Small Fractal Antennas

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(December 22, 2003)

1 Problem

Antennas for hand-held communication devices are necessarily small, and typically use wavelengths λ that are large compared to the size of the antenna. This typically implies that the magnitude of the antenna reactance X (= imaginary part of the antenna impedance) is large compared to that of its radiation resistance $R_{\rm rad}$ (which is related to the time-average radiated power P and the peak current I_0 at the feedpoint by $P = I_0^2 R_{\rm rad}/2$), so that it is challenging to build an effective impedance matching circuit between the feedline and the antenna. Furthermore, small antennas use small conductors so it may be that the Ohmic resistance $R_{\rm Ohm}$ of the antenna is significant compared to its radiation resistance,[1] which lowers the antenna efficiency, defined as,

$$\text{Antenna Efficiency} = \frac{R_{\rm rad}}{R_{\rm rad} + R_{\rm Ohm}}\,. \tag{1}$$

It is possible to lower the reactance of an antenna by changing the shape of its conductors without increasing the overall size of the antenna. If the length and complexity of the shape of the antenna conductors is increased while keeping the overall area of the antenna constant, we create what is sometimes called a fractal antenna.[2] Details of the antenna reactance of fractal antennas are best calculated with a numerical code such as NEC4. Here, you are asked to use relatively simple analytic arguments to discuss the radiation (and Ohmic) resistance of a planar fractal antenna that fits within a square of edge length $a \ll \lambda$.

Show that the radiation resistance of a fractal loop antenna is smaller than that of a simple loop antenna of the same extent a. Show that the radiation resistance of a dipole antenna based on a dense (Hilbert) fractal pattern is essentially identical to that of a simple linear dipole antenna of the same total height $a \ll \lambda$, even if the total length L of the conductor is of order λ.

Then, since the Ohmic resistance of a fractal antenna is necessarily larger than that of a simple dipole or loop antenna of the same overall extent, the efficiency (1) of a fractal antenna is lower than that of the simpler antenna. Nonetheless, in some cases the lower reactance of the fractal antenna may provide a useful advantage in simplifying the feed electronics of the antenna system.

2 Solution

We first discuss small fractal antennas as receiving antennas. This discussion will be somewhat qualitative, so we follow it with more quantitative discussion of their behavior as

[1] The antenna impedance is $Z = R_{\rm rad} + R_{\rm Ohm} + iX$, where $i = \sqrt{-1}$.
[2] Fractal antennas are an outgrowth of meander antennas [1].

broadcast antennas. The antenna reciprocity theorem[3] guarantees that a good broadcast antenna is also a good receiving antenna.

2.1 Remarks about Receiving Antennas

A receiving antenna can be considered as a 2-terminal device whose purpose is to produce a voltage (that can be amplified externally, and demodulated to produce an audio signal, *etc.*) in response to an electromagnetic wave. If the conductor of an antenna fits within a square of edge a that is small compared to the wavelength λ of the electromagnetic wave that is to be detected, then the electric and magnetic fields $\mathbf{E}$ and $\mathbf{B}$ have negligible spatial variation over the antenna at any moment in time.

If the receiving antenna is a dipole, then it responds primarily to the electric field of the wave. Clearly, the largest voltage drop across the antenna, is just the field strength E times the largest spatial dimension of the antenna. That is,

$$V_{\rm max} = \sqrt{2} a E \qquad \text{(small dipole antenna)}, \tag{2}$$

independent of the detailed arrangement of the conductor within the square of edge a.[4] We immediately infer that a small fractal dipole antenna cannot be superior to an ordinary small dipole antenna if their overall spatial extents are the same.

In practice, the signal from a small dipole antenna is more like 1/2 of the maximal voltage (2). This is because a signal in a dipole antenna is based on the induced electric dipole moment $p = qd$, which depends on the distance d between the centers of each arm of the antenna, which is typically half the distance between the tips.

