϵ is the permittivity of the medium. Then, we find $K = \sigma C/\epsilon$.

When considering propagation of waves with time dependence $e^{-i\omega t}\ (= e^{j\omega t})$ (which tacitly assumes there is no distortion), the permittivity can be written as $\epsilon = \epsilon' + i\epsilon''$, where nonzero ϵ'' characterizes energy loss in the oscillating dipoles of the medium. In this case the relation between the total current $\mathbf{J}$ in the medium and the electric field $\mathbf{E}$ includes the polarization current $\partial \mathbf{D}/\partial t$, according to $\mathbf{J} = \mathbf{J}_{\text{cond}} + \mathbf{J}_{\text{pol}} = \sigma \mathbf{E} + Re(-i\omega \mathbf{D}) = (\sigma + \omega \epsilon'')\mathbf{E} \equiv \sigma_T \mathbf{E}$, so that $\sigma_T = \sigma + \epsilon''\omega$. Then, the (real) conductance K can be written as,
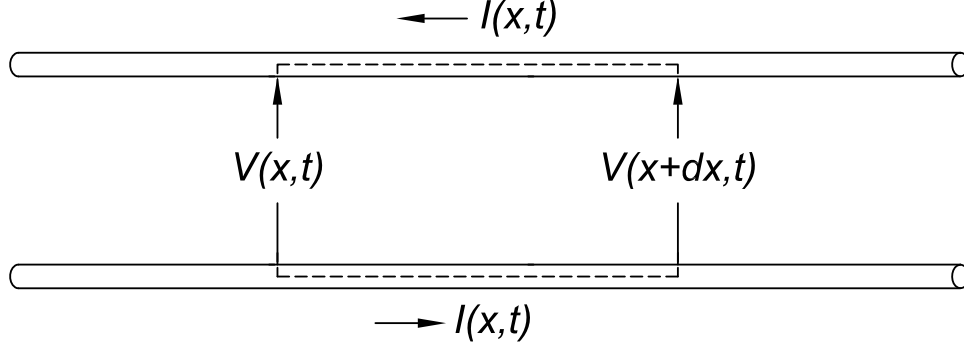
$$K = \frac{C(\sigma + \omega \epsilon'')}{\epsilon'} \equiv \omega C \tan \delta, \qquad \text{where} \qquad \tan \delta \equiv \frac{\sigma + \omega \epsilon''}{\omega \epsilon'} \tag{3}$$

is the so-call loss tangent. The conductance K is constant at low frequencies but varies linearly with ω at high frequencies.
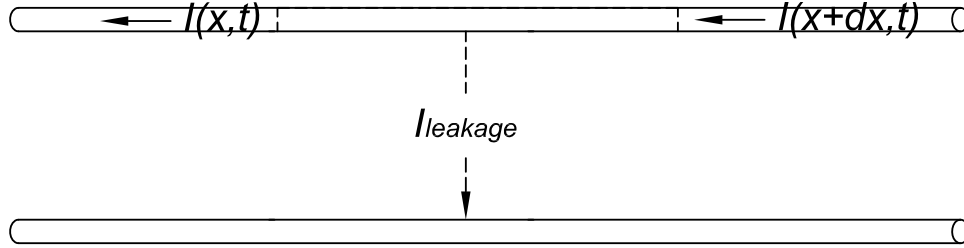
2 Solution

Referring to the sketch below, Kirchhoff's rule for the circuit of length dz (shown by dashed lines) tells us that,

$$V(x) - I(R\,dx) - V(x+dx) - (L\,dx)\frac{\partial I}{\partial t} = 0, \qquad \text{or} \qquad -\frac{\partial V}{\partial x} = L\frac{\partial I}{\partial t} + IR. \qquad (5)$$

Next, the charge dQ that accumulates on length dx of the upper wire during time dt is $(C\,dx)\,V$ in terms of the difference V in voltage between the wires, which also can be written in terms of currents as,

$$Q = (C\,dx)\,dV = [I(x) - I(x+dx) - I_{\text{leakage}}\,dx]\,dt, \qquad \text{so} \qquad -\frac{\partial I}{\partial x} = C\frac{\partial V}{\partial t} + KV. \qquad (6)$$

Together these imply the desired wave equation,

$$\frac{\partial^2 I}{\partial x^2} = LC\frac{\partial^2 I}{\partial t^2} + (RC + KL)\frac{\partial I}{\partial t} + KRI. \qquad (7)$$

We seek solutions of the form,

$$I = e^{-\gamma x} f(x - vt), \qquad (8)$$

for which,

$$\frac{\partial I}{\partial t} = -ve^{-\gamma x} f', \qquad \text{and} \qquad \frac{\partial^2 I}{\partial t^2} = v^2 e^{-\gamma x} f'', \qquad (9)$$

while,

$$\frac{\partial I}{\partial x} = -\gamma e^{-\gamma x} f + e^{-\gamma x} f', \qquad \text{so} \qquad \frac{\partial^2 I}{\partial x^2} = \gamma^2 e^{-\gamma x} f - 2\gamma e^{-\gamma x} f' + e^{-\gamma x} f''. \qquad (10)$$

