Excitation of a Rectangular Electromagnetic Cavity by a Passing, Relativistic Electron

Kirk T. McDonald
Joseph Henry Laboratories, Princeton University, Princeton, NJ 08544

(June 30, 2011)

1 Problem

Deduce the strength of the lowest mode of a rectangular electromagnetic cavity when excited by a relativistic electron of charge $-e$ and speed $v \approx c$, where c is the speed of light in vacuum. The rectangular cavity has dimensions $d_x \ge d_y \ge d_z$, and the electron moves parallel to the z-axis. The cavity walls may be taken as perfect conductors, and the interior of the cavity is vacuum.

2 Solution

To a good approximation, the fields excited in the cavity by a passing, relativistic charge are independent of the possible prior presence of fields in the cavity. If such fields are present, the electron will gain (or lose) energy as it traverses the cavity, but its trajectory is essentially unchanged if $v \approx c$. Then, the energy gained by the electron is equal and opposite to the change in the electromagnetic field energy of the cavity due to the additional excitation of the cavity [1]. From this, we can deduce the strength of the excitation due to the electron.

Our argument is a kind of reciprocity relation: the energy of excitation of a mode of a cavity by an electron is related to the (maximum) energy that mode, if previously excited, can transfer to the electron.

2.1 E and B Fields of the Cavity Modes

The cavity has extent $0 < x < d_x$, $0 < y < d_y$, and $0 < z < d_z$. The electric field must be everywhere perpendicular to the (perfectly conducting) walls, such that for time dependence $e^{-i\omega t}$ there exists a set of modes with non-negative integer indices $\{l, m, n\}$ of the form,

$$E_x = E_0\, e_x \cos k_x x \sin k_y y \sin k_z z\, e^{-i\omega t}, \tag{1}$$

$$E_y = E_0\, e_y \sin k_x x \cos k_y y \sin k_z z\, e^{-i\omega t}, \tag{2}$$

$$E_z = E_0\, e_z \sin k_x x \sin k_y y \cos k_z z\, e^{-i\omega t}, \tag{3}$$

where $\hat{\mathbf{e}} = (e_x, e_y, e_z)$ is a unit vector, the wave vector $\mathbf{k}$ is given by,

$$\mathbf{k} = (k_x, k_y, k_z) = \pi \left(\frac{l}{d_x}, \frac{m}{d_y}, \frac{n}{d_z} \right), \tag{4}$$

and at most only one of indices l, m, or n is zero (see, for example, [2]). These fields obey the free-space wave equation,

$$\nabla^2 \mathbf{E} = \frac{1}{c^2} \frac{\partial^2 \mathbf{E}}{\partial t^2} = -\frac{\omega^2}{c^2} \mathbf{E}, \tag{5}$$

where c is the speed of light in vacuum, which implies that,

$$\omega = kc = \pi c \sqrt{\frac{l^2}{d_x^2} + \frac{m^2}{d_y^2} + \frac{n^2}{d_z^2}}. \tag{6}$$

The first (free-space) Maxwell equation, $\boldsymbol{\nabla} \cdot \mathbf{E} = 0$ implies that $\hat{\mathbf{e}} \cdot \mathbf{k} = 0$, so that there are two orthogonal "polarizations" $\hat{\mathbf{e}}$ for each set of indices $\{l, m, n\}$.[1]

The magnetic field is related to the electric field by Faraday's law (in Gaussian units),

$$\boldsymbol{\nabla} \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial ct} = ik\mathbf{B}, \tag{8}$$

such that,

$$B_x = iE_0\, b_x \sin k_x x \cos k_y y \cos k_z z\, e^{-i\omega t}, \tag{9}$$

$$B_y = iE_0\, b_y \cos k_x x \sin k_y y \cos k_z z\, e^{-i\omega t}, \tag{10}$$

$$B_z = iE_0\, b_z \cos k_x x \cos k_y y \sin k_z z\, e^{-i\omega t}, \tag{11}$$

where $\hat{\mathbf{b}}$ is the unit vector,

$$\hat{\mathbf{b}} = \hat{\mathbf{e}} \times \hat{\mathbf{k}} = \frac{1}{k}(e_y k_z - e_z k_y, e_z k_x - e_x k_z, e_x k_y - e_y k_x). \tag{12}$$

