What is the "Flux Rule"?

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This note is a kind of Abstract for the longer discussion in [1].

A recent paper advocated a nonstandard definition of $\mathcal{EMF}$ (electromotive force), eq. (15) of [2], and argued that to deduce a "flux rule" from this "we must abandon special relativity", and in any case the "flux rule" is "not a physical law". All this serves to confuse, rather than clarify, the issues, which seem to be somewhat controversial.

We take the view that the "flux rule" is simply the integral form of Faraday's law (whose differential form is $\boldsymbol{\nabla} \times \mathbf{E} = -\partial \mathbf{B}/\partial t$), with the help of Stokes' theorem,

$$\oint_{\text{closed loop}} \mathbf{E} \cdot d\mathbf{l} = \int \boldsymbol{\nabla} \times \mathbf{E} \cdot d\mathbf{Area} = -\int \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{Area} = -\frac{d}{dt} \int \mathbf{B} \cdot d\mathbf{Area} = -\frac{d\Phi_{\mathbf{B}}}{dt}, \tag{1}$$

where in the case of a moving loop, bringing the time derivative outside the area integral changes the partial derivative to a total derivative. This is a consequence of one of Maxwell's equations, which are compatible with special relativity. Indeed, use of both Maxwell's second and fourth equations for a moving, rigid circuit leads to inconsistencies when using Galilean relativity, which are resolved by the Lorentz transformation of the electromagnetic fields, as discussed in Appendix C of [3].

Maxwell described the first integral in eq. (1) as the "electromotive force" ($\mathcal{EMF}$) in the first and last sentences of Art. 598 of [4].[1] With this usage, it is common to write the "flux rule" as,

$$\mathcal{EMF} = \oint_{\text{closed loop}} \mathbf{E} \cdot d\mathbf{l} = -\frac{d\Phi_{\mathbf{B}}}{dt}. \tag{2}$$

In Art. 598 of [4], Maxwell started from the integral form of Faraday's law, that the (scalar) *electromotive force* $\mathcal{E}$ in a circuit is related to the rate of change of the magnetic flux through it by his eqs. (1)-(2) (of Art. 598),

$$\mathcal{E} = -\frac{d\Phi_{\mathbf{B}}}{dt} = -\frac{d}{dt} \int \mathbf{B} \cdot d\mathbf{Area} = -\frac{d}{dt} \oint \mathbf{A} \cdot d\mathbf{l} = -\oint \left(\frac{\partial \mathbf{A}}{\partial t} + (\mathbf{v} \cdot \boldsymbol{\nabla})\mathbf{A} \right) \cdot d\mathbf{l}, \tag{3}$$

where $\mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A}$ and the last form, involving the convective derivative, holds for a circuit that moves with velocity $\mathbf{v}$ with respect to the lab frame.[2] In his discussion leading to eq. (3) of Art. 598, Maxwell argued for the equivalent of use of the vector-calculus identity,

$$\boldsymbol{\nabla}(\mathbf{v} \cdot \mathbf{A}) = (\mathbf{v} \cdot \boldsymbol{\nabla})\mathbf{A} + (\mathbf{A} \cdot \boldsymbol{\nabla})\mathbf{v} + \mathbf{v} \times (\boldsymbol{\nabla} \times \mathbf{A}) + \mathbf{A} \times (\boldsymbol{\nabla} \times \mathbf{v}), \tag{4}$$

[1]In the third edition of Maxwell's *Treatise*, edited by J.J. Thomson after Maxwell's death, the term "electromotive force" in Arts. 598-600 was changed to "electromotive intensity".

[2]In Maxwell's notation, $E = \mathcal{E}$, $p = \Phi_{\mathbf{B}}$, $(F, G, H) = \mathbf{A}$, $(F\,dx/ds + G\,dy/ds + H\,dz/ds)\,ds = \mathbf{A} \cdot d\mathbf{l}$, $(dx/dt, dy/dt, dz/dt) = \mathbf{v}$, and $(a, b, c) = \mathbf{B}$. Note that we interpret Maxwell's $(d/dt)(F, G, H)$ as $\partial \mathbf{A}/\partial t$.

which implies for the present case,

$$(\mathbf{v}\cdot\boldsymbol{\nabla})\mathbf{A} = -\mathbf{v}\times(\boldsymbol{\nabla}\times\mathbf{A}) + \boldsymbol{\nabla}(\mathbf{v}\cdot\mathbf{A}) = -\mathbf{v}\times\mathbf{B} + \boldsymbol{\nabla}(\mathbf{v}\cdot\mathbf{A}), \tag{5}$$

$$\mathcal{E} = \oint\left(\mathbf{v}\times\mathbf{B} - \frac{\partial\mathbf{A}}{\partial t}\right)\cdot d\mathbf{l}, \tag{6}$$

since $\oint \boldsymbol{\nabla}(\mathbf{v}\cdot\mathbf{A})\cdot d\mathbf{l} = 0$.

Maxwell noted that a term of the form $-\boldsymbol{\nabla}\Psi$, where Ψ is a scalar field such as the electric scalar potential, could be added to the integrand of our eq. (6), his eq. (4) of Art. 598 of [4], with no effect on the integral. In his eq. (10) of Art. 599, Maxwell described the integrand as the *electromotive force* $\mathfrak{E} = \mathbf{v}\times\mathbf{B} - \partial\mathbf{A}/\partial t - \boldsymbol{\nabla}\Psi$. This has the implication that if the point with velocity $\mathbf{v}$ were occupied by an electric charge q it would experience force $\mathbf{F} = q(\mathbf{v}\times\mathbf{B} + \mathbf{E})$, where $\mathbf{E} = -\partial\mathbf{A}/\partial t - \boldsymbol{\nabla}\Psi$. That is, Maxwell had, in effect, stated the "Lorentz" force law in of Art. 599 [4], though this largely went unrecognized at that time (while it is acknowledged in [2].)

