Feynman Cylinder Paradox

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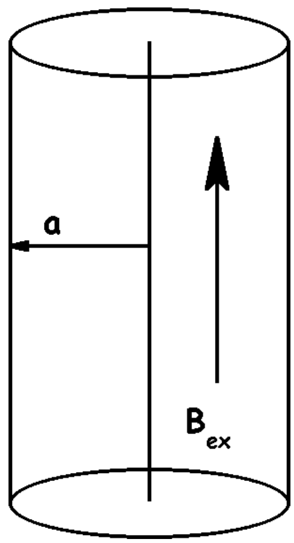
1 Problem

An infinitely long wire with linear charge density $-\lambda$ lies along the z axis. An insulating cylindrical shell of radius a and moment of inertia I per unit length is concentric with the wire, and can rotate freely about the z axis. The areal charge density on the cylinder is $\sigma = \lambda/2\pi a$ and is uniformly distributed.



The cylinder is immersed in an external magnetic field $B_{\rm ex}\hat{\mathbf{z}}$, and is initially at rest.

Starting at $t = 0$ the external magnetic field is slowly reduced to zero over a time $T \gg a/c$, where c is the speed of light. What is the final angular velocity ω of the cylinder?

2 Solution

This problem is a version of the Feynman disk paradox [1]-[25] that is particularly easy to analyze.[1] However, it avoids a subtle point related to the return flux of the external magnetic field, as discussed in sec. 2.4. This problem is based on earlier discussions by McKenna [26, 27] and by Romer [2].

[1]The cylinder paradox had its origins in a discussion by J.J. Thomson, p. 348 of [23], the paper that introduced the concept of electromagnetic field angular momentum. See also [24].

2.1 Solution Via Conservation of Angular Momentum

The initial angular momentum $\mathbf{L}_i$ (per unit length) of the system is entirely due to the electromagnetic field,

$$\mathbf{L}_{i,\mathrm{field}} = \int \mathbf{r} \times \mathbf{p} \, d\mathrm{Area} = 2\pi \int_0^\infty \mathbf{r} \times \frac{\mathbf{E} \times \mathbf{B}}{4\pi c} r dr, \tag{1}$$

recalling that the field momentum density is the Poynting vector $\mathbf{S} = c\mathbf{E} \times \mathbf{B}/4\pi$ (in Gaussian units) divided by c^2, where c is the speed of light.

In the present problem, an electric field exists only for $r < a$, since the charge density σ on the cylinder has been chosen to cancel the field from the charged wire for $r > a$. From Gauss' law we obtain,

$$\mathbf{E} = -\frac{2\lambda}{r} \hat{\mathbf{r}} \qquad (r < a), \tag{2}$$

and hence the field momentum density in a cylindrical coordinate system (r, ϕ, z) is,

$$\mathbf{p} = \frac{\lambda B_{\mathrm{ex}}}{2\pi c r} \hat{\boldsymbol{\phi}} \qquad (r < a). \tag{3}$$

The initial angular momentum is therefore[2]

$$\mathbf{L}_{i,\mathrm{field}} = 2\pi \frac{\lambda B_{\mathrm{ex}}}{2\pi c} \int_0^a \frac{\mathbf{r} \times \hat{\boldsymbol{\phi}}}{r} r dr = \frac{\lambda a^2 B_{\mathrm{ex}}}{2c} \hat{\mathbf{z}}. \tag{5}$$

The angular momentum when the external magnetic field is zero is due to the rotation of the cylinder at angular velocity ω. There is now the mechanical angular momentum $I\omega$ as well as the field angular momentum due to the solenoidal magnetic field inside the rotating, charged cylinder. The azimuthal current (per unit length) is,

$$J_\phi = \frac{Q}{T} = 2\pi a \sigma \frac{\omega}{2\pi} = \frac{\lambda \omega}{2\pi} \, . \tag{6}$$

