Conducting Ellipsoid and Circular Disk

Kirk T. McDonald
Joseph Henry Laboratories, Princeton University, Princeton, NJ 08544
(September 21, 2002; updated December 27, 2019)

1 Problem

Show that the surface charge density σ on a conducting ellipsoid,

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1, \tag{1}$$

can be written,

$$\sigma_{\rm ellipsoid} = \frac{Q}{4\pi abc\sqrt{\frac{x^2}{a^4} + \frac{y^2}{b^4} + \frac{z^2}{c^4}}}\,, \tag{2}$$

where Q is the total charge.

Show that if the charge distribution of the ellipsoid is projected onto any of its symmetry planes, the result is independent of the extent of the ellipse perpendicular to the plane of projection (*i.e.*, σ_{xy}, the projection of $\sigma_{\rm ellipsoid}$ on the x-y plane, is independent of parameter c). Thus, the projection of the charge distribution of a conducting oblate or prolate spheroid onto its equatorial plane is the same as the projected charge distribution of a conducting sphere.

By considering a thin conducting circular disk of radius a as a special case of an ellipsoid, show that its surface charge density (summed over both sides) can be written as,

$$\sigma_{\rm circular\ disk} = \frac{V_0}{2\pi^2\sqrt{a^2 - r^2}}, \tag{3}$$

where the electric potential V_0 of the disk is related by $V_0 = \pi Q/2a$.

Show also that if the charge distribution on the conducting ellipsoid is projected onto any of the coordinate axes, the result is uniform (*i.e.*, the charge distribution projected onto the x-axis is $\sigma_x = Q/2a$). In particular, we expect that the charge distribution along a conducting needle will be uniform, since the needle can be considered as the limit of a conducting ellipsoid, two of whose three axes have shrunk to zero.

2 Solution

The charge distribution (3) on a conducting ellipsoid can be deduced in a variety of ways [1]-[7]. We record here a highly geometric derivation following Thomson (1869) [2].[1,2]

[1]The charge distribution on the surface of a conducting, prolate spheroid was deduced by Green (1828), pp. 68-69 of [8], by noting that the equipotentials of a uniformly charged needle are spheroids. He stated that his results "agree with what has been long known".

[2]Thomson's method was stated in a textbook by Murphy in 1833 [1] (with no equations), which suggests that it was already well known.

The starting point is the "elementary" result that the electric field is zero in the interior of a spherical shell of any thickness that has a uniform volume charge density between the inner and outer surfaces of the shell. A well-known geometric argument (due to Newton, book 1, prop. 70, p. 218 of [9]) for this is illustrated in Fig. 1.

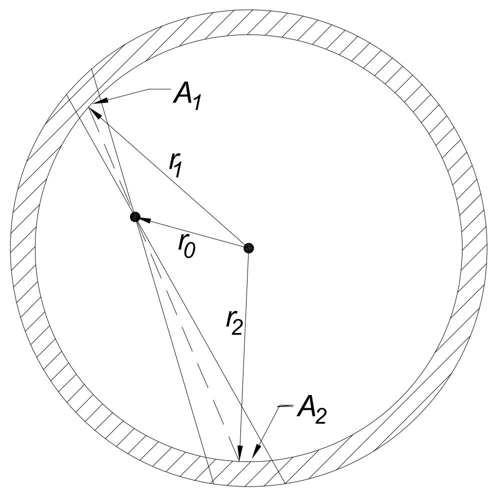

Figure 1: For any point $\mathbf{r}_0$ in the interior of a uniformly charged shell of charge, the axis of a narrow bicone intercepts the inner surface of the shell at points $\mathbf{r}_1$ and $\mathbf{r}_2$. The corresponding areas on the inner surface of the shell intercepted by the bicone are A_1 and A_2. In the limit of small areas, $A_1/R_{01}^2 = A_2/R_{02}^2$.

The electric field at point $\mathbf{r}_0$ in the interior of the shell due to a lamina of thickness δ and area A_1 centered on point $\mathbf{r}_1$ that lies within a narrow cone whose vertex is point 0 is given by,

$$\mathbf{E}_1 = \frac{\rho\, d\mathrm{Vol}_1}{R_{01}^2}\hat{\mathbf{R}}_{01}, \tag{4}$$

where ρ is the volume charge density, $d\mathrm{Vol}_1 = A_1\delta$, $\mathbf{R}_{01} = \mathbf{r}_0 - \mathbf{r}_1$, and the center of the sphere is taken to be at the origin. Likewise, the electric field from a lamina of area A_2 centered on point $\mathbf{r}_2$ defined by the intercept with the shell of the same narrow cone extended in the opposite direction (forming a bicone) is given by,

$$\mathbf{E}_2 = \frac{\rho\, d\mathrm{Vol}_2}{R_{02}^2}\hat{\mathbf{R}}_{02}, \tag{5}$$