Our eq. (6) corresponds to Maxwell's eq. (4) of Art. 598 of [4], which is eq. (1) of [2]. It appears to be not gauge invariant, and as such could be called "nonphysical". However, Maxwell's eq. (10) of Art. 599 of [4] is gauge invariant, and better written with $\mathbf{E}$ rather than $-\partial\mathbf{A}/\partial t - \boldsymbol{\nabla}\Psi$. Then, we understand that Maxwell deduced an alternative form of our eq. (2),[3]

$$\mathcal{EMF} = \oint(\mathbf{E} + \mathbf{v}\times\mathbf{B})\cdot d\mathbf{l} = \mathcal{EMF}_{\text{fixed loop}} + \mathcal{EMF}_{\text{motional}}, \tag{7}$$

where,

$$\mathcal{EMF}_{\text{fixed loop}}(t) = -\frac{\partial}{\partial t}\int_{\text{loop at time }t}\mathbf{B}\cdot d\mathbf{Area} = -\int_{\text{loop}}\frac{\partial\mathbf{B}}{\partial t}\cdot d\mathbf{Area} = \oint_{\text{loop}}\mathbf{E}\cdot d\mathbf{l}, \tag{8}$$

and,

$$\mathcal{EMF}_{\text{motional}} = \oint_{\text{loop}}\mathbf{v}\times\mathbf{B}\cdot d\mathbf{l}, \tag{9}$$

in which $\mathbf{v}$ is the velocity (in the inertial lab frame of the calculation) of an element $d\mathbf{l}$ of the loop (which may or may not be conducting).

The two methods, eq. (2) and eq. (7), of computing induced $\mathcal{EMF}$s, give the same results (when correctly computed), although for examples with moving circuit elements, the method of eq. (7) is generally easier to apply.

The "flux rule" (2) is rather abstract for an arbitrary closed loop, and only has "practical" significance if the closed loop is an electrically conducting path. Even then, if the conductors

[3]The first clear statement of the equivalence of eqs. (2) and (7) may be in sec. 86 of the text of Abraham (1904) [5], which credits Hertz (1890) [6] for inspiration on this. Boltzmann understood this equivalence in 1891 [7], but did not express it very clearly. An early verbal statement of this in the American literature was by Steinmetz (1908), pp. 1352-53 of [8], with a more mathematical version given by Bewley (1929) in Appendix I of [10]. Textbook discussions in English include that by Becker, pp. 139-142 of [9], by Sommerfeld, pp. 286-288 of [11], by Panofsky and Phillips, pp. 160-163 of [12], and by Zangwill, sec. 14.4 of [13].

along the loop are moving, the interpretation of eq. (2) is difficult,[4] such that many people, including Feynman [15], advocate use of our eqs. (7)-(9), with the view that the "flux rule" does not take motion of the loop into account, and is just our eq. (8),

$$\mathcal{EMF}_{\rm fixed\ loop}(t) = -\int_{\rm loop\ at\ time,\ t} \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{Area}. \qquad (10)$$

A review (with over 500 references) of the debate over the versions of the "flux rule" is given in [1], sec. 2.4 of which includes six examples of circuits with moving parts that can be analyzed via our eq. (2) as well as our eqs. (7)-(9).[5]

It was stated in eq. (1) of [2] that in Maxwell's eq. (4) of Art. 599, our eq. (6), the velocity $\mathbf{v}$ is not that of the line element but of the conduction charges at that point. This is not what Maxwell said, and while eq. (1) of [2] should "rapidly fall into oblivion", Maxwell's version is "alive and well" in the form of our eqs. (7)-(9) that are (sensibly) advocated by many as the best way to analyze the $\mathcal{EMF}$ induced in moving circuits.

References

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[4]An interesting qualification to the "flux rule" (2) was given by Carter (1967) on p. 170 of his "engineering" textbook [14]: *The equation $\mathcal{E} = -d\Phi/dt$ always gives the induced e.m.f. correctly, provided the flux-linkage is evaluated for a circuit so chosen that at no point are particles of the material moving across it.* That is, a valid path through the interior of a material must be at rest with respect to that material.

I believe this statement should also include the proviso: *..., and at no time is there a discontinuous change in the linked flux.*

Note that mention of *the material* implies the concept of $\mathcal{EMF}$ was only considered by Carter for conducting circuits, and not for "imaginary" closed curves. Further discussion of "Carter's Rule" is given in secs. 2.3-4 of [1].

[5]In circuit analyses, which rarely involve moving circuits, one speaks of the $\mathcal{EMF}$ of batteries and inductors (among other circuit elements), where this notion of $\mathcal{EMF}$ does not apply to a closed loop, but rather to a 2-terminal device. For terminals at points a and b, one often considers that $\mathcal{EMF}_{ab} = \int_a^b \mathbf{E} \cdot d\mathbf{l}$ along some path between a and b, typically defined by the circuit element of interest. But, in general, the integral depends on the path, such that this $\mathcal{EMF}$ is not well defined in a mathematical sense. A review of this intricate issue is given in [16].

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