The magnetic field is everywhere tangential to the walls of the cavity (which motivated the use of the cosine functions in the electric field (1)-(3)). Thus, $\hat{\mathbf{b}} \cdot \mathbf{k} \propto \det(\hat{\mathbf{e}}, \mathbf{k}, \mathbf{k}) = 0$, consistent with the third Maxwell equation, $\boldsymbol{\nabla} \cdot \mathbf{B} = 0$. Also, $\hat{\mathbf{e}} \cdot \hat{\mathbf{b}} \propto \det(\hat{\mathbf{e}}, \hat{\mathbf{e}}, \mathbf{k}) = 0$, such that for each mode the vectors $\mathbf{E}$, $\mathbf{B}$ and $\mathbf{k}$ form a mutually orthogonal triad, with,

$$\hat{\mathbf{e}} = \hat{\mathbf{b}} \times \hat{\mathbf{k}} = \frac{1}{k}(b_y k_z - b_z k_y, b_z k_x - b_x k_z, b_x k_y - b_y k_x). \tag{13}$$

The lowest cavity mode (often called TM_{110}) has indices $(l, m, n) = (1, 1, 0)$, wave vector,

$$\mathbf{k} = \pi \left(\frac{1}{d_x}, \frac{1}{d_y}, 0 \right), \qquad k = \frac{\omega}{c} = \frac{\pi \sqrt{d_x^2 + d_y^2}}{d_x d_y}, \tag{14}$$

[1] The electric field can be regarded as the superposition of eight plane waves,

$$\begin{aligned} \mathbf{E} = -\frac{E_0}{8} \Big[& (e_x, e_y, e_z)\, e^{i(k_x x + k_y y + k_z z - \omega t)} + (-e_x, e_y, e_z)\, e^{i(-k_x x + k_y y + k_z z - \omega t)} \\ & -(e_x, -e_y, e_z)\, e^{i(k_x x - k_y y + k_z z - \omega t)} + (-e_x, -e_y, e_z)\, e^{i(-k_x x - k_y y + k_z z - \omega t)} \\ & -(e_x, e_y, -e_z)\, e^{i(k_x x + k_y y - k_z z - \omega t)} - (-e_x, e_y, -e_z)\, e^{i(-k_x x + k_y y - k_z z - \omega t)} \\ & +(e_x, -e_y, -e_z)\, e^{i(k_x x - k_y y - k_z z - \omega t)} + (-e_x, -e_y, -e_z)\, e^{i(-k_x x - k_y y - k_z z - \omega t)} \Big], \end{aligned} \tag{7}$$

with a similar relation holding for the magnetic field.

electric polarization unit vector $\mathbf{e} = (0, 0, 1)$, magnetic polarization unit vector,

$$\mathbf{b} = \frac{\pi}{k}\left(\frac{1}{d_y}, -\frac{1}{d_x}, 0\right). \tag{15}$$

The electric field of the lowest mode is,

$$E_x = E_y = 0, \qquad E_z = E_0 \sin k_x x \sin k_y y \, e^{-i\omega t}, \tag{16}$$

and the stored electromagnetic energy is,

$$U = \int \frac{|E|^2 + |B|^2}{16\pi} \, d\mathrm{Vol} = \int \frac{|E|^2}{8\pi} \, d\mathrm{Vol} = \frac{E_0^2 d_x d_y d_x}{32\pi}. \tag{17}$$

2.2 Maximum Energy Gain by a Passing, Relativistic Electron

We suppose that the electron moves parallel to the z-axis and reaches the midplane of the cavity $z = d_z/2$, when the electric field is minimal, $E(x, y) = -E_0 \sin k_x x \sin k_y y$ at the x-y position of the electron. The transit time of the relativistic electron is $\Delta t = d_z/c$, so the maximum energy gain of the electron is,

$$\begin{aligned}
\Delta U &= -eE(x, y) \int_{-d_z/2c}^{d_z/2c} \cos \omega t \, c dt = \frac{2ceE_0}{\omega} \sin k_x x \sin k_y y \sin \frac{\omega d_z}{2c} \\
&= \frac{2eE_0}{k} \sin k_x x \sin k_y y \sin \frac{k d_z}{2}.
\end{aligned} \tag{18}$$

2.3 Excitation of the Lowest Mode by the Electron

As the electron passes through the cavity it leads to time-dependent charges and currents on the interior surface of the cavity, which generates additional electromagnetic fields (sometimes called transition radiation or wakefields). These fields can be decomposed as a sum of the modes of the cavity, including the lowest mode. Of course, the total field inside the cavity is the sum of the excited modes and the initial field in the lowest mode. The total energy in the cavity is the sum of the electromagnetic energies in the various (orthogonal) modes.