The resulting final magnetic field is along the z axis, with strength,

$$\mathbf{B}_f = \frac{4\pi J_\phi}{c} \hat{\mathbf{z}} = \frac{2\lambda\omega}{c} \hat{\mathbf{z}} \qquad (r < a), \tag{7}$$

independent of radius for $r < a$ according to Ampere's law. Since this field is in the same sense as the original field, we can immediately use eq. (5) to find the final field angular momentum:

$$\mathbf{L}_{f,\mathrm{field}} = \frac{\lambda a^2 B_f}{2c} \hat{\mathbf{z}} = \frac{\lambda^2 a^2 \omega}{c^2} \hat{\mathbf{z}}. \tag{8}$$

[2]As discussed, for example, in [28], the field angular momentum in quasistatic examples can also be computed via,

$$\mathbf{L}_{i,\mathrm{field}} = \int \mathbf{r} \times \frac{\rho \mathbf{A}^{(\mathrm{C})}}{c} \, d\mathrm{Vol} = a \, \hat{\mathbf{r}} \times \lambda \frac{a B_{\mathrm{ex}} \hat{\boldsymbol{\phi}}}{2c} = \frac{\lambda a^2 B_{\mathrm{ex}}}{2c} \hat{\mathbf{z}}, \tag{4}$$

noting that the vector potential of the external magnetic field is $\mathbf{A}^{(\mathrm{C})} = r B_{\mathrm{ex}} \, \hat{\boldsymbol{\phi}}/2$.

The total angular momentum in the final state is therefore,

$$\mathbf{L}_f = \mathbf{L}_{f,\mathrm{mechanical}} + \mathbf{L}_{f,\mathrm{field}} = \left(I + \frac{\lambda^2 a^2}{c^2} \right) \omega \hat{\mathbf{z}}. \tag{9}$$

Since there is no frictional torque in this problem (and we ignore radiation), angular momentum is conserved. Hence,

$$\omega = \frac{\lambda a^2 B_{\mathrm{ex}}}{2cI(1 + \lambda^2 a^2/c^2 I)} \approx \frac{\lambda a^2 B_{\mathrm{ex}}}{2cI} \left(1 - \frac{\lambda^2 a^2}{c^2 I} \right). \tag{10}$$

The presence of c^2 in the denominator of the last term of eq. (10) indicates the presence of relativistic effects in this problem.

2.2 Solution Via Faraday's Law

As the magnetic field drops, its time derivative $\dot{\mathbf{B}}$ results in an induced electric field in the azimuthal direction. According to Faraday's law, we have,

$$E_\phi(r) = -\frac{r\dot{B}_z}{2c}. \tag{11}$$

This field acts on the charged cylindrical shell to produce an azimuthal torque (per unit length) of,

$$N_\phi = aE_\phi(a) 2\pi a\sigma = -\frac{\lambda a^2 \dot{B}_z}{2c} = \frac{dL_{\mathrm{mechanical}}}{dt} = I\frac{d\omega}{dt}. \tag{12}$$

We integrate to find the final angular velocity,

$$\omega = \int_0^\infty \frac{d\omega}{dt} dt = -\frac{\lambda a^2}{2cI} \int_0^\infty \dot{B}_z dt = \frac{\lambda a^2 (B_{\mathrm{ex}} - B_f)}{2cI}, \tag{13}$$

Again, we must note that the final magnetic field is not zero, but is given by eq. (7). With this, eq. (13) becomes,

$$\omega = \frac{\lambda a^2 (B_{\mathrm{ex}} - 2\lambda\omega/c)}{2cI}, \tag{14}$$

which again leads to eq. (10).

2.3 Another Relativistic Correction

This section was written April, 2002.

In addition to the above accounting of angular momentum, there is a small amount of initial angular momentum associated with the motion of the conduction current that produces the field B_{ex}. Furthermore, in the final state the cylinder of radius a is rotating angular velocity ω, so its moment of inertia increases by the factor $\gamma = 1/\sqrt{1 - a^2\omega^2/c^2}$ due to the relativistic increase of mass.[3]

[3]The related issue of the relation between mechanical kinetic energy of electrical currents and magnetic field energy is considered in [29].