A loop antenna responds primarily to the magnetic field of the broadcast wave, via Faraday's law. That is, the 2-terminal signal voltage is proportional to time rate of change of the magnetic flux through the antenna, which is proportional to the area of the antenna,

$$V \propto \frac{d\Phi}{dt} \propto \omega B \text{Area} \quad \text{(small loop antenna)}, \tag{3}$$

where $\omega = 2\pi f$ is the angular frequency of the (carrier) wave.[5] Thus, if a loop antenna fits within a square of edge a, the signal will be strongest if the shape is simply a square of edge a. A fractal shape for the conductor reduces the area of the antenna (provided it still fits within a square of edge a), and hence reduces its effectiveness as a small loop antenna.

The power extracted from the incident wave by an antenna depends on the effective impedance Z of the combination of the antenna plus receiving circuit, according to $P = Re(V^2/2Z)$. If the total impedance of a small antenna + receiving circuit can be made small, the small antenna can extract just as much power from the incident wave as the large antenna. Hence, understand of antenna reactance is important for receiving as well as broadcast antennas. This note, however, limits its further discussion to the real part of the antenna impedance.

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[3]See, for example, [2, 3].

[4]The maximal signal voltage can be achieved only with proper alignment of the antenna with respect to the electric field of the wave; *i.e.*, the arms of the dipole should be parallel to the electric field vector $\mathbf{E}$.

[5]The maximal signal voltage in a loop antenna is achieved when the axis of the loop is parallel to the magnetic field $\mathbf{B}$ of the wave.

We now turn to a discussion of small antennas as broadcast devices.

2.2 Radiation Resistance of Small Linear and Loop Antennas

A simple measure of the performance of a broadcast antenna is its radiation resistance $R_{\rm rad}$, which relates the (time-averaged) radiation power P to the peak current I_0 that drives the antenna, according to,

$$P = \frac{1}{2} I_0^2 R_{\rm rad}. \tag{4}$$

A higher radiation resistance is better, in that more power is radiated compared to the power $I_0^2 R_{\rm Ohm}/2$ lost to heating the antenna due to the ordinary resistance $R_{\rm Ohm}$ of its conductor.

2.2.1 Small Center-Fed Linear Dipole Antenna

Recall that the radiation resistance of a center-fed, linear dipole antenna of length $a \ll \lambda$ is,

$$R_{\rm rad} = \left(\frac{a}{\lambda}\right)^2 197\ \Omega, \qquad \text{(center-fed linear dipole)}, \tag{5}$$

assuming that the current drops linear between the center of the antenna (the feed point) and the tips (where the current must be zero)[4]. The radiation resistance of a small linear dipole antenna of length a falls off as $(a/\lambda)^2$.

2.2.2 Small Loop Antenna

Likewise, the radiation resistance of a small loop antenna of area A is [5],

$$R_{\rm rad} = \left(\frac{A}{\lambda^2}\right)^2 31,170\ \Omega, \qquad \text{(loop)}, \tag{6}$$

independent of the shape of the loop provided its longest diameter (or diagonal) is small compared to λ. The radiation resistance of a small, square, loop antenna of edge a falls off as $(a/\lambda)^4$. For $a \lesssim \lambda/12$, a loop antenna has lower radiation resistance than that of a linear dipole antenna.

2.3 Small Fractal Antennas

Turning now to the question of the merits of a fractal antenna whose largest dimension a is still small compared to the wavelength λ, we note that this condition implies that phase differences are negligible between the radiation from different parts of the antenna. In this case, it suffices to analyze the radiation in the dipole approximation. That is, all details of the radiation pattern follow from knowledge of the electric and magnetic dipole moments of the charge and current distributions in the antenna.

2.3.1 Small Fractal Antennas with Conductor Length $\ll \lambda$

If the total length of the conductor in the antenna is also small compared to λ, an additional simplification holds. For a loop antenna, the (instantaneous) current I is uniform throughout the antenna, so the magnetic moment is simply IA, and eq. (6) still holds for the radiation resistance of the loop antenna.

The area of a fractal loop antenna is less than that of the geometric figure on which the fractal loop is based, as shown in Fig. 1. Therefore, the radiation resistance of a fractal loop antenna is lower than that of the corresponding simple loop antenna.