We assume that the initial field in the lowest mode is much larger than the strengths of the various excitations by the electron. In this approximation, the total field energy in the cavity is just the initial energy plus the "interference" energy between the initial lowest mode and the further excitation of this mode by the electron. Then, the energy (18) gained (18) by the electron must be equal and opposite to the interference energy.

Denoting the electric field of the lowest mode as excited by the electron by,

$$\Delta U = -eE(x, y) \int_{-d_z/2c}^{d_z/2c} \cos \omega t \, dt = E_{\mathrm{ex}} \sin k_x x \sin k_y y \, e^{-i\omega t}, \tag{19}$$

where E_x is a complex number, then the interference energy is,

$$U_{\mathrm{int}} = \int \frac{E_0 Re(E_{\mathrm{ex}}) \sin^2 k_x x \sin^2 k_y y}{4\pi} \, d\mathrm{Vol} = \frac{E_0 Re(E_{\mathrm{ex}}) \, d_x d_y d_z}{16\pi}. \tag{20}$$

Equating this to the negative of eq. (18), we have that,

$$Re(E_{\rm ex}) = -\frac{32\pi e}{kd_x d_y d_z} \sin k_x x \sin k_y y \sin \frac{kd_z}{2} = -\frac{32e}{\sqrt{d_x^2 + d_y^2} d_z} \sin k_x x \sin k_y y \sin \frac{kd_z}{2}\,. \qquad (21)$$

This result is independent of the initial field E_0, and must hold even if $E_0 = 0$.

In principle, the imaginary part of $E_{\rm ex}$ is not determined, but the spirit of the present argument is that there is no physical role for the imaginary part, so I assume it is negligible. That is, the magnitude of the excitation of the lowest mode by the passing electron is,

$$E_{\rm ex} = \frac{32\pi e}{kd_x d_y d_z} \left| \sin k_x x \sin k_y y \sin \frac{kd_z}{2} \right| = \frac{32e}{\sqrt{d_x^2 + d_y^2}\, d_z} \left| \sin k_x x \sin k_y y \sin \frac{kd_z}{2} \right|. \qquad (22)$$

This problem has been considered by a very different approach in sec. 14.2 of [3],[2] for the particular case that $d_x = d_y \equiv a = \lambda/\sqrt{2}$ and $d_z \equiv b = \lambda/2$, with $x = y = a/2\sqrt{2}$. For this case, eq. (22) gives $E_{\rm ex} = 64e/\lambda^2$, and the energy of this excitation is (recalling eq. (17)) $U_{\rm ex} = 32e^2/\pi\lambda$, in agreement with eq. (14.76) of [3].

As discussed in [1], the excitation of other low modes of the cavity by the electron results in fields of similar order to that of eq. (22). If we are interested in the excitation of a particular mode, we can first suppose that the cavity has an initial field in this mode only, calculate the energy gain by the passing electron, and deduce the excitation of this mode by equating the final interference energy to the negative of the energy gain. We see that if a mode does not result in an energy gain by a passing electron, there will be no excitation of this mode by the passing electron.

For modes whose force on the electron is only transverse to its trajectory, we can calculate the impulse $\Delta P_\perp$ of transverse momentum imparted to the electron of initial total energy U_0, and deduce the (small) change in the electron's energy as $\Delta U \approx c^2 \Delta P_\perp^2/2U_0$. This contrasts with the case of an impulse $\Delta P_\parallel$ along the electron's velocity, for which the energy gain is $\Delta U \approx c\Delta P_\parallel$. Since $\Delta P \propto E_0$, the strength of the excitation of a mode that only gives a transverse "kicks" to a passing electron varies as the initial strength E_0 of that mode, such that if this mode is not initially present it will not be excited by the electron. It remains that a mode such as that with indices $(l,m,n) = (1,1,1)$ that gives both longitudinal and transverse "kicks" to a passing electron will be excited even if not initially present, and could then impart transverse "kicks" to subsequent electrons.[3]

[2]The calculation of [3] proceeds via use of the vector potential for the cavity in the Lorenz gauge. While I believe the result of their argument to be correct, the vector potential used there obeys the Lorenz-gauge condition, but is not the usual Lorenz-gauge retarded potential. For further discussion, see sec. 2.2.3 of [2].