To characterize the initial mechanical angular momentum, we suppose the magnetic field $\mathbf{B}_{\rm ex}$ is produced by a long cylinder of radius $b > a$, which must therefore carry azimuthal current (per unit length along the z axis),

$$I_{\rm ex} = \frac{c}{4\pi} B_{\rm ex}. \tag{15}$$

This current is due to an areal number density n_e of conduction electrons that we take to have velocity v_e. Then, the current $I_{\rm ex}$ is also related by,

$$I_{\rm ex} = -e n_e v_e, \tag{16}$$

writing $e > 0$ as the magnitude of the charge of the electron. Hence,

$$n_e v_e = -\frac{c}{4\pi} \frac{B_{\rm ex}}{e}. \tag{17}$$

The initial mechanical angular momentum (per unit length) associated with conduction electrons is,

$$\mathbf{L}_{i,\rm mech} = 2\pi b n_e \gamma_e m_e v_e b\, \hat{\mathbf{z}} = -\gamma_e \frac{m_e c}{2e} b^2 B_{\rm ex}\, \hat{\mathbf{z}}, \tag{18}$$

where the total number of conduction electrons per unit length is $2\pi b n_e$, m_e is the rest mass of the electron, and $\gamma_e = 1/\sqrt{1 - v_e^2/c^2} \approx 1$. Combining this with eq. (5), the total initial angular momentum is,

$$\mathbf{L}_i = \frac{\lambda a^2 B_{\rm ex}}{2c} \left(1 - \gamma_e \frac{m_e c^2}{\lambda e} \frac{b^2}{a^2}\right) \hat{\mathbf{z}} = \frac{\lambda a^2 B_{\rm ex}}{2c} \left(1 - \gamma_e \frac{e}{\lambda r_e} \frac{b^2}{a^2}\right) \hat{\mathbf{z}}, \tag{19}$$

where $r_e = e^2/m_e c^2$ is the classical electron radius. The last term in eq. (19) is not necessarily small, since e/r_e corresponds to $\approx 10^{13}$ electrons/cm.

Reviewing the argument of sec. 2.2, we see that in eq. (12) the derivative $d\omega/dt$ should really be $d\gamma\omega/dt$, with the moment of inertia I being calculated using the rest mass of the cylinder. However, eq. (7) for the final magnetic field remains the same, so eq. (14) becomes,

$$\gamma\omega = \frac{\lambda a^2 (B_{\rm ex} - 2\lambda\omega/c)}{2cI}, \tag{20}$$

Expanding γ as approximately $1 + a^2\omega^2/2c^2$, we find,

$$\omega \approx \frac{\lambda a^2 B_{\rm ex}}{2cI} \left(1 - \frac{\lambda^2 a^2}{c^2 I} - \frac{\lambda^2 a^6 B_{\rm ex}^2}{2c^4 I^2}\right). \tag{21}$$

2.4 A Subtle Point

This section was updated May 2015.

This example, and near equivalents [2, 7, 26], are crafted so as to avoid a complication associated with the return flux of the magnetic field.

To see the difficulty, suppose instead that the linear charge density on the central wire were λ_0, while that on the cylinder of radius a is still called λ. Then, according to eq. (1), the initial field angular momentum would be,

$$\mathbf{L}_{i,\mathrm{field}} = -\frac{B_{\mathrm{ex}}}{2c}[\lambda_0 b^2 + \lambda(b^2 - a^2)]\hat{\mathbf{z}} = \frac{B_{\mathrm{ex}}}{2c}[\lambda a^2 - (\lambda + \lambda_0)b^2]\hat{\mathbf{z}}. \tag{22}$$

where b is the radius of the solenoid that provides the external field. Here, we make the usual (but as we will see, unwarranted) assumption that the field of a long solenoid is essentially zero outside the solenoid.