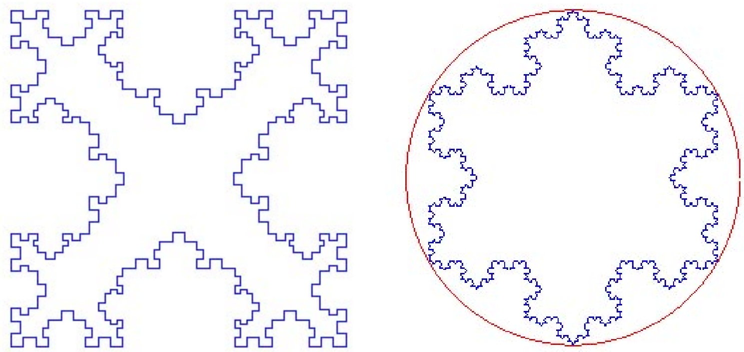

Figure 1: Two examples of the pattern of the conductor in fractal loop antennas.

In a center-fed dipole antenna whose conductor has total length L that is small compared to λ, the current distribution falls off linearly with distance l along the conductor from the central feed points to the tips (at $l = \pm L/2$)of the antenna. That is,

$$I(l,t) = I_0 \left(1 - \frac{2\,|l|}{L}\right) e^{-i\omega t}, \tag{7}$$

taking distance l (which is measured along the conductor, from the central feed point) to be positive on one arm of the antenna and negative along the other.

The equation of continuity for charges and currents (charge conservation) can be written in general as $\nabla \cdot \mathbf{J} = -\partial\rho/\partial t$, where $\mathbf{J}$ is the current density and ρ is the charge density. For the case of a dipole antenna made from a pair of wires, the equation of continuity becomes,[6]

$$\rho(l,t) = \frac{i}{\omega}\frac{\partial \rho}{\partial t} = -\frac{i}{\omega}\frac{\partial I}{\partial l} = \pm \frac{2iI_0}{\omega L} e^{-i\omega t}. \tag{8}$$

The instantaneous current distribution is uniform in each arm of the antenna, but with opposite signs in the two arms. The total charge $Q_\pm$ on each of the arms is,

$$Q_\pm(t) = \pm \frac{iI_0}{\omega} e^{-i\omega t}, \tag{9}$$

[6]Equation (8) holds only if the radius of curvature of the "kinks" in the fractal pattern is larger than the diameter of the conductor. This condition will always be met in any practical application of the fractal antenna concept.

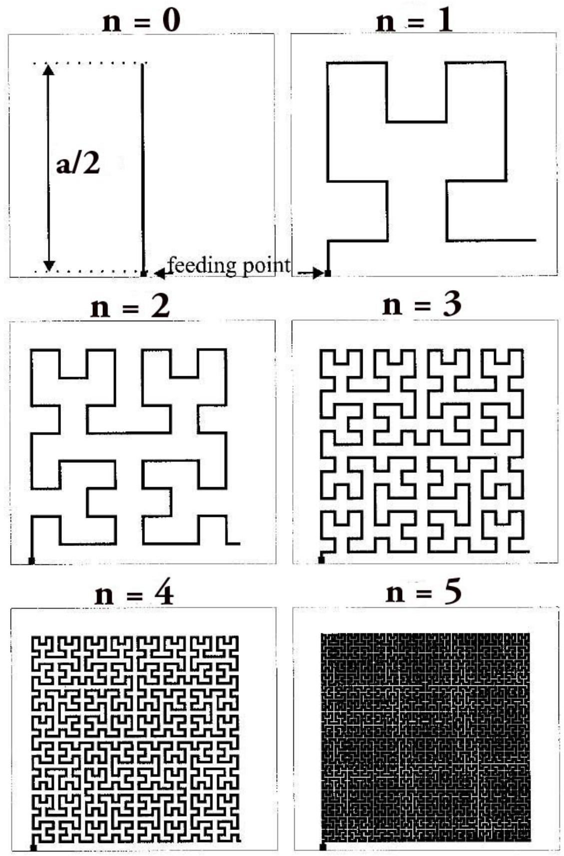

Figure 2: The pattern of one arm of a Hilbert fractal dipole antenna. From [6]. For the calculations in the text, the origin is taken at the feed point, the x axis is horizontal and the y axis is vertical. The patterns of fractal dipole antennas are obtained by reflecting the patterns in the figure about the x axis.

