Engelhardt's Electromagnetic Spaceship

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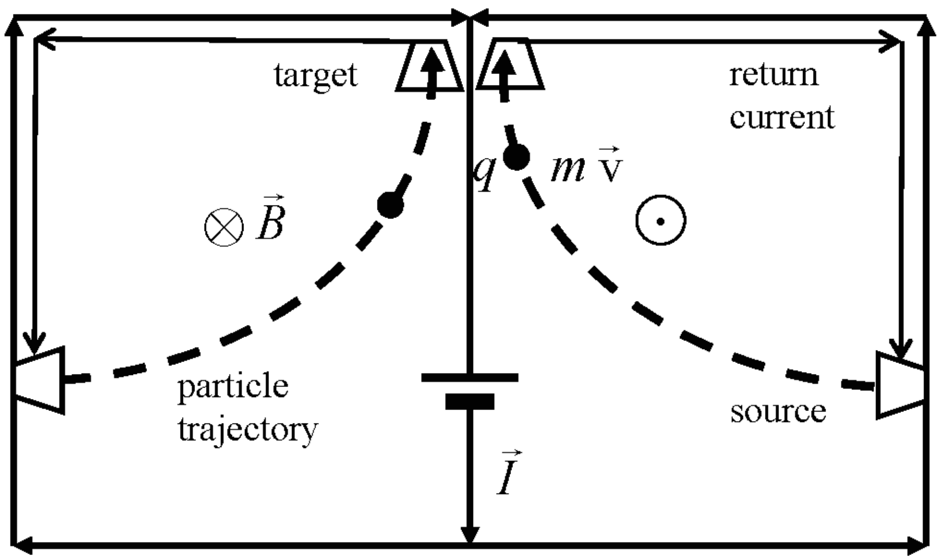
1 Problem

In 2008, Engelhardt [1] proposed a "bootstrap spaceship" as sketched below.[1,2]



Positive ions are emitted "horizontally" and inwards from sources around a ring on the "vertical" wall of a conducting cylinder, and deflected upwards by the azimuthal magnetic field generated by a "downward" electrical current (driven by a "battery") in a wire along the axis of the cylinder. When the ions are collected at the (conducting) "top" plate of the cylinder, the latter experiences an impulsive "upward" force.

If this were the only "vertical" force on the cylinder,[3] the system would indeed be a "bootstrap spaceship".[4]

Can this be so?

2 Solution

There are other forces on the cylinder in this example, as discussed below.

First, we note that if the ions are positive charges, when they are collected at the "top" plate they do not move further with respect to the cylinder. They are neutralized by electrons, such that a buildup of positive electric charge at the collector/target is avoided. But,

[1]For a review that there are no bootstrap spaceships, see [2].

[2]Thanks to V. Onoochin for pointing the author to [1].

[3]In the rest of this note, the term "cylinder" refers to the system consisting of the central wire, "top" and "bottom" plates, as well as the outer cylindrical surface.

[4]Thrusters in which emitted ions or electrons go free have found application in satellite maneuvers [3].

there is a net transfer of mass from the ion sources to the "top" plate, such that the center of mass of the system relative to the cylinder moves "upwards".

The center of mass/energy of an isolated system, such as the present example, must remain at rest if initially at rest.[5] So, we infer that the final position of the cylinder, in the lab frame, is actually "below" its initial position after the flow of ions has stopped.[6]

2.1 Downward Magnetic Forces on the Conducting Cylinder

The magnetic forces on the "vertical" currents in the central wire and outer wall are "horizontal", and sum to zero by symmetry.

However, the forces on currents in the "top" and "bottom" plates of the system include "vertical" components. The magnetic field due to the flowing ions is largest when the ions are closest to the "top" plate, for which the $\mathbf{I} \times \mathbf{B}_{\rm ions}$ force on the cylinder has a "downward" component. And, this "downward" force is larger when the velocity of the ions is larger.

2.2 Downward Electric Forces on Charges Induced on the Conducting Cylinder by the Ions

In addition, when the (positive) ions are in flight, they induce negative charge on the conducting cylinder.

