Are Christoffel Symbols Needed to Analyze Transient Phenomena in Faraday's Dynamo?

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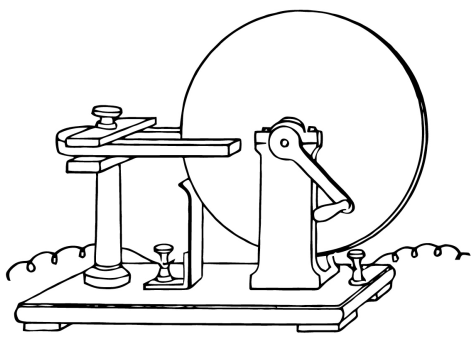
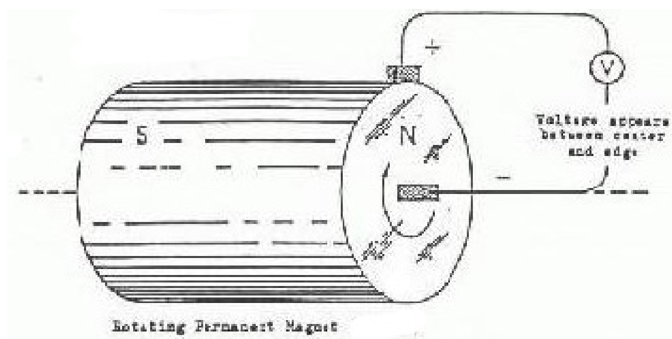
1 Problem

Suppose the rotor of a Faraday disk dynamo (1831, sec 4 of [1]) has radius a, mechanical moment of inertia $I_{\rm mech}$ about its axis, and initial angular velocity ω_0, when immersed in a uniform, axial magnetic field B. The dynamo is connected to a load of electrical resistance R via frictionless contacts at its edge and on its axis, as sketched in the figures below.[1] Assuming dissipation of energy is negligible by means other than the Joule heating of the resistor, deduce the time dependence $\omega(t)$ of the angular velocity.





In several papers [4]-[7], Kron argued that if one chooses to analyze problems of rotating electrical machinery in the rotating frame, then one must employ much of the mathematical apparatus of general relativity, including the Christoffel symbols. Is this really necessary?

2 Solution

The larger issue is that while analysis of mechanical systems in accelerated frames is often convenient and insightful, this is not the case for electromagnetism for which one must deal with "fictitious" charge and current densities in accelerated frames of reference (as well as such mathematical formalism as Christoffel symbols).[2] If one desires knowledge of some particular electromagnetic quantity at some place and time in an accelerated frame, it is typically preferable to deduce this quantity in the lab frame, and then transform it to the instantaneous, inertial comoving frame (although the result is not strictly the value in the accelerated frame).

[1]Faraday noted in 1831, and again in 1851 [2] that the external magnetic field could be replaced by the permanent magnetic field of a conducting disk, as sketched on the right above. See also [3].

[2]No Christoffel symbols are needed to express electromagnetism is a uniformly rotating frame, but the presence of the "fictitious" charges and currents render this exercise somewhat counterintuitive. See, for example, [8].

In the present example the angular velocity is desired in the (inertial, to a good approximation) lab frame, and consideration of the transient rotating frame of the Faraday disk is not worth while.

The strategy is to relate the electrical current I that flows through the dynamo and the load resistor R to an electromotive force $\mathcal{E}$ that can be said to drive the current. For this, it is better not to use some version of Faraday's law (and to avoid the famous question of whether the "lines" of magnetic field rotate with the rotating magnetic in the second variant of the dynamo sketched on p. 1). The crisper argument (valid for both variants of the Faraday dynamo) is to consider a charge carrier e that moves radially with respect to the rotating disk (as part of the conduction current I) but which is dragged azimuthally with angular velocity ω by its electromagnetic interaction with the "lattice" of the disk. In this view, the Lorentz force [9] on the charge e in the magnetic field $\mathbf{B}$ (rather than $\mathbf{H}$ [10]) is (in Gaussian units),