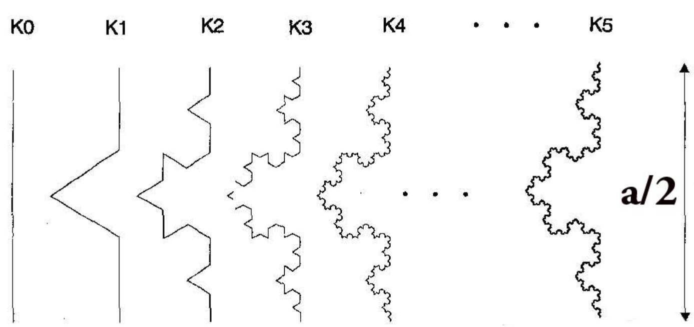

Figure 3: The pattern of one arm of a Koch fractal dipole antenna. From [7].

which is independent of the length L of the conductor.

We take the plane of the fractal dipole antenna to be the x-y plane, with the feed point at the origin. The pattern of the conductor is symmetric about the x axis. Hence, the antisymmetry of the charge distribution (8) implies that the x component of the electric dipole moment vanishes. For a fractal dipole antenna pattern such as the Hilbert pattern shown in Fig. 2, for which the conductor is in effect uniformly distributed along the y axis and whose total extent along the y axis is a, the y component of the electric dipole moment is,

$$p_y(t) = \sum_i Q_i y_i = Q_+ \left(\frac{a}{4}\right) + Q_- \left(-\frac{a}{4}\right) = \frac{iI_0 a}{2\omega} e^{-i\omega t}, \tag{10}$$

which is identical to the result for a short linear dipole antenna of length $a \ll \lambda$ [4]. The time-averaged radiated power P is therefore (in Gaussian units),

$$P = \frac{|\ddot{p}_y|^2}{3c^3} = \frac{I_0^2 a^2}{12c} \frac{\omega^2}{c^2} = \frac{I_0^2}{2} \frac{2\pi^2}{3c} \frac{a^2}{\lambda^2} \equiv \frac{I_0^2}{2} R_{\text{rad}}. \tag{11}$$

Noting that $1/c = 30\ \Omega$, the radiation resistance of a small fractal dipole antenna of total length $L \ll \lambda$ is,

$$R_{\text{rad}} = \left(\frac{a}{\lambda}\right)^2 197\ \Omega, \qquad \text{(fractal center-fed dipole, } L \ll \lambda), \tag{12}$$

which is identical to that of a small linear dipole antenna, as given by eq. (5).

For a dipole antenna based on the Koch fractal, shown in Fig. 3), we see that the distribution of segments is not uniform along the y axis. However, the fractal pattern in each arm is symmetric about the midheight of each arm, so the dipole moment of each arm is still the total charge on the arm times the height of the midpoint of the arm, as in eq. (10). Hence, the result (12) holds for the radiation resistance of a Koch dipole antenna as well.

In sum, we have found that the radiation resistance, and hence also the antenna efficiency (1), of small fractal dipole and loop antennas is not better than that of simple dipole and loop antennas of the same overall extent, provided the total length L of the conductor is also small compared to the wavelength λ.

2.3.2 Small Fractal Dipole Antennas with Conductor Length $\approx \lambda$

Since the area of a fractal loop antenna of extent a is little changed from the area of a simple loop of extent a, the radiation resistance of the fractal loop antenna is little different (although always smaller) than that of the simple loop antenna, even when the conductor length L of the fractal antenna becomes large compared to a. Hence, we do not pursue this case further.