In the limit of bicones of small half angle, the two parts of the bicone as truncated by the shell are similar, so that,

$$\frac{A_1}{R_{01}^2} = \frac{A_2}{R_{02}^2}, \qquad \frac{d\mathrm{Vol}_1}{R_{01}^2} = \frac{d\mathrm{Vol}_2}{R_{02}^2}, \tag{6}$$

and, of course, $\hat{\mathbf{R}}_{02} = -\hat{\mathbf{R}}_{01}$. Hence $\mathbf{E}_1 + \mathbf{E}_2 = 0$. Since this construction can be applied to all points in the material of the spherical shell, and for all pairs of surface elements subtended by (narrow) bicones, the total electric field in the interior of the shell is zero.

We now reconsider the above argument after arbitrary scale transformations have been applied to the rectangular coordinate axes,

$$x \to k_1 x, \qquad y \to k_2 y, \qquad z \to k_3 z. \tag{7}$$

A spherical shell of radius s is thereby transformed into an ellipsoid,

$$\frac{x^2}{s^2} + \frac{y^2}{s^2} + \frac{z^2}{s^2} = 1 \qquad \to \qquad \frac{x^2}{s^2/k_1^2} + \frac{y^2}{s^2/k_2^2} + \frac{z^2}{s^2/k_3^2} = 1. \tag{8}$$

As parameter s is varied, one obtains a set of similar ellipsoids, centered on the origin.

A small volume element obeys the transformation,

$$d\mathrm{Vol} = dxdydz \to k_1 k_2 k_3 \; dxdydz = k_1 k_2 k_3 \; d\mathrm{Vol}. \tag{9}$$

The three points 0, 1, and 2 in Fig. 1 lie along a line, so that,

$$\mathbf{R}_{01} = \mathbf{r}_0 - \mathbf{r}_1 = C\mathbf{R}_{02} = C(\mathbf{r}_0 - \mathbf{r}_2), \tag{10}$$

where C is a (negative) constant. This relation is invariant under the scale transformation (7), so that together with eq. (9) the relation,

$$\frac{d\mathrm{Vol}_1}{R_{01}^2} = \frac{d\mathrm{Vol}_2}{R_{02}^2}\,, \tag{11}$$

is also invariant. Hence, if the ellipsoidal shell, which is the transform of the spherical shell of Fig. 1, contains a uniform volume charge density, the relation $\mathbf{E}_1 + \mathbf{E}_2 = 0$ remains true at the vertex of any bicone in the interior of the shell, which implies that the total electric field is zero there.[3]

This proof is based on the premise that the ellipsoidal shell is bounded by two similar ellipsoids, and that the volume charge density in the shell is uniform.

If we let the outer ellipsoid of the shell approach the inner one, always remaining similar to the latter, we reach a configuration that is equivalent to a thin, conducting ellipsoid, since in both cases the electric field is zero in the interior. Hence, the surface charge distribution on a thin, conducting ellipsoid must the same as the projection onto its surface of a uniform charge distribution between that surface and a similar, but slightly larger ellipsoidal surface.

The charge σ per unit area on the surface of a thin, conducting ellipsoid is therefore proportional to the thickness, which we write as δd, of the ellipsoidal shell formed by that surface and a similar, but slightly larger ellipsoid,

$$\sigma = \rho \; \delta d, \tag{12}$$

where constant ρ is to be determined from a knowledge of the total charge Q on the conducting ellipsoid.

[3]This result was obtained by Newton (for the gravtitational attraction) at points within spheroids shells, book 1, prop. 91, cor. 3, p. 239 of [9]. The results for ellipsoidal shells may have first been given by Ivory, p. 364 of [10] (1809).

The thickness δd of a thin ellipsoidal shell at some point on its inner surface is the distance between the plane that is tangent to the inner surface at the specified point, and the plane that is tangent to the outer surface at the point similar to the specified point. These planes are parallel since the ellipsoids are parallel. In particular, if the semimajor axes of the inner ellipsoid are called a, b, and c, then those of the outer ellipsoid can be written $a+\delta a$, $b+\delta b$ and $c+\delta c$. Let the (perpendicular) distance from the plane tangent to the specified point on the inner ellipsoid to its center be called d, and the corresponding distance from the outer tangent plane be $d+\delta d$, so that δd is the desired thickness of the shell at the specified point. Then, the condition of similarity is that,

$$\frac{\delta a}{a} = \frac{\delta b}{b} = \frac{\delta c}{c} = \frac{\delta d}{d}\,. \tag{13}$$