The final magnetic field is still given by eq. (7), so the final field angular momentum would be,

$$\mathbf{L}_{f,\mathrm{field}} = \frac{\lambda\omega}{c^2}[\lambda a^2 - (\lambda + \lambda_0)b^2]\hat{\mathbf{z}}. \tag{23}$$

The total final angular momentum would be,

$$\mathbf{L}_f = \left(I + \frac{\lambda}{c^2}[\lambda a^2 - (\lambda + \lambda_0)b^2]\right)\omega\hat{\mathbf{z}}. \tag{24}$$

Equating (22) and (24) the final angular velocity would be,

$$\omega = \frac{B_{\mathrm{ex}}[\lambda a^2 - (\lambda + \lambda_0)b^2]}{2c\{I + [\lambda a^2 - (\lambda + \lambda_0)b^2]/c^2\}}. \tag{25}$$

However, the argument in sec. 2.2 based on Faraday's law is exactly the same as before, which again implies that the final angular velocity is given by eq. (10).

The argument based on Faraday's law seems the more robust, so I conclude that eq. (10) is correct for any value of the charge density λ_0 on the central wire.

The field angular momentum calculations must be in error.

Real solenoids have only finite length, and the magnetic field is not quite zero outside the solenoid since all of the magnetic flux inside the solenoid must be returned on paths outside the solenoid. As discussed in [28], computations of field angular momentum associated with long solenoids are more reliably made using eq. (4) than eq. (1). We see that the form (4) predicts a field angular momentum that is independent of the charge density λ_0 along the axis, which restores agreement with the argument based on Faraday's law.[4]

2.5 Do the Electric and Magnetic Field Lines Rotate When the Cylinder Rotates?

This section is based on a query by Michael Romalis, May 20, 2015.

Suppose the charged cylinder were rotating with angular velocity ω in the absence of any external magnetic field. The electric field is again given by eq. (2), and the magnetic field is given by eq. (7). Do the lines of these electric and magnetic fields also rotate with angular velocity ω?

[4]In examples where the source of the magnetic field has only a finite extent, an analysis in spherical coordinates is possible [1, 25] using both eqs. (1) and (4), given the same results.

An appealing view of electric field lines is that they begin/end on electric charges, such that if charges are in motion so are the electric fields lines associated with them. Hence, when the charged cylinder rotates with final angular velocity ω we interpret the radial electric field lines of eq. (2) as rotating with this angular velocity.

In contrast, magnetic field lines always form close loop, as magnetic charges do not exist (as far as we know). Hence, it is less clear that the magnetic field lines rotate along with the charged cylinder. Indeed, Faraday's view (secs. 218 and 220 of [30], and sec. 3090 of [31]) was that the magnetic field lines do not rotate in this case.[5]

If we follow Einstein [33] in supposing that the density $u = (E^2 + B^2)/8\pi$ of energy in the electric and magnetic fields corresponds to density u/c^2 of effective mass, and also suppose that this energy density rotates along with the charged cylinder, then there are densities of momentum and angular momentum associated with the fields. In particular, the angular momentum per unit length associated with the rotating electric field lines is

$$\mathbf{L}_E = \int \mathbf{r} \times \left(\frac{E^2}{8\pi c} \mathbf{r} \times \boldsymbol{\omega} \right) d\text{Area} = \int_0^a \frac{r^2 \omega}{8\pi c} \left(\frac{2\lambda}{r} \right)^2 2\pi r\, dr\, \hat{\mathbf{z}} = \frac{\lambda^2 a^2 \omega}{2c^2}\, \hat{\mathbf{z}}, \qquad (26)$$

and that associated with the magnetic field (if it rotates) is,

$$\mathbf{L}_B = \int \mathbf{r} \times \left(\frac{B^2}{8\pi c} \mathbf{r} \times \boldsymbol{\omega} \right) d\text{Area} = \int_0^a \frac{r^2 \omega}{8\pi c} \left(\frac{2\lambda\omega}{c} \right)^2 2\pi r\, dr\, \hat{\mathbf{z}} = \frac{\lambda^4 a^2 \omega^3}{2c^4}\, \hat{\mathbf{z}},. \qquad (27)$$