$$\mathbf{F} = e\frac{\mathbf{v}}{c} \times \mathbf{B} = \frac{e\omega r B}{c}\,\hat{\mathbf{r}} \equiv e\mathbf{E}_{\mathrm{eff}} \qquad (0 < r < a), \tag{1}$$

where the effective (radial, lab-frame) electric field inside the rotating disk is,

$$\mathbf{E}_{\mathrm{eff}} = \frac{\omega r B}{c}\,\hat{\mathbf{r}}, \tag{2}$$

and c is the speed of light in vacuum. We identify a "motional" electromotive force,

$$\mathcal{E} = \int_0^a \mathbf{E}_{\mathrm{eff}} \cdot d\mathbf{r} = \frac{\omega a^2 B}{2c}\,, \tag{3}$$

with the effect of the Lorentz force (1), such that the electrical current is given by,

$$I = \frac{\mathcal{E}}{R} = \frac{\omega a^2 B}{2cR}\,. \tag{4}$$

The rate of dissipation of energy U by the load resistor is then,

$$\frac{dU}{dt} = -\frac{\mathcal{E}^2}{R} = -I^2 R = -\frac{\omega^2 a^4 B^2}{4c^2 R}\,. \tag{5}$$

The stored energy U consists of the rotational kinetic energy plus the magnetic field energy, where the latter is tiny compared to the former. And, we assume that dissipation of energy by friction, and by electromagnetic radiation, is negligible, such that,

$$U \approx \frac{I_{\mathrm{mech}}\omega^2}{2}\,, \qquad \frac{dU}{dt} \approx I_{\mathrm{mech}}\omega\dot{\omega} = -\frac{\omega^2 a^4 B^2}{4c^2 R}\,. \tag{6}$$

and hence,

$$\frac{\dot{\omega}}{\omega} \approx -\frac{a^4 B^2}{4c^2 R I_{\mathrm{mech}}}\,, \qquad \omega(t) \approx \omega_0\, e^{-a^4 B^2\, t/4c^2 R I_{\mathrm{mech}}}. \tag{7}$$

A Appendix: Steady Rotation

The preceding discussion used the convenient fiction of an effective electric field (2) inside the rotating disk. However, this field is not a true electric field $\mathbf{E}$ that obeys Faraday's law,

$$\oint \mathbf{E} \cdot d\mathbf{l} = -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} . \tag{8}$$

To clarify the character of the actual electric field $\mathbf{E}$ in Faraday's dynamo we consider the case of steady rotation, for which we expect,

$$\oint \mathbf{E} \cdot d\mathbf{l} = 0 \qquad \text{(steady state)}. \tag{9}$$

It is important to note that any real dynamo has a small internal resistance R'. Indeed, if the disk were a perfect conductor, the external magnetic field could not penetrate into the disk and there would be no Lorentz force on the charge charges, and no dynamo effect.

The current in the system is actually,

$$I = \frac{\mathcal{E}}{R + R'} . \tag{10}$$

An electric field $\mathbf{E}$ exists inside the (resistive) wire and the resistor to drive the current there,[3] and line integral of the electric field along the external wires and the resistor R is,

$$\int_{\text{ext}} \mathbf{E} \cdot d\mathbf{l} = IR = \frac{\mathcal{E} R}{R + R'} . \tag{11}$$

Inside the disk the current $\mathbf{J}$ can be related to the conductivity σ' of the disk by the generalization of Ohm's law,

$$\mathbf{J} = \sigma' \left(\mathbf{E} + \frac{\mathbf{v}}{c} \times \mathbf{B} \right) . \tag{12}$$

We suppose the radial current flows in a tube of small area A such that[4]

$$J = \frac{I}{A} = \sigma' \left(E_r + \frac{\omega r B}{c} \right) , \qquad \text{and} \qquad R' = \frac{a}{A\sigma'} . \tag{13}$$

The radial electric field inside the tube is,

$$E_r = -\frac{\omega r B}{c} + \frac{I}{A\sigma'} = -\frac{\omega r B}{c} + \frac{IR'}{a} . \tag{14}$$

[3]This electric field is shaped by static surface charges along the wire and resistor, which charges flowed from the disk during the transient spin-up phase.