Since the volume of an ellipsoid with semimajor axes a, b, and c is $4\pi abc/3$, the volume of the ellipsoidal shell is $4\pi(a+\delta a)(b+\delta b)(c+\delta c)/3 - 4\pi abc/3 = 4\pi abc(\delta d/d)$, using eq. (13). As the constant ρ has an interpretation as the uniform charge density within the material of the ellipsoidal shell, we find that the total charge Q on the conducting ellipsoid is related by,

$$Q = \rho\ \mathrm{Vol}_{\mathrm{shell}} = \frac{4\pi abc}{d}\rho\ \delta d, \tag{14}$$

and hence,

$$\sigma = \rho\ \delta d = \frac{Qd}{4\pi abc}\,. \tag{15}$$

It remains to find an expression for the distance d to the tangent plane. If we write the equation for the ellipsoid in the form,

$$f(x,y,z) = \sqrt{\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}} - 1 = 0, \tag{16}$$

then the gradient of f is perpendicular to the tangent plane. Thus, the vector $\mathbf{d}$ from the center of the ellipsoid to the tangent plane is proportional to $\boldsymbol{\nabla} f$. That is,

$$\mathbf{d} \propto \boldsymbol{\nabla} f = \left(\frac{x}{a^2}, \frac{y}{b^2}, \frac{z}{c^2}\right), \tag{17}$$

noting that $\sqrt{x^2/a^2 + y^2/b^2 + z^2/c^2} = 1$ on the ellipsoid. The unit vector $\hat{\mathbf{d}}$ is therefore,

$$\hat{\mathbf{d}} = \frac{\left(\frac{x}{a^2}, \frac{y}{b^2}, \frac{z}{c^2}\right)}{\sqrt{\frac{x^2}{a^4} + \frac{y^2}{b^4} + \frac{z^2}{c^4}}}. \tag{18}$$

The magnitude d of the vector $\mathbf{d}$ is related to the vector $\mathbf{r} = (x,y,z)$ of the specified point on the ellipse by,[4]

$$d = \mathbf{r}\cdot\hat{\mathbf{d}} = \frac{\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}}{\sqrt{\frac{x^2}{a^4} + \frac{y^2}{b^4} + \frac{z^2}{c^4}}} = \frac{1}{\sqrt{\frac{x^2}{a^4} + \frac{y^2}{b^4} + \frac{z^2}{c^4}}}\,. \tag{19}$$

[4] *Nov. 15, 2019.* It was pointed out in [11] that $d = 1/(\hat{\mathbf{d}}\cdot\boldsymbol{\nabla} f) = 1/df/du = \int df\,\delta(f)/df/du = \int du\,\delta(f)$, where u is a coordinate perpendicular to surface of the ellipsoid, with $u = 0$ on the surface, and $\delta(f)$ is the Dirac delta function.

At length, we have found the charge density on the surface of a conducting ellipsoid to be,

$$\sigma_{\text{ellipsoid}} = \frac{Q}{4\pi abc\sqrt{\frac{x^2}{a^4}+\frac{y^2}{b^4}+\frac{z^2}{c^4}}}\,, \tag{20}$$

where Q is the total charge.

To deduce the projection of the charge distribution of the conducting ellipse onto one of its symmetry planes, say the x-y plane, note that a projected area element $dx\,dy$ corresponds to area $d\mathbf{A}$ on the surface of the ellipsoid that is related by,

$$dx\ dy = d\mathbf{A}\cdot\hat{\mathbf{z}}\ = dA\ d_z, \tag{21}$$

where the unit vector $\hat{\mathbf{d}}$ that is normal to the surface of the ellipsoid at the point (x, y) is given by eq. (19). Thus,

$$dA = \frac{dx\,dy}{\hat{d}_z} = \frac{c^2\ dx\,dy}{z}\sqrt{\frac{x^2}{a^4}+\frac{y^2}{b^4}+\frac{z^2}{c^4}}. \tag{22}$$

The charge projected onto the x-y plane from both $z > 0$ and $z < 0$ is,

$$dQ_{xy} = 2\sigma_{\text{ellipsoid}}dA = \frac{Qc\ dx\,dy}{2\pi abz} = \frac{Q\ dx\,dy}{2\pi ab\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}}, \tag{23}$$

combining eqs. (1), (20) and (22). The projected charge density (due to both halves of the ellipsoid), $\sigma_{xy} = dQ_{xy}/dx\,dy$, is independent of the parameter c that specifies the size of the ellipsoid in z. For example, the charge distributions of a sphere, a disk, and both an oblate and prolate spheroid, all of the same equatorial diameter, are the same when projected onto the equatorial plane.

If the project the charge dQ_{xy} of eq. (23) onto the x axis, the result is,

$$dQ_x = \frac{Q\ dx}{2\pi a}\int_{-\sqrt{b^2-\frac{b^2x^2}{a^2}}}^{\sqrt{b^2-\frac{b^2x^2}{a^2}}}\frac{dy}{\sqrt{b^2-\frac{b^2x^2}{a^2}-y^2}} = \frac{Q}{2a}dx, \tag{24}$$

which is uniform in x! In particular, the uniform charge distribution on a conducting sphere projects to a uniform charge distribution on any diameter; and the charge distribution is uniform along a conducting needle that is the limit of conducting ellipsoid.