[3]I believe that for the $(1,1,1)$ mode eq. (22) would be modified by replacing the factor $\sin kd_z/2$ with,

$$\frac{k^2}{2(k^2 - k_z^2)} \sin \frac{kd_z}{2} (1 + \cos k_z d_z) - \frac{kk_z}{2(k^2 - k_z^2)} \cos \frac{kd_z}{2} \sin k_z d_z. \qquad (23)$$

For the particular example of sec. 14.2 of [3], the excitation of the (1,1,1) mode is 0.86 times that of the (1,1,0) mode.

2.4 Excitation While the Charge Is Inside the Cavity

The analysis of secs. 2.2-3 holds only after the charge has exited the cavity. If we suppose the charge enters the cavity at time $t = 0$, then the energy it has gained by time t is,

$$\begin{aligned}\Delta U(t) &= -eE(x,y)\int_0^t \cos\omega(t - d_z/2c)\,cdt = \frac{ceE_0}{\omega}\sin k_x x \sin k_y y\left(\sin\omega(t - d_z/2c) + \sin\frac{kd_z}{2}\right)\\ &= \frac{eE_0}{k}\sin k_x x \sin k_y y\left(\sin\omega t\cos\frac{kd_z}{2} + (1+\cos\omega t)\sin\frac{kd_z}{2}\right). \end{aligned} \tag{24}$$

This energy gain is balanced by a reduction of the field energy in the cavity. We could proceed as in sec. 2.3 and identify the strength of the mode as that for which the interference energy is the negative of eq. (24),

$$E_{\rm ex}(t) \approx \frac{16e}{\sqrt{d_x^2 + d_y^2}\,d_z}\sin k_x x \sin k_y y\left(\sin\omega t\cos\frac{kd_z}{2} + (1-\cos\omega t)\sin\frac{kd_z}{2}\right). \tag{25}$$

This result depends on the tacit assumption that even when the charge is still inside the cavity the electromagnetic fields can be represented as a sum of orthogonal modes. The contribution to the fields from the charge occupy only a sphere of radius ct about the entry point of the charge (which entered at time $t = 0$). To represent this sphere of fields by a sum of modes requires contributions of modes of arbitrarily high frequency.

However, metal cavities are transparent to waves of frequency above the plasma frequency, which is in the far UV (about 16 eV for copper, I believe). This high-frequency cutoff implies that the result (25) is only an approximation.

2.5 High-Frequency Cutoff

The procedure of secs. 2.2-3 can be applied to any mode, and if that mode can transfer energy to a passing electron, the electron will excite that mode. Since the number of modes is infinite, this suggests that a single electron might transfer an infinite energy to the cavity, as claimed in sec. 3.3 of [11]. However, an electron cannot transfer more than its kinetic energy to the cavity, so there must be a cutoff for higher-order (higher-frequency) modes.

To get a sense of this cutoff, it is useful to consider how the electron transfers energy to the cavity, which is via its transition radiation, *i.e.*, the radiation associated with the time-dependent surface charge distribution induced on the cavity walls when the electron is inside the cavity. Useful insight into transition radiation (and other radiation processes of relativistic charges) is obtained in the Weizsäcker-Williams approximation, which can be summarized [4] as: α photons are radiated per formation length $L_0(\omega)$ per unit bandwidth, up to a critical (angular) frequency ω_C,

$$\frac{dn_\omega}{dl} \approx \frac{\alpha}{L_0(\omega)}\frac{d\omega}{\omega} \times \begin{cases} 1 & (\omega < \omega_C), \\ e^{-\omega/\omega_C} & (\omega \geq \omega_C), \end{cases} \tag{26}$$

where $\alpha \approx 1/137$ is the fine-structure constant. The critical frequency arises because there will always be some minimum impact parameter, $b_{\rm min}$ between the passing charge and the medium that it perturbs, below which radiation is suppressed. For a charge with Lorentz factor $\gamma = 1/\sqrt{1 - v^2/c^2}$, the critical frequency is given by,

$$\omega_C \approx \gamma \frac{c}{b_{\min}}, \qquad b_{\min} \approx \gamma \lambda_C, \tag{27}$$

which reflects that the transverse scale of the electric field of a relativistic charge varies with γ.[4] The formation length is the distance over which a photon moves ahead of the charge by one wavelength, and so depends on the angle and frequency of the photon,[5]

$$L_0 \approx \frac{2\lambda}{\theta^2 + 1/\gamma^2} \qquad L_0 \approx \gamma^2 \lambda \qquad\qquad (\lambda \approx \lambda_C). \tag{28}$$