The "downward" momentum imparted to the cylinder by the magnetic and electric forces cancels the "upward kick" associated with collection of the ions at the "top" plate.[7]

2.3 Action and Reaction

During its flight, an ion of (positive) electric charge q and velocity $\mathbf{v}$ experiences the Lorentz force (in Gaussian units),

$$\begin{aligned} \mathbf{F}_{\rm on\ ion} &= q\left(\mathbf{E}_{\rm cyl} + \frac{\mathbf{v}}{c} \times \mathbf{B}_{\rm cyl}\right) = q \int d{\rm Area} \left(\frac{\sigma \mathbf{r}}{r^3} + \frac{\mathbf{v}}{c} \times \frac{\mathbf{K} \times \mathbf{r}}{c r^3}\right) \\ &= q \int d{\rm Area} \left(\frac{\sigma \mathbf{r}}{r^3} + \frac{(\mathbf{v} \cdot \mathbf{r})\mathbf{K} - (\mathbf{v} \cdot \mathbf{K})\mathbf{r}}{c^2 r^3}\right), \end{aligned} \tag{1}$$

where $\mathbf{E}_{\rm cyl}$ is the electric field due to induced surface charge density σ on the walls of the cylinder, and $\mathbf{B}_{\rm cyl}$ is the magnetic field due to the surface current density $\mathbf{K}$ on those walls (which includes the current associated with moving, induced charges), and c is the speed

[5] For a review of this principle, see Appendix A of [2].

[6] If the ion sources were replaced by electron sources, then the final mass distribution in the system would be the same as the initial distribution, and the cylinder would end up at its initial location.

[7] Note that in the absence of the magnetic field due to the current in the central wire, the ions are still deflected by the electric forces, which would be "downward" for sources located "below" the midplane of the cylinder, as in the figure on the previous page. The ions, with a "downward" component to their momenta, would eventually be collected by the conducting cylinder, resulting in a "downward" thrust on the latter. But, the "upward" electric force of the ions in flight on the induced, negative charge, mainly on the "bottom" plate, led to an "upward" motion of the cylinder, that is canceled by the final "downward" thrust of the collected ions.

of light in vacuum. Meanwhile, the induced surface charge density σ on the walls of the cylinder, and the surface current density $\mathbf{K}$ experience force,[8]

$$\begin{aligned}\mathbf{F}_{\rm cyl} &= \int d\text{Area}\left(\sigma(\mathbf{E}_{\rm ion}+\mathbf{E}_{\rm cyl})+\frac{\mathbf{K}}{c}\times(\mathbf{B}_{\rm ion}+\mathbf{B}_{\rm cyl})\right)=\int d\text{Area}\left(\sigma\mathbf{E}_{\rm ion}+\frac{\mathbf{K}}{c}\times\mathbf{B}_{\rm ion}\right)\\ &= -q\int d\text{Area}\left(\frac{\sigma\mathbf{r}}{r^3}+\frac{\mathbf{K}}{c}\times\frac{\mathbf{v}\times\mathbf{r}}{cr^3}\right)=-q\int d\text{Area}\left(\frac{\sigma\mathbf{r}}{r^3}+\frac{(\mathbf{K}\cdot\mathbf{r})\mathbf{v}-(\mathbf{v}\cdot\mathbf{K})\mathbf{r}}{c^2r^3}\right), \quad (2)\end{aligned}$$

where $\mathbf{E}_{\rm ion}$ and $\mathbf{B}_{\rm ion}$ are the electromagnetic fields due to the ion, and we note that the total force on the cylinder due to its own electromagnetic fields is zero.

The total electromagnetic force on the cylinder plus ion is magnetic,

$$\mathbf{F}_{\rm total}=\mathbf{F}_{\rm cyl}+\mathbf{F}_{\rm on\ ion}=q\int d\text{Area}\frac{(\mathbf{v}\cdot\mathbf{r})\mathbf{K}-(\mathbf{K}\cdot\mathbf{r})\mathbf{v}}{c^2r^3}=q\int d\text{Area}\frac{\mathbf{r}\times(\mathbf{K}\times\mathbf{v})}{c^2r^3}. \quad (3)$$

The integrand of eq. (3) is not zero, as it would be if the Lorentz force obeyed Newton's third law. However, the integral is zero, in accordance with the center-of-energy theorem.[9]

Thus, as the ions are deflected "upwards", the cylinder is pulled "downwards", such that the "downwards" momentum of the cylinder is equal and opposite to the "upwards" momentum of the ions just before they collide with the cylinder. The collision brings the momentum of the cylinder back to zero: no "bootstrap spaceships".

2.4 Electromagnetic Field Momentum

There are some subtleties which do not affect the above discussion, but are perhaps of interest.