Inserting these into the wave equation we find,

$$\gamma^2 f - 2\gamma f' + f'' = KRf - v(RC + KL)f' + v^2 LC f''. \tag{11}$$

This should be true for an arbitrary function f, so the coefficients of each derivative of f must separately be equal,

$$\gamma = \sqrt{KR}, \qquad 2\frac{\gamma}{v} = RC + KL = 2\sqrt{RCKL}, \qquad \text{and} \qquad v = \sqrt{\frac{1}{LC}}, \tag{12}$$

where we have used the first and third relations in obtaining the second form of the second. In general, $a + b \neq 2\sqrt{ab}$; this only holds when $a = b$. So, we deduce the desired condition,

$$RC = KL, \tag{13}$$

for distortionless telegraphy.[2] With this condition, we can re-express γ as $\gamma = R\sqrt{C/L}$.

Finally, we consider the impedance $Z = V/I$, assuming the line to be driven sinusoidally with time dependence $e^{-i\omega t} (= e^{j\omega t})$. For this, we suppose that $V = V_0 \, e^{-\gamma x} e^{i(kx - \omega t)}$ and $I = V/Z$, where the wave number k is related by $k = \omega/v = \omega\sqrt{LC}$. Using these forms[3] in eqs. (5) and (6) we find that,

$$Z = \sqrt{\frac{R - i\omega L}{K - i\omega C}} = \sqrt{\frac{L}{C} \frac{(RC/KL)K^2 + \omega^2 C^2}{K^2 + \omega^2 C^2} + i\omega \frac{RC - KL}{K^2 + \omega^2 C^2}} \left(= \sqrt{\frac{R + j\omega L}{K + j\omega C}} \right). \tag{14}$$

Not only is the line impedance purely real, with value $Z = \sqrt{L/C}$, when both R and K are zero, but also when condition (13) for distortionless transmission is satisfied.

Remarks: A typical cable has $RC \gg KL$ at low frequencies, where K is constant. It costs a lot to reduce RC, although this was the direction of industry prior to Heaviside. He noted that one shouldn't even try to reduce leakage K, so long as the signal is not attenuated until it is undetectable – and the distortion-free condition makes it much easier to detect small signals. Rather one should increase the inductance, or leakage, or both! This counterintuitive result did not sit well with industry leaders, who, needless to say, were little guided by partial differential equations.

At high frequencies the conductance K rises linearly with frequency, so there is always a particular (high!) frequency at which condition (13) is satisfied, and Z is purely real, without inductive loading of the line.

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[2]The condition (13) is equivalent to the statement that the resistive losses per unit length, $|I|^2 R/2$, are equal to the leakage-current (dielectric) losses per unit length, $|I_{\text{leakage}}|^2 /2K = |V^2| K/2$.

[3]These forms are "exact" solutions to the wave equation (7) only if the distortionless condition (13) holds, although eq. (14) follows in general from the first-order differential equations (5)-(6) when using these forms. That is, the notion of an impedance Z associated with waves of a pure angular frequency ω is a good approximation for lines shorter than $1/\gamma$, for which the wave equation (7) is well satisfied by waves of the form $V = V_0 e^{-\gamma x} e^{i(kx - \omega t)}$ and $I = V/Z$.

A   Appendix: Historical Note

A wave equation for charge and current on a single wire was first given by Kirchhoff [1, 2] (1857) in the context of Weber's electrodynamics [3]. The consideration of a single wire by Kirchhoff followed the practice of telegraphy at the time [4]-[8], in which only a single wire appeared to be involved, and the role of the ground/earth as a return path to "complete the circuit" was not yet recognized,[4] as were neither the capacitance between the wire and ground/earth nor the self inductance of the wire + ground circuit.

Weber followed Neumann (1845) [9] who had introduced the concept of mutual inductance M of two circuits (but not the self inductance L of a single circuit), as well as the vector potential $\mathbf{A}$. For example, the magnetic flux Φ_{12} through circuit 1 due to current I_2 in circuit 2 is related by $\Phi_{12} = M_{12}I_2 = \int \mathbf{B}_2 \cdot d\mathbf{Area}_1 = \oint \mathbf{A}_2 \cdot d\mathbf{l}_1$. We would now add that the integral form of Faraday's law then tells us that if current I_2 varies with time, a (scalar) $\mathcal{EMF}$ is induced in circuit 1 related by $\mathcal{E}_{12} = -\dot{\Phi}_{12} = -M_{12}\dot{I}_2$. However, Neumann, Weber and Kirchhoff did not say this, but rather emphasized the (vector) electromotive) force (*electromotorische Kraft*) of one current element on another, and that the (scalar) $\mathcal{EMF}$ on current element 1 due to changes in current element 2 is related by $\mathcal{E}_{12} = -k_{12}\dot{I}_2$, where k_{12} is a geometric factor.[5]