The minimum relevant transverse scale, $b_{\min}$, for transition radiation is the plasma wavelength $\lambda\!\!\!^{-}_p = c/\omega_p$, so the critical frequency is $\omega_C \approx \gamma\omega_p$, according to eq. (27). This is well into the x-ray regime ($\hbar\omega_p \approx 16$ eV for copper).[6] Hence, the total energy $U_{\rm rad}$ of transition radiation by a passing charge in a cavity of length $L > L_0$ is,

$$U_{\rm rad} \approx \int_0^{\omega_C} \hbar\omega \frac{2\alpha\, d\omega}{\omega} = 2\alpha\hbar\omega_C = 2\gamma\alpha\hbar\omega_p \quad (\approx 0.24\gamma \text{ eV for copper}), \tag{29}$$

adding the energies of the transition radiation at the entrance and exit surfaces of the cavity.

A Appendix: Right Circular Cylinder Cavity

The modes of a right circular cylinder cavity may be characterized as transverse electric (TE) or transverse magnetic (TM) as described, for example, in sec. 8.7 of [7]. The field patterns of a few of the lowest modes are sketched below (from [8]).

The lowest mode, TM$_{010}$, for a cavity of radius R, with the z-axis being that of the cavity, has fields in cylindrical coordinates (r, ϕ, z),

$$\begin{aligned} E_z &= E_0\, J_0(kr)\, e^{-i\omega t}, & (30)\\ B_\phi &= -iE_0\, J_1(kr)\, e^{-i\omega t}, & (31)\end{aligned}$$

where the resonant frequency is,

$$\omega = kc = \frac{2.405c}{R}\,, \tag{32}$$

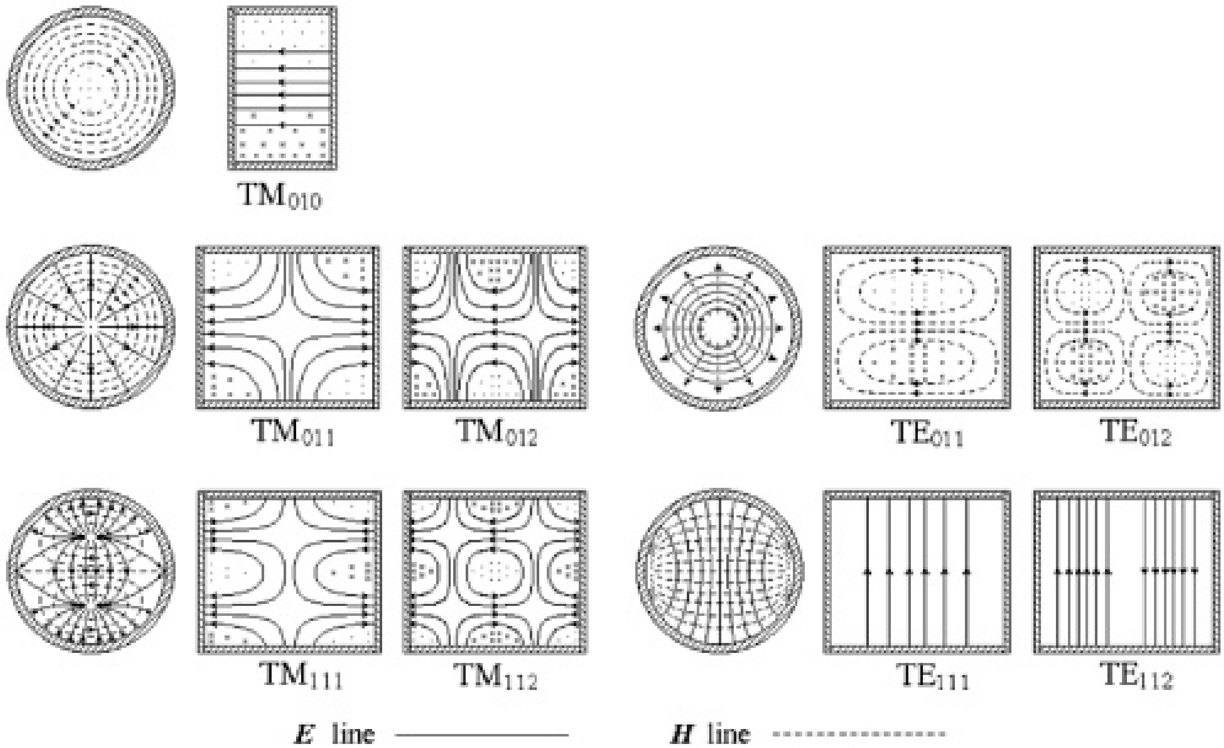
such that $J_0(kR) = 0$ so the tangential electric field is zero at $r = R$, and c is the speed of light in vacuum. The magnetic field is related to the electric field by Faraday's law,