There exists a nonzero electromagnetic field momentum in this example,

$$\mathbf{P}_{\rm EM}=\int d\text{Vol}\,\frac{\mathbf{E}\times\mathbf{B}}{4\pi c}=\int d\text{Vol}\,\frac{\varrho\mathbf{A}^{(\rm C)}}{c}, \quad (4)$$

where ϱ is the electric charge density and $\mathbf{A}^{(\rm C)}$ is the magnetic vector potential in the Coulomb gauge.[10]

The magnetic field is largely due to the current I in the central wire, driven by the "battery", for which the vector potential is,

$$\mathbf{A}^{(\rm C)}=\frac{2I}{c}\ln\frac{r}{R}\,\hat{\mathbf{z}}. \quad (5)$$

in a cylindrical coordinate system with $\hat{\mathbf{z}}$ "upwards" and the ions sources located at $r = R$, taking the field momentum to be zero when the ions are all still in their source. Then, using the second form of eq. (4), the induced charge on the cylinder $r = R$ does not contribute to the field momentum. The charge induced by an ion on the "top" and "bottom" plates is less in magnitude than the ion charge q, so (when ions are in flight), the total charge with $r < R$ is positive, and the field momentum is "downwards" (in the negative z-direction).

[8]For the low velocities of the present example, we ignore effects of retardation.

[9]See, for example, Appendix A of [2].

[10]The first form of eq. (4) is the basic form of electromagnetic field momentum, and the second form holds for quasistatic examples. See [5].

2.5 "Hidden" Mechanical Momentum

In addition, there exists a "hidden" mechanical momentum [4] in the currents flowing on the surface of the cylinder, associated with the variations in the "relativistic mass" of the conduction electrons in different parts of the conducting cylinder. For quasistatic examples, such as the present case, the "hidden" mechanical momentum and the field momentum are equal and opposite,[11] Therefore, the field momentum and the "hidden" mechanical momentum together have little effect on the "overt" mechanical momentum considered in secs. 2.1-3 above.

In greater detail, the total momentum of the system, which is zero with respect to the lab frame in which system was initially at rest, can be written as,

$$\begin{aligned} \mathbf{P}_{\rm total} &= \mathbf{P}_{\rm mech} + \mathbf{P}_{\rm EM} = \mathbf{P}_{\rm mech,cyl} + \mathbf{P}_{\rm mech,ion} + \mathbf{P}_{\rm EM} \\ &= \mathbf{P}_{\rm mech,cyl,overt} + \mathbf{P}_{\rm mech,cyl,hidden} + \mathbf{P}_{\rm mech,ion} + \mathbf{P}_{\rm EM} = 0. \end{aligned} \tag{6}$$

We also have in the present, quasistatic system,

$$\mathbf{P}_{\rm mech,cyl,hidden} + \mathbf{P}_{\rm EM} = 0, \tag{7}$$

such that eq. (6) implies,

$$\mathbf{P}_{\rm mech,cyl,overt} + \mathbf{P}_{\rm mech,ion} = 0, \tag{8}$$

as considered in sec. 2.3.

The equation of motion of an ion is,

$$\frac{d\mathbf{P}_{\rm mech,ion}}{dt} = \mathbf{F}_{\rm on\ ion}, \tag{9}$$

while the equation of motion of the cylinder is ,

$$\frac{d\mathbf{P}_{\rm mech,cyl}}{dt} = \frac{d\mathbf{P}_{\rm mech,cyl,overt}}{dt} + \frac{d\mathbf{P}_{\rm mech,cyl,hidden}}{dt} = \frac{d\mathbf{P}_{\rm mech,cyl,overt}}{dt} - \frac{d\mathbf{P}_{\rm EM}}{dt} = \mathbf{F}_{\rm cyl}, \tag{10}$$

using the time derivative of eq. (7). Hence,

$$\frac{d\mathbf{P}_{\rm mech,cyl,overt}}{dt} = \mathbf{F}_{\rm cyl} + \frac{d\mathbf{P}_{\rm EM}}{dt}, \tag{11}$$

for what it's worth.

References

[1] W. Engelhardt, *The Lorentz Rocket* (May, 2008), http://kirkmcd.princeton.edu/examples/EM/engelhardt_08.pdf

[2] K.T. McDonald, *No Bootstrap Spaceships* (Aug. 9, 2018), kirkmcd.princeton.edu/examples/bootstrap.pdf

[11] See sec. 4.1.4 of [2].

[3] K.T. McDonald, *Electron Trajectories in a Hall Thruster* (Feb. 27, 2008), kirkmcd.princeton.edu/examples/thrusterp.pdf

[4] K.T. McDonald, *On the Definition of "Hidden" Momentum* (July 9, 2012), kirkmcd.princeton.edu/examples/hiddendef.pdf

[5] K.T. McDonald, *Four Expressions for Electromagnetic Field Momentum* (Apr. 10, 2006), kirkmcd.princeton.edu/examples/pem_forms.pdf