In contrast, the field angular momentum per unit length computed according to eq. (1) is,

$$\mathbf{L}_{\text{field}} = \int \mathbf{r} \times \mathbf{p}\, d\text{Area} = \int_0^\infty \mathbf{r} \times \frac{\mathbf{E} \times \mathbf{B}}{4\pi c} 2\pi r dr = \int_0^a \frac{r}{4\pi c} \frac{2\lambda}{r} \frac{2\lambda\omega}{c} 2\pi r\, dr\, \hat{\mathbf{z}} = \frac{\lambda^2 a^2 \omega}{c^2}\, \hat{\mathbf{z}}. \qquad (28)$$

The supposed contribution (27) to the field angular momentum due to the possibly rotating magnetic field lines does not have the same functional form as the "standard" result (28), which reinforces Faraday's view that the magnetic fields lines are not actually rotating in this case.

On the other hand, the result (26) obtained by assuming that the electric field lines do rotate is 1/2 of the "standard" result (28). This suggests that there is some validity to regarding the rotating electric field as carrying momentum and angular momentum with it.

We noted above that the most reliable computation of the field angular momentum associated with a long/infinite solenoid is via the vector potential,

$$\mathbf{L}_{\text{field}} = \int \mathbf{r} \times \frac{\rho \mathbf{A}^{(\text{C})}}{c}\, d\text{Vol} = a\, \hat{\mathbf{r}} \times \lambda \frac{a B(r<a) \hat{\boldsymbol{\phi}}}{2c} = \frac{\lambda^2 a^2 \omega}{c^2} \hat{\mathbf{z}}, \qquad (29)$$

independent of the value of the charge density $-\lambda_0$ on the wire.

In particular, if $\lambda_0 = 0$, then the electric field is zero for $r < a$, and $E_r = 2\lambda/r$ for $r > a$, and the field angular momentum associated with the rotating electric field lines is infinite,

$$\mathbf{L}_E = \int \mathbf{r} \times \left(\frac{E^2}{8\pi c} \mathbf{r} \times \boldsymbol{\omega} \right) d\text{Area} = \int_a^\infty \frac{r^2 \omega}{8\pi c} \left(\frac{2\lambda}{r} \right)^2 2\pi r\, dr\, \hat{\mathbf{z}} = \frac{\lambda^2 (\infty^2 - a^2) \omega}{2c^2}\, \hat{\mathbf{z}}. \qquad (30)$$

[5]For a review of this issue, see sec. 2 of [32].

Note also that the velocity of rotation of the electric field lines is $v = \omega r$ at radius r, which exceeds the speed of light for $r > c/\omega$. Hence, the interpretation of the rotating field energy density $u = E^2/8\pi$ as being associated with an effective, rotating mass density $E^2/8\pi c^2$ is doubtful for $r > c/\omega$.

We conclude that the form (1), or better (4), should be used for computation of the field angular momentum, rather than supposing that the rotating electric field lines can be associated with a rotating, effective mass density $E^2/8\pi c^2$.

2.6 Could We Use $\mathbf{S} = V\mathbf{J}$ for the Poynting Vector? (Sept. 29, 2022)

As reviewed in secs. 3.3 and 3.5 of [34], an alternative form of the Poynting vector, namely $\mathbf{S} = V\mathbf{J}$, was advocated by Livens (1917), which has some ongoing popularity in the engineering community for static examples as it realtes the flow of electromagnetic energy to the flow of electric current, which is naïvely appealing (the "plumbing analogy").