The case of a thin, conducting elliptical disk in the x-y plane can be obtained from eq. (20) by letting c go to zero.[5] For this we note that eq. (16) for a general ellipsoid permits us to write,

$$c\sqrt{\frac{x^2}{a^4}+\frac{y^2}{b^4}+\frac{z^2}{c^4}} = \sqrt{c^2\left(\frac{x^2}{a^4}+\frac{y^2}{b^4}\right)+1-\frac{x^2}{a^2}-\frac{y^2}{b^2}} \to \sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}. \tag{25}$$

[5]For a rather different method of solution, see [12].

The charge density on each side of a conducting elliptical disk is therefore,

$$\sigma_{\text{elliptical disk}} = \frac{Q}{4\pi ab\sqrt{1 - \frac{x^2}{a^2} + \frac{y^2}{b^2}}}\,. \tag{26}$$

The charge density on each side of a conducting circular disk of radius a follows immediately as,

$$\sigma_{\text{circular disk}} = \frac{Q}{4\pi a\sqrt{a^2 - r^2}}\,, \tag{27}$$

where $r^2 = x^2 + y^2$. Such a disk has potential V_0, which can be found by calculating the potential at the center of the disk according to,

$$V_0 = V(r = 0, z = 0) = \int_0^a \frac{2\sigma(r)}{r} 2\pi r\, dr = \frac{Q}{a}\int_0^a \frac{dr}{\sqrt{a^2 - r^2}} = \frac{\pi Q}{2a}\,. \tag{28}$$

Hence, a conducting disk of radius a at potential V_0 has charge density,

$$\sigma_{\text{circular disk}} = \frac{V_0}{2\pi^2\sqrt{a^2 - r^2}}. \tag{29}$$

References

[1] R. Murphy, *Elementary Principles of the Theories of Electiricity, Heat and Molecular Actions*, Part 1 (Pitt Press, 1833), pp. 68-70,
http://kirkmcd.princeton.edu/examples/EM/murphy_electricity_33.pdf

[2] W. Thomson (Lord Kelvin), *Papers on Electricity and Magnetism*, 2nd ed. (Macmillan, 1884), pp. 7 and 178-179,
http://kirkmcd.princeton.edu/examples/EM/thomson_electrostatics_magnetism.pdf

[3] Lord Kelvin and P.G. Tait, *Treatise on Natural Philosophy*, Vol. 2 (Cambridge U. Press, 1903), secs. 519-520, http://kirkmcd.princeton.edu/examples/mechanics/thomson_tait_treatise_v2.pdf

[4] O.D. Kellogg, *Foundations of Potential Theory* (Springer, 1929), pp. 188-191,
http://kirkmcd.princeton.edu/examples/mechanics/kellogg_potential_29.pdf

[5] E.J. Routh, *A Treatise on Analytical Statics with Illustrations Taken from the Theories of Electricity and Magnetism*, Vol. 2 (Cambridge U. Press, 1922), Arts. 71 and 376,
http://kirkmcd.princeton.edu/examples/mechanics/routh_statics_v2_22.pdf

[6] E. Durand, *Électrostatique*, Vol. 2 (Masson, 1966), pp. 415-417,
http://kirkmcd.princeton.edu/examples/EM/durand_statique_v2.pdf

[7] W.R. Smythe, *Static and Dynamic Electricity*, 3rd ed. (McGraw-Hill, 1968), sec. 5.02,
http://kirkmcd.princeton.edu/examples/EM/smythe_50.pdf

[8] G. Green, *Mathematical Papers* (Macmillan, 1871),
http://kirkmcd.princeton.edu/examples/EM/green_papers_71.pdf

[9] I. Newton, *Philosophiæ Naturalis Principia Mathematica* (London, 1726), http://kirkmcd.princeton.edu/examples/mechanics/newton_principia.pdf

[10] J. Ivory, *On the Attractions of homogeneous Ellipsoids*, Phil. Trans. Roy. Soc. London **99**, 345 (1809), http://kirkmcd.princeton.edu/examples/mechanics/ivory_ptrsl_99_345_09.pdf

[11] T.L. Curtright *et al.*, *Charge Densities for Conducting Ellipsoids*, Eur. J. Phys. **41** 035204 (2020), http://kirkmcd.princeton.edu/examples/EM/curtright_ejp_41_035204_20.pdf

[12] E.T. Copson, *On the problem of the electrified disc*, Proc. Edin. Math. Soc. **8**, 14 (1947), http://kirkmcd.princeton.edu/examples/EM/copson_pems_8_14_47.pdf