$$\boldsymbol{\nabla} \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial ct} = ik\mathbf{B}. \tag{33}$$

[4]See, for example, sec. 1 of [4].

[5]See, for example, sec. 1.1 of [4].

[6]While the characteristic angle of transition radiation is $1/\gamma$, there is only a power-law falloff at larger angles, and the optical transition radiation from an intense beam of charged particles can be used to measure the spot size to accuracy of a few optical λ [5, 6].



The energy stored in the TM_{010} mode of a cavity of axial length L is,

$$\begin{aligned} U &= \int \frac{|E|^2 + |B|^2}{16\pi}\, d\mathrm{Vol} = \int \frac{|E|^2}{8\pi}\, d\mathrm{Vol} = \frac{E_0^2 L}{4} \int_0^R J_0^2(kr)\, r dr = \frac{E_0^2 L R^2}{4} \int_0^1 J_0^2(kRx)\, x dx \\ &= \frac{E_0^2 L R^2 J_1^2(kR)}{8}\,, \end{aligned} \tag{34}$$

using 6.521.1 of [9].[7]

An electron that passes through the cavity at radius r at speed $v \approx c$ gains a maximum energy,

$$\begin{aligned} \Delta U &= -eE(r) \int_{-L/2c}^{L/2c} \cos \omega t\, c dt = \frac{2ceE_0 J_0(kr)}{\omega} \sin \frac{\omega L}{2c} \\ &= \frac{2eE_0 J_0(kr)}{k} \sin \frac{kL}{2}\,, \end{aligned} \tag{36}$$

recalling the argument of sec. 2.2

[7]The time-average energy stored in the magnetic field equals that stored in the electric field, so we infer that, for $J_0(kR) = 0$,

$$\int_0^R J_0^2(kr)\, r dr = \int_0^R J_1^2(kr)\, r dr. \tag{35}$$

The interference energy associated with the additional excitation of the TM_{010} mode is, following sec. 2.3 and using eq. (34),

$$U_{\rm int} = \frac{E_0 E_{\rm ex} L R^2 J_1^2(kR)}{4}, \tag{37}$$

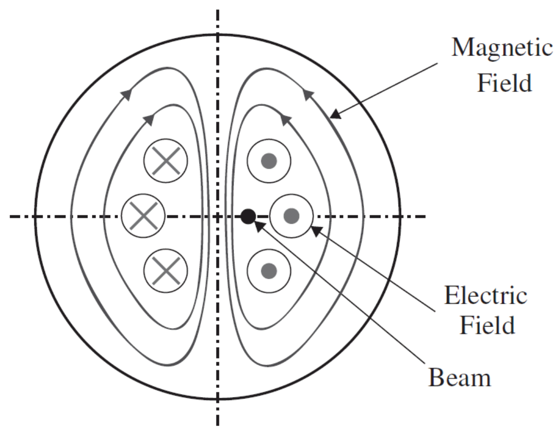
Equating this to the eq. (36), we have that the magnitude of the excited electric field is,

$$E_{\rm ex} = \frac{8eJ_0(kr)}{(kR)LRJ_1^2(kR)}\left|\sin\frac{kL}{2}\right| = \frac{12.3eJ_0(kr)}{LR}\left|\sin\frac{kL}{2}\right|, \tag{38}$$

noting that $J_1(2.405) = 0.519$.

The excitation of cylindrical cavities has been extensively considered for particle accelerators. See, for example, [10, 11], which uses a Green-function method due to Condon [12] (that appears to be very similar to the method of Schwinger [3]). Equation (38) is, I believe, the same as eq. (A8) (in SI units) of [10] with $p = 0$, $g = L$, $r = 0$ and $ct > g$; i.e., for times after the charge has exited the cavity.