To analyze a fractal dipole antenna whose conductor has total length $L \lesssim \lambda$, we need a model of the current distribution $I(l,t)$. The current distribution will be symmetric about $l = 0$, and will vanish at the tips of the antenna: $I(\pm L/2, t) = 0$. Noting these constraints, we can make a Fourier analysis of the current distribution based on the functions $\sin[nkL/2(1 - 2|l|/L)]$, $n = 1, 2, 3, \ldots$, where $k = \omega/c = 2\pi/\lambda$. Thus,

$$I(l,t) = I_0 \frac{\sum_n A_n \sin\left[\frac{nkL}{2}\left(1 - \frac{2|l|}{L}\right)\right]}{\sum_n A_n \sin\left(\frac{nkL}{2}\right)} e^{-i\omega t}, \tag{13}$$

where I_0 is the current at the feed point if $L \le \lambda/2$. (For $L > \lambda/2$ the peak current does not occur at the feed point.) The case $kL = \pi$ ($L = \lambda/2$) could be called a fractal half-wave antenna. If the fractal segments of the antenna have length L/n, Fourier coefficients with $n \lesssim N$ may be important. When modeling a linear dipole antenna, the usual approximation [4] is to take $A_1 = 1$ and set all other Fourier coefficients to zero.

We do not have a simple method to evaluate the Fourier coefficients A_n, but it turns out that we will not need to know these coefficients if the antenna has a dense fractal pattern!

Because the extent a of the fractal antenna is still small compared to λ, we can continue to calculate in the dipole approximation. For this we need the charge distribution corresponding to eq. (13), which we obtain following the logic of eq. (8),

$$\rho(l,t) = -\frac{i}{\omega}\frac{\partial I}{\partial l} = \pm \frac{iI_0}{c}\frac{\sum_n nA_n \cos\left[\frac{nkL}{2}\left(1 - \frac{2|l|}{L}\right)\right]}{\sum_n A_n \sin\left(\frac{nkL}{2}\right)} e^{-i\omega t}. \tag{14}$$

We need the electric dipole moment of this charge distribution. Again $p_x = 0$, while,

$$\begin{aligned} p_y(t) &= \frac{iI_0}{c\sum_n A_n \sin\left(\frac{nkL}{2}\right)} e^{-i\omega t} \int_{-a/2}^{a/2} \pm y\, dy\, \sum_n nA_n \cos\left[\frac{nkL}{2}\left(1 - \frac{2\,|l(y)|}{L}\right)\right] \\ &= \frac{2iI_0}{c\sum_n A_n \sin\left(\frac{nkL}{2}\right)} e^{-i\omega t} \int_0^{a/2} y\, dy\, \sum_n nA_n \cos\left(\frac{nkL}{2} - nkl(y)\right). \end{aligned} \tag{15}$$

For a high-order Hilbert fractal pattern, the function $l(y)$ for $y > 0$ takes on essentially all values between 0 and $L/2$ with equal probability. Hence, we can approximate $\cos(nkL/2 - nkl)$ by its average on the interval $[0, L/2]$, *i.e.*, by,

$$\frac{2}{L}\int_0^{L/2} dl\ \cos(nkL/2 - nkl) = \frac{\sin(nkL/2)}{nkL/2}. \tag{16}$$

We must also note that the function $l(y)$ is multiple valued in the case of a Hilbert fractal pattern; at each height y, there are L/a segments of the fractal. Hence, in eq. (15) we replace $\cos(nkL/2 - nkl)$ by L/a times the average value (16), *i.e.*, by $\sin(nkL/2)/(nka/2)$. The Fourier series in the numerator and denominator of eq. (15) are now identical, so we obtain,

$$p_y(t) \approx \frac{iI_0 a}{2\omega} e^{-i\omega t} \qquad \text{(Hilbert fractal dipole, } L \approx \lambda\text{)}. \tag{17}$$

This is a remarkable result. Use of a dense (Hilbert) fractal pattern of total length $L \approx \lambda$ for the dipole antenna leads to a radiation resistance that is essentially identical to that of a simple linear dipole antenna of the same total height a, assuming that the antenna is small ($a \ll \lambda$).

This conclusion is based on the assumption of a Hilbert fractal pattern (Fig. 2), which permitted the approximation (16). In the case of a Koch fractal pattern (Fig. 3), this approximation does not hold, so perhaps slight improvements over simple linear dipoles are possible here when $L \approx \lambda$. Further, one may choose to use a low-order fractal pattern, rather than a high-order one (which is hard to construct). There may be slight advantages in appropriately chosen low-order fractal dipoles over a simple linear dipole antenna [8].