[4]This assumption is only approximately correct, as some current flows throughout the disk and the velocity of the charge carriers is not simply $\boldsymbol{\omega} \times \mathbf{r}$ as due to the rotation of the disk. In this approximation, $\mathbf{J} = \sigma'(\mathbf{E} + \omega r B\,\hat{\mathbf{r}}/c)$, and since $\boldsymbol{\nabla} \cdot \mathbf{J} = 0$ in the disk, we have that $4\pi\rho = \boldsymbol{\nabla} \cdot \mathbf{E} = -2\omega B/c$, and there is a constant, nonzero charge density ρ throughout the disk.

Note that the electric field (14) is opposite in direction to the effective field (2).[5] The line integral of the electric field along the tube is,

$$\int_{\rm tube} \mathbf{E} \cdot d\mathbf{l} = -\frac{\omega a^2 B}{2c} + IR' = -\mathcal{E} + IR'. \tag{15}$$

When the disk is not connected to the external resistor, and $I = 0$, this line integral is the so-called Hall voltage [11].

The line integral around the entire circuit is,

$$\oint \mathbf{E} \cdot d\mathbf{l} = \int_{\rm tube} \mathbf{E} \cdot d\mathbf{l} + \int_{\rm ext} \mathbf{E} \cdot d\mathbf{l} = -\mathcal{E} + I(R + R') = 0, \tag{16}$$

as expected in a steady-state.

It may also be instructive to consider the energy balance in the dynamo, which delivers power,

$$P = \mathcal{E} I = \frac{\mathcal{E}^2}{R + R'}\,, \tag{17}$$

to Joule heating of the electrical resistance.

It is often said that the magnetic Lorentz force drives the current, but it is also true that in many situations magnetic fields do no work.[6]

Not only does the magnetic Lorentz force drive charge radially, but it leads to a torque that opposes the rotation of the disk. In the approximation that the current flows only radially in the small tube,

$$\boldsymbol{\tau}_B = \int_{\rm tube} \mathbf{r} \times \left(\frac{I\,\hat{\mathbf{r}}}{c} \times \mathbf{B} \right) dr, \qquad \tau_{B,z} = -\frac{Ir^2 B}{2c}\,, \tag{18}$$

taking the z-axis along the direction of the angular velocity $\boldsymbol{\omega}$. For steady motion, this torque must be opposed by a mechanical torque $\tau_{\rm mech}$ that delivers power,

$$P_{\rm mech} = \tau_{\rm mech}\, \omega = \frac{I\omega\, r^2 B}{2c} = \mathcal{E} I = P\,. \tag{19}$$

We can say that the mechanical agent does work against the azimuthal Lorentz force, and the radial Lorentz force does work opposite to the that of the azimuthal Lorentz force (such that the total work done by the Lorentz force is zero). Hence, the radial Lorentz force does work equal to that by the mechanical agent, and equal to the power P of eq. (17),

$$P_B = \sum_{\rm charges} \mathbf{v} \cdot e\frac{\mathbf{v}}{c} \times \mathbf{B} = 0 = \sum v_r F_{B,r} + \sum v_\phi F_{B,\phi} = P_{B,r} + P_{B,\phi}, \tag{20}$$

$$P_{B,r} = -P_{B,\phi} = P_{\rm mech} = P = \mathcal{E} I. \tag{21}$$

The usual summary of all this is that the (radial) Lorentz force does the work that drives the current.

[5] This is similar to the case of a battery where one sometimes speaks of a nonelectrostatic (nonconservative) field inside the battery that is opposite to the actual electric field there.

[6] This statement requires careful interpretation, as discussed, for example, in [12].

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