We can then follow Poincaré [35] in identifying the momentum density in the electromagnetic field as $\mathbf{p}_{\rm EM} = \mathbf{S}/c^2$, leading to

$$\mathbf{p}_{\rm EM}^{\rm (F)} = \frac{V\mathbf{J}}{c^2}\,, \tag{31}$$

as first explicitly argued by Furry [36]. This further leads us to suppose that the field angular momentum might be computed as,

$$\mathbf{L}_{\rm EM}^{\rm (F)} = \int \mathbf{r} \times \mathbf{p}_{\rm EM}^{\rm (F)}\, d{\rm Vol} = \int \mathbf{r} \times \frac{V\mathbf{J}}{c^2}\, d{\rm Vol}. \tag{32}$$

To apply this to the present example, we must identify the potential V associated with the charged wire + cylinder and the current density $\mathbf{J}$ associated with the "uniform" magnetic field $\mathbf{B}$. However, the electric potential outside the infinite, charged cylinder can be taken as zero, while the source currents of the magnetic field lie outside the cylinder. Then, eq. (32) implies that there would be no field angular momentum, and there would be no resolution to the cylinder paradox.

This is consistent with a general result, reviewed on p. 7 of [37], that the Furry form (32) is not equal to the form,

$$\mathbf{L}_{\rm EM}^{\rm (P)} = \int \mathbf{r} \times \frac{\mathbf{E} \times \mathbf{B}}{4\pi c}\, d{\rm Vol}, \tag{33}$$

based on the standard form of the Poynting vector, $\mathbf{S} = (c/4\pi)\mathbf{E} \times \mathbf{B}$.

Thus, the Feynman cylinder paradox provides an (initially) static example where the identification of the Poynting vector as $\mathbf{S} = V\mathbf{J}$ is unsatisfactory.

References

[1] R.P. Feynman, R.B. Leighton and M. Sands, *The Feynman Lectures on Physics*, Vol. II, Secs. 17-4, 27-5,6 (Addison Wesley, 1964),

http://www.feynmanlectures.caltech.edu/II_17.html#Ch17-S4
http://www.feynmanlectures.caltech.edu/II_27.html#Ch27-S5
http://www.feynmanlectures.caltech.edu/II_27.html#Ch27-S6

[2] R.H. Romer, *Angular Momentum of Static Electromagnetic Fields*, Am. J. Phys. **34**, 772 (1966), http://kirkmcd.princeton.edu/examples/EM/romer_ajp_34_772_66.pdf
Electromagnetic Angular Momentum, Am. J. Phys. **35**, 445 (1967), http://kirkmcd.princeton.edu/examples/EM/romer_ajp_35_445_67.pdf
Electromagnetic Angular Momentum, Am. J. Phys. **53**, 15 (1985), http://kirkmcd.princeton.edu/examples/EM/romer_ajp_53_15_85.pdf

[3] E.M. Pugh and G.E. Pugh, *Physical Significance of the Poynting Vector in Static Fields*, Am. J. Phys. **35**, 153 (1967), http://kirkmcd.princeton.edu/examples/EM/pugh_ajp_35_153_67.pdf

[4] E. Corinaldesi, *Angular momentum of a static electromagnetic field*, Am. J. Phys. **48**, 83 (1980), http://kirkmcd.princeton.edu/examples/EM/corinaldesi_ajp_48_83_80.pdf

[5] J.M. Aguirregabiria and A. Hernández, *The Feynman paradox revisited*, Eur. J. Phys. **2**, 168 (1981), http://kirkmcd.princeton.edu/examples/EM/aguirregabiria_ejp_2_168_81.pdf

[6] G.G. Lombardi, *Feynman's disk paradox*, Am. J. Phys. **51**, 213 (1983), http://physics.princeton.edu/~mcdonald/examples/EM/lombardi_ajp_51_213_83.pdf