2.4 Antenna Reactance

The preceding discussion has emphasized only the radiation resistance (which is the real part of the antenna impedance if we ignore the Ohmic resistance R_{Ohm}). In general, antennas present a nonzero reactance (*i.e.*, imaginary part of the antenna impedance) to their power source. If the magnitude of the reactance is large compared to the radiation resistance, as is typically the case for small antennas, the rf power supply voltage must be larger than would be the case were the reactance equal to zero.

In practice, there is a preference for antennas whose reactance is small compared to their radiation resistance. The ideal case of zero reactance has come to be called "resonance". The lowest resonant frequency for a center-fed dipole antenna of length L occurs when the wavelength is roughly $L/2$. At lower frequencies, the reactance of the dipole antenna is capacitive. Hence, it is favorable to add an inductive reactance in series with a short dipole antenna to bring the total reactance close to zero.

Rather than using an external inductor, it is possible to modify the shape of the antenna so as to increase its inductive reactance, and correspondingly lower the (lowest) resonant frequency of the antenna.

A possible interest in fractal antennas is that they tend to have lower total reactance than a dipole antenna of the same overall size, and hence their resonant frequencies are lower. For example, a 3rd-order Hilbert fractal antenna can have resonant frequencies only 1/6 those of a dipole antenna of the same size [6]. This reduction in antenna reactance is achieved by the used of much longer conductors in the antenna, so the ordinary resistance of a fractal antenna can become significant if small diameter wires are used, thereby reducing the efficiency of the antenna.

Whether fractal antennas offer practical advantage over classic methods of controlling the antenna reactance (see, for example, secs. 21-2 and 21-5 of [2]) is a matter for detailed study, either in the laboratory or on a computer with a numerical electromagnetic code (NEC).

A Appendix: Designer Near Fields for "Small" Antennas

For "small" antennas, whose size is much less than a wavelength, the far-field radiation pattern can only be that of a Hertzian dipole [9, 10]. More complex far-field radiation patterns arise only if the size of the antenna is comparable to (or larger than) a wavelength, such that effects of retardation between different components of the antenna become important.

Here, we restrict our attention to "small" antennas, and consider what amount of variation of near fields is possible, consistent with the same far field radiation pattern.

We shall distinguish two subregions of the near field. If the antenna has characteristic length a, and radiates waves of length $\lambda \gg a$, the radiation fields become larger than the quasistatic fields only for distances $\gtrsim \lambda$ from the antenna. The "near zone" is the region in which the radiation fields are not yet prominent, and so is the region within distance λ of the antenna.

Close to the conductors of the antenna, the details of the fields are very dependent on the geometry of the conductors. However, at distances $\gtrsim 2a$ from the antenna the fields take on the form of an ideal Hertzian dipole radiator.

In designing the near fields of an antenna, we therefore should consider separately what forms are possible in the region $\lesssim 2a$ from the antenna, and the region from $\approx 2a$ to $\approx \lambda$ from the antenna.

The options in the latter region are much more restricted than in the former, so we consider the latter case first.

We recall that there are two forms of Hertzian dipole radiators, electric dipoles and magnetic dipoles.[7]

Electric dipole radiators that broadcast at angular frequency ω are characterized by their electric dipole moment $\mathbf{p}e^{-i\omega t}$ where vector $\mathbf{p}$ is constant in time but can have complex components. Similarly, magnetic dipole radiators are characterized by their magnetic dipole moment $\mathbf{m}e^{-i\omega t}$, where the constant vector $\mathbf{m}$ can have complex components.