Modes that can deflect electrons transversely, such as TM_{110} shown below, give no net axial acceleration to an axially symmetric beam of electrons (increasing the energy of electrons at some azimuths while decreasing the energy at others). Hence, the excitation of these modes by such a beam is negligible.



If a bunch of n electrons passes through the cavity such that all lengths scales of the bunch are small compared to the wavelength of a mode, then the excitations of the various electrons add coherently. As an example (taken from the accelerating cavities of a so-called neutrino factory [13]), we consider a bunch of $n = 10^{12}$ muons passing through a right circular cavity with $L = 30$ cm, $R = 15$ cm, for which the excitation of the fundamental mode, with $kR = kL/2 = 2.405$, has $E_{\rm ex} \approx 5.4 \times 10^5$ V/m, noting that in SI units eq. (38) includes a factor $1/4\pi\epsilon_0 \approx 9 \times 10^{-11}$.

References

[1] M.S. Zolotorev, S. Chattopadhyay and K.T. McDonald, *A Maxwellian Perspective on Particle Acceleration*, (Sept. 7, 1999), http://kirkmcd.princeton.edu/examples/vacuumaccel.pdf

[2] K.T. McDonald, *Potentials for a Rectangular Electromagnetic Cavity*, (Mar. 4, 2011), http://kirkmcd.princeton.edu/examples/cavity.pdf

[3] K.A. Milton and J. Schwinger, *Electromagnetic Radiation: Variational Methods, Waveguides and Accelerators* (Springer, 2006), http://kirkmcd.princeton.edu/examples/EM/milton_em.pdf

[4] M.S. Zolotorev and K.T. McDonald, *Classical Radiation Processes in the Weizsäcker-Williams Approximation*, (Aug. 25, 1999), http://kirkmcd.princeton.edu/examples/weizsacker.pdf

[5] X. Artru *et al.*, *Experimental investigations on geometrical resolution of optical transition radiation (OTR)*, Nucl. Instr. and Meth. A **410**, 148 (1998), http://kirkmcd.princeton.edu/examples/accel/artru_nim_a410_148_98.pdf *Resolution power of optical transition radiation: Theoretical considerations*, *ibid.* B **145**, 160 (1998), http://kirkmcd.princeton.edu/examples/accel/artru_nim_b145_160_98.pdf

[6] P. Catravas *et al.*, *Beam Profile Measurement at 30 GeV Using Optical Transition Radiation*, Proc. PAC'99, WEA100, http://accelconf.web.cern.ch/accelconf/p99/PAPERS/WEA100.PDF http://kirkmcd.princeton.edu/examples/accel/catravas_pac99_2111.pdf

[7] J.D. Jackson, *Classical Electrodynamics*, 3$^{\rm rd}$ ed. (Wiley, 1999), http://kirkmcd.princeton.edu/examples/EM/jackson_ce3_99.pdf

[8] K. Zhang and D. Li, *Electromagnetic Theory for Microwaves and Optoelectronics* (Springer, 2007).

[9] I.S. Gradshteyn and I.M. Ryzhik, *Table of Integrals, Series, and Products*, 5$^{\rm th}$ ed. (Academic Press, 1994), http://kirkmcd.princeton.edu/examples/EM/gradshteyn_80.pdf

[10] T. Weiland and B Zotter, *Wake Potenitals of a Relativistic Current in a Cavity*, Part. Accel. **11**, 143 (1981), http://kirkmcd.princeton.edu/examples/accel/weiland_pa_11_143_81.pdf

[11] K.L.F. Bane, P.B. Wilson and T. Weiland, *Wake Fields and Wake Field Acceleration*, AIP Conf. Proc. **127**, 875 (1985), http://kirkmcd.princeton.edu/examples/accel/bane_aipcp_127_875_85.pdf

[12] E.U. Condon, *Electronic Generation of Electromagnetic Oscillations*, J. Appl. Phys. **11**, 502 (1940), http://kirkmcd.princeton.edu/examples/EM/condon_jap_11_502_40.pdf
Forced Oscillations in Cavity Resonators, J. Appl. Phys. **12**, 129 (1941), http://kirkmcd.princeton.edu/examples/EM/condon_jap_12_129_41.pdf

[13] M. Apollonio *et al.*, *Accelerator design concept for future neutrino facilities*, JINST **14**, p07001 (2009), http://kirkmcd.princeton.edu/examples/accel/apollonio_jinst_4_p07001_09.pdf