[7] F.L. Boos, Jr, *More on the Feynman's Disk Paradox*, Am. J. Phys. **52**, 756 (1984), http://kirkmcd.princeton.edu/examples/EM/boos_ajp_52_756_84.pdf

[8] T. Bahder and J. Sak, *Elementary solution to Feynman's disk paradox*, Am. J. Phys. **53**, 495 (1985), http://kirkmcd.princeton.edu/examples/EM/bahder_ajp_53_495_85.pdf

[9] O.B. Keyes, *Comments on "Feynman's Disk Paradox"*, Am. J. Phys. **52**, 680 (1984), http://kirkmcd.princeton.edu/examples/EM/keyes_ajp_52_680_84.pdf

[10] T.-C.E. Ma, *Field angular momentum in Feynman's disk paradox*, Am. J. Phys. **54**, 949 (1986), http://kirkmcd.princeton.edu/examples/EM/ma_ajp_54_949_86.pdf

[11] H.S.T. Driver, *Angular momentum in static electric and magnetic fields: A simple case*, Am. J. Phys. **55**, 755 (1987), http://kirkmcd.princeton.edu/examples/EM/driver_ajp_55_755_87.pdf

[12] N.L. Sharma, *Field versus action at a distance in a static situation*, Am. J. Phys. **56**, 420 (1988), http://kirkmcd.princeton.edu/examples/EM/sharma_ajp_56_420_88.pdf

[13] J. Higbie, *Angular momentum in the field of an electron*, Am. J. Phys. **56**, 378 (1988), http://kirkmcd.princeton.edu/examples/EM/higbie_ajp_56_378_88.pdf

[14] D.J. Griffiths, *Note on "Field versus action-at-a-distance in a static situation" by N.L. Sharma*, Am. J. Phys. **57**, 558 (1989), http://kirkmcd.princeton.edu/examples/EM/griffiths_ajp_57_558_89.pdf

[15] A.S. de Castro, *Electromagnetic angular momentum for a rotating charged shell*, Am. J. Phys. **59**, 180 (1991), http://kirkmcd.princeton.edu/examples/EM/decastro_ajp_59_180_91.pdf

[16] F.S. Johnson, B.J. Cragin and R.R. Hodges, *Electromagnetic momentum density and the Poynting vector in static fields*, Am. J. Phys. **62**, 33 (1994),
http://kirkmcd.princeton.edu/examples/EM/johnson_ajp_62_33_94.pdf

[17] O. Rojo, *On the Energy Flow Vector of the E-M Field*, Rev. Bras. Ens. Fis. **19**, 396 (1997), http://kirkmcd.princeton.edu/examples/EM/rojo_rbef_19_396_97.pdf

[18] T. Padmanabhan, *Angular Momentum of Electromagnetic Field*, Phys. Ed. **23**, 285 (2006), http://kirkmcd.princeton.edu/examples/EM/padmanabhan_pe_23_285_06.pdf
See also Chap. 22, p. 241 of T. Padmanabhan, *Sleeping Beauties in Theoretical Physics* (Springer, 2015), http://kirkmcd.princeton.edu/examples/mechanics/padmanabhan_15.pdf

[19] B.Y.-K. Hu, *Introducing electromagnetic field momentum*, Eur. J. Phys. **33**, 873 (2012),
kirkmcd.princeton.edu/examples/EM/hu_ejp_33_873_12.pdf

[20] G. Pantazis and L. Perivolaropoulos, *A general realistic treatment of the disk paradox*, Eur. J. Phys. **38**, 015207 (2017),
http://kirkmcd.princeton.edu/examples/EM/pantazis_ejp_38_015204_17.pdf

[21] K. Fukushima and S. Pu, *Relativistic Decomposition of the Orbital and the Spin Angular Momentum in Chiral Physics and Feynman's Angular Momentum Paradox*, Lect. Notes Phys. **987**, 381 (2021),
http://kirkmcd.princeton.edu/examples/EM/fukushima_lnp_987_381_21.pdf

