Why Do We Speak of Magnetic Flux but not Electric Flux?

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"Magnetic flux"[1] Φ_B is defined for a closed loop and some surface bounded by that loop as

$$\Phi_B = \int\int_{\rm surface} \mathbf{B} \cdot d\mathbf{Area}, \tag{1}$$

where $\mathbf{B}$ is the magnetic field and $d\mathbf{Area} = \hat{\mathbf{n}}\, d\text{Area}$ and $\hat{\mathbf{n}}$ is a unit vector perpendicular to the element dArea of surface area.

Similarly, we could define the "electric flux" Φ_E associated with the same loop and surface bounded by it as

$$\Phi_E = \int_{\rm surface} \mathbf{E} \cdot d\mathbf{Area}, \tag{2}$$

where $\mathbf{E}$ is the electric field.

However, the magnetic flux (1) is more universal than the electric flux (2), which is one reason why the electric flux (2) is less mentioned than the magnetic flux.

To see this, consider two surfaces (1 and 2) bounded by the same loop, and the volume enclosed by these two surfaces. The magnetic flux across the closed surface defined by surfaces 1 and 2 is the difference between the fluxes across surfaces 1 and 2,

$$\int_{\rm closed\ surface} \mathbf{B} \cdot d\mathbf{Area} = \int_{\rm surface\ 1} \mathbf{B} \cdot d\mathbf{Area} - \int_{\rm surface\ 2} \mathbf{B} \cdot d\mathbf{Area}. \tag{3}$$

Then, Gauss' law tells us that

$$\int_{\rm closed\ surface} \mathbf{B} \cdot d\mathbf{Area} = \int_{\rm enlosed\ volume} \boldsymbol{\nabla} \cdot \mathbf{B}\, d\text{Vol} = 0, \tag{4}$$

and so the magnetic flux is the same across any two surfaces bounded by a given closed loop. This result depends on the relation (a Maxwell equation) that $\boldsymbol{\nabla} \cdot \mathbf{B} = 0$, *i.e.*, that there are no magnetic charges/poles.

In contrast, another Maxwell equation tells us that $\boldsymbol{\nabla} \cdot \mathbf{E} = 4\pi\rho$ (in Gaussian units), where ρ is the volume density of electric charge. Thus, while the electric flux (2) is well defined for any particular surface bounded by the closed loop of interest, different surfaces bounded by the same closed loop are associated with different electric fluxes.

[1]The term "flux" might seem to be associated with motion of some quantity. However, the term "flux" first appeared in the literature of electromagnetism in Arts. 12-13 of Maxwell's *Treatise* [1], where he defined any surface integral of a vector field such as $\mathbf{B}$, of the form of eq. (1), to be a "flux". Maxwell's terminology seems to have been inspired by a vector description of heat flow.

It may be that the term "magnetic flux" is sometimes taken to mean the magnetic field.

Maxwell's equation for Faraday's law, $\boldsymbol{\nabla} \times \mathbf{E} = -\partial \mathbf{B}/\partial ct$, can be expressed in integral form for some closed loop as

$$\mathcal{EMF} = \oint_{\text{closed loop}} \mathbf{E} \cdot d\mathbf{l} = \int_{\text{surface}} \boldsymbol{\nabla} \times \mathbf{E} \cdot d\mathbf{Area} = -\frac{1}{c} \int_{\text{surface}} \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{Area} = -\frac{1}{c} \frac{d\Phi_B}{dt}\,, \tag{5}$$

which is now known as Faraday's "flux rule"[2] for the "electromotive force" $\mathcal{EMF}$, although Faraday never used the term "flux".

Similarly, we could convert Maxwell's generalization of Ampère's law, $\boldsymbol{\nabla} \times \mathbf{B} = 4\pi\mathbf{J}/c + \partial \mathbf{E}/\partial ct$, into an integral form for some closed loop as[3]

$$\begin{aligned} \mathcal{MMF} &= \oint_{\text{closed loop}} \mathbf{B} \cdot d\mathbf{l} = \int_{\text{surface}} \boldsymbol{\nabla} \times \mathbf{B} \cdot d\mathbf{Area} \\ &= \frac{4\pi}{c} \int_{\text{surface}} \mathbf{J} \cdot d\mathbf{Area} + \frac{1}{c} \int_{\text{surface}} \frac{\partial \mathbf{E}}{\partial t} \cdot d\mathbf{Area} = \frac{4\pi I}{c} + \frac{1}{c} \frac{d\Phi_E}{dt}\,, \end{aligned} \tag{6}$$

where $\mathcal{MMF}$ is the "magnetomotive force" that would act on a magnetic charge/pole p (if this existed) lying on the closed loop, $\mathbf{J}$ is the electric current density, and $I = \int \mathbf{J} \cdot d\mathbf{Area}$ is the total current that passes through the loop.[4] Maxwell considered this to be a "true (electric) current", although this is no longer taken to be so.

However, since magnetic charges/poles do not exist, eq. (6) (which might be the main application of the concept of "electric flux") is seldom discussed.

This note was inspired by e-discussions with Allan Walstad.

References

[1] J.C. Maxwell, *A Treatise on Electricity and Magnetism*, Vol. 1 (Clarendon Press, 1873), http://kirkmcd.princeton.edu/examples/EM/maxwell_treatise_v1_73.pdf

[2] K.T. McDonald, *What is the "Flux Rule"?* (May 16, 2023), http://kirkmcd.princeton.edu/examples/flux_rule.pdf

[3] K.T. McDonald, *Is Faraday's Disk Dynamo a Flux-Rule Exception?* (July 27, 2019), kirkmcd.princeton.edu/examples/faradaydisk.pdf

[4] J.C. Maxwell, *On Physical Lines of Force. Part III.—The Theory of Molecular Vortices applied to Statical Electricity*, Phil. Mag. **23**, 12 (1862), http://kirkmcd.princeton.edu/examples/EM/maxwell_pm_23_12_62.pdf

[2]Discussions by the author of the "flux rule" are at [2, 3].

[3]The differential equation for $\boldsymbol{\nabla} \times \mathbf{B}$ above my eq. (6) was first given by Maxwell in his eq. (112) of [4].

[4]In his verbal discussion of my eq. (6), Maxwell called $(1/4\pi)\, d\Phi_E/dt$ the "displacement current", based on his view that the field $\mathbf{D} = \epsilon\mathbf{E}$, where ϵ is the (relative) dielectric constant of the medium, is some kind of strain/displacement of the medium/ether.

In this note, we have tacitly assumed that $\epsilon = 1$.