Rather impressively, Kirchhoff [2] deduced a wave equation for the current and charge on current elements (conductors), finding the wavespeed to be $c = 1/\sqrt{\epsilon_0\mu_0}$, where the constants ϵ_0 and μ_0 can be determined from electro- and magnetostatic experiments (Weber and Kohlsrausch (1856) [11]), and the value of c was close to the speed of light as then known. However, as Weber's electrodynamics was based on action-at-a distance, and was not a field theory in the sense of Faraday and Maxwell, Weber and Kirchhoff did not infer that, since electric waves on wire moved at light speed, light must be an electromagnetic phenomenon.

We now consider that electromagnetic waves associated with conductors are almost entirely outside the conductors, and that the wavespeed of surface charge and current densities matches the wave speed in the medium outside the conductor, *i.e.*, c in case of vacuum. Kirchhoff's argument was the first demonstration of the latter result, which holds for waves propagating parallel to the surface of a conductor of "any" shape. In this sense, Kirchhoff did not deduce the "telegrapher's equation" for transmission lines based on two, parallel conductors, for which the wave speed is $v = 1/\sqrt{LC}$, where L and C are the inductance and capacitance per unit length. This behavior is consistent with Kirchhoff's result because of a general (geometrical) "theorem" that $LC = \epsilon_0\mu_0 = 1/c^2$ for a one-dimensional transmission line, if dielectric effects can be ignored and the current flows only on the surface of the conductors. See, for example, [12, 13, 14].

The term impedance in circuit analysis was introduced by Heaviside in 1886 [15].

The kind of derivation of the "telegrapher's equation" found in textbooks today, us-

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[4]Kirchhoff mentioned the earth on p. 406 of [2] (English version), but only as a reference at zero potential, and not as a conductor.

[5]The first analysis to relate an $\mathcal{EMF}$ to a (self) inductance of a circuit was given in 1851 by Helmholtz [10], who gave a model of propagation of electric signals on nerves as an L-R circuit.

ing Kirchhoff's circuit laws, was first given by Heaviside in 1876 [16].[6,7] See also [18, 19] (where $p = d/dt$ in the latter), which mentioned Kirchhoff's derivation as transcribed by J.J. Thomson, p. 138, June 25, (1886) of [20].[8]

Discussion of a "distortionless" transmission line was made by Heaviside in 1887 [21],[9] who argued that long-distance telegraph lines (including trans-Atlantic cables) should be designed to be "distortionless". Previous cables were fairly far from this ideal, and compatible with a theory of W. Thomson (Lord Kelvin) [22] that ignored the effect of inductance (and so considered telegraphy to be a diffusion phenomenon rather than a wave phenomenon). However, long cables are expensive so there was considerable hesitation to abandon the large existing capital investment and implement the proposed improvements.[10] Large-scale implementation of "distortionless" telegraphy occured only after 1900 following vigorous advocacy by M. Pupin [23], for whom the physics building of Columbia U. is named.

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[6]Heaviside denoted the resistance per unit length of the wire by r, its capacitance per unit length by c, and its inductance per unit length by s. On p. 143 of [16], Heaviside noted that in addition to the current in the wire, there is a reverse current in the earth.

[7]The term telegrapher's equation was coined by Poincaré [17] (1893): *d'équation des télégraphistes*.

[8]Heaviside's struggles to follow Kirchhoff's argument appear on pp. 279-280 of [18] and on pp. 81-82 of [19].

[9]Heaviside indicated his dissatisfaction with Kirchhoff's argument on p. 310 of [21]: *In treatments of electromagnetism by the German methods, a* current element *and its properties of attraction, repulsion, etc., occupy an important place. It is, however, quite an abstraction and devoid of physical significance when by itself. But the current element in our theory above, ..., is a physical reality. It is a complete electromagnetic system of itself, with the electric currents closed. To fix ideas most simply, the two conductors may a wire in an enveloping tube separated by a dielectric...*

In the last sentence above, Heaviside invented the concept of a coaxial cable.

His main point was perhaps that Kirchhoff considered only a single wire, which if straight would not form a complete circuit, so the meaning of Kirchhoff's wave equation was suspect. Nonetheless, Kirchhoff's wave equation is valid when applied to two-conductor transmission lines in which there is only one wire plus "ground" as the second conductor (and no dielectric). So, Heaviside's point should have been that Kirchhoff did not realize that his "circuit" had a second conductor.

[10]Indeed, the editor of the journal that published Heaviside's papers was fired for being too sympathetic to Heaviside's views that were initially quite unpopular with industry. Heaviside, who was unemployed for most of his life, could not be fired!

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