[22] L. Jiménez, I. Campos and J.A.E. Roa-Neri, *The Feynman paradox and hidden momentum*, Eur. J. Phys. **43**, 055202 (2022),
http://kirkmcd.princeton.edu/examples/EM/jimenez_ejp_43_055202_22.pdf

[23] J.J. Thomson, *On Momentum in the Electric Field*, Phil. Mag. **8**, 331 (1904),
http://kirkmcd.princeton.edu/examples/EM/thomson_pm_8_331_04.pdf

[24] K.T. McDonald, *J.J. Thomson and "Hidden" Momentum* (Apr. 30, 2014),
http://kirkmcd.princeton.edu/examples/thomson.pdf

[25] See also Prob. 11 of Ph501 Set 5, which is based on [10],
http://kirkmcd.princeton.edu/examples/ph501set5.pdf

[26] R.P. McKenna, Analog **75**, No. 1, 14 (Mar. 1965),
http://kirkmcd.princeton.edu/examples/EM/mckenna_analog_75_1_14_65.pdf

[27] K.T. McDonald, *McKenna's Paradox: Charged Particle Exiting the Side of A Solenoid Magnet* (Apr. 12, 2006), http://kirkmcd.princeton.edu/examples/mckenna.pdf

[28] K.T. McDonald, *Electromagnetic Field Angular Momentum of a Charge at Rest in a Uniform Magnetic Field* (Dec. 21, 2014),
http://kirkmcd.princeton.edu/examples/lfield.pdf

[29] K.T. McDonald, *How Much of Magnetic Energy is Kinetic Energy?* (Sept. 12, 2009),
http://kirkmcd.princeton.edu/examples/kinetic.pdf

[30] M. Faraday, *Experimental Researches in Electricity.–Second Series*, Phil. Trans. Roy. Soc. London **122**, 163 (1832), http://kirkmcd.princeton.edu/examples/EM/faraday_ptrsl_122_163_32.pdf

[31] M. Faraday, *Experimental Researches in Electricity.–Twenty-eighth Series*, Phil. Trans. Roy. Soc. London **142**, 25 (1852),
http://kirkmcd.princeton.edu/examples/EM/faraday_ptrsl_142_25_52.pdf

[32] A.I. Miller, *Unipolar Induction: a Case Study of the Interaction between Science and Technology*, Ann. Sci. **38**, 155 (1981),
http://kirkmcd.princeton.edu/examples/EM/miller_as_38_155_81.pdf

[33] A. Einstein, *Ist die Trägheit eines Körpers von seinem Energieinhalt abhängig?*, Ann. Phys. **18**, 639 (1905), http://kirkmcd.princeton.edu/examples/EM/einstein_ap_18_639_05.pdf
Translation: *Does the Inertia of a Body Depend upon its Energy-Content?*,
http://kirkmcd.princeton.edu/examples/EM/einstein_ap_18_639_05_english.pdf

[34] K.T. McDonald, *Alternative Forms of the Poynting Vector* (July 13, 2018),
http://kirkmcd.princeton.edu/examples/poynting_alt.pdf

[35] H. Poincaré, *La Théorie de Lorentz et la Principe de Réaction*, Arch. Neer. **5**, 252 (1900), http://kirkmcd.princeton.edu/examples/EM/poincare_an_5_252_00.pdf.
Translation: The Theory of Lorentz and the Principle of Reaction,
http://kirkmcd.princeton.edu/examples/EM/poincare_an_5_252_00_english.pdf

[36] W.H. Furry, *Examples of Momentum Distributions in the Electromagnetic Field and in Matter*, Am. J. Phys. **37**, 621 (1969),
http://kirkmcd.princeton.edu/examples/EM/furry_ajp_37_621_69.pdf

[37] K.T. McDonald, *Four Expressions for Electromagnetic Field Momentum* (Apr. 10, 2006), kirkmcd.princeton.edu/examples/pem_forms.pdf