The electromagnetic fields of these electric and magnetic dipole radiators are, for distances $\gtrsim 2a$ from the radiator (whose size is a) are (in Gaussian units) [10],

$$\begin{aligned}
\mathbf{E} &= k^2[(\hat{\mathbf{r}} \times \mathbf{p}) \times \hat{\mathbf{r}} - \hat{\mathbf{r}} \times \mathbf{m}]\frac{e^{i(kr-\omega t)}}{r} - ik\{[3(\mathbf{p}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{p}] - \hat{\mathbf{r}} \times \mathbf{m}\}\frac{e^{i(kr-\omega t)}}{r^2} \\
&\quad +[3\mathbf{p}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{p}]\frac{e^{i(kr-\omega t)}}{r^3}\,, \qquad (18) \\
\mathbf{B} &= k^2[(\hat{\mathbf{r}} \times \mathbf{m}) \times \hat{\mathbf{r}} + \hat{\mathbf{r}} \times \mathbf{p}]\frac{e^{i(kr-\omega t)}}{r} - ik\{[3(\mathbf{m}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{m}] - \hat{\mathbf{r}} \times \mathbf{p}\}\frac{e^{i(kr-\omega t)}}{r^2} \\
&\quad +[3\mathbf{m}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{m}]\frac{e^{i(kr-\omega t)}}{r^3}\,, \qquad (19)
\end{aligned}$$

where $\hat{\mathbf{r}} = \mathbf{r}/r$ is the unit vector from the center of the dipole to the observer,

The only flexibility we have in the design of these fields are our choices as to the magnitudes, directions and phases of the magnetic moments $\mathbf{p}$ and $\mathbf{m}$.

In the near field, where $r < \lambda$, the terms in eqs. (18) and (19) that vary as $1/r^3$ are the largest. That is,

$$\begin{aligned}
\mathbf{E}_{\rm near}(2a \lesssim r \lesssim \lambda) &\approx [3(\mathbf{p}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{p}]\frac{e^{i(kr-\omega t)}}{r^3}\,, \qquad (20) \\
\mathbf{B}_{\rm near}(2a \lesssim r \lesssim \lambda) &\approx [3(\mathbf{m}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{m}]\frac{e^{i(kr-\omega t)}}{r^3}\,. \qquad (21)
\end{aligned}$$

These fields have the shape of static dipole fields multiplied by the traveling wave $e^{i(kr-\omega t)}$, and thereby have components both parallel to and transverse to the radial direction, in contrast to the radiation fields that are purely transverse. Note that the electric field in the near zone is, in the first approximation, due only to the electric dipole antenna, while the magnetic field in the near zone is due only to the magnetic dipole antenna. Hence, no combination of small electric and magnetic dipole antennas can eliminate the nonradiating fields in the near zone, as may be a goal of enthusiasts for "crossed-field" antennas.

[7]Actually, there is a third possible form of small antennas, the so-called helical toroidal dipole antenna [11], aspects of which may be (unknowingly) incorporated into the design of "cross-field" antennas such as that of [12]. However, unless helical toroidal antennas involve counter windings, they are in effect single-turn loop antennas, as considered here.

If we desire the electric and magnetic fields (20)-(21) to be equal in magnitude in the near zone to a first approximation, then we need $|\mathbf{m}| = |\mathbf{p}|$.[8]

If in addition, we desire the electric and magnetic fields to be 90° out of phase in the near zone, we need $\mathbf{m} = i\,|\mathbf{p}|\,\hat{\mathbf{m}}$, where the directions $\hat{\mathbf{m}}$ and $\hat{\mathbf{p}}$ are arbitrary.

It is not possible to satisfy the preceding constraints and have the electric and magnetic fields everywhere at right angles to one another in the near field. If these fields were at right angles, their scalar product,

$$\mathbf{E}_{\rm near} \cdot \mathbf{B}_{\rm near} \propto 3(\mathbf{m} \cdot \hat{\mathbf{r}})(\mathbf{p} \cdot \hat{\mathbf{r}}) + \mathbf{m} \cdot \mathbf{p}, \tag{22}$$

should vanish. Consider a coordinate system with $\mathbf{p}$ *along the* z*-axis. Then, vector* $\mathbf{m}$ *points along angles* (θ_m, ϕ_m) *in spherical coordinates, and has rectangular coordinates,*

$$\mathbf{m} = m(\sin\theta_m \cos\phi_m, \sin\theta_m \sin\phi_m, \cos\theta_m). \tag{23}$$

The radial unit vector has components,

$$\hat{\mathbf{r}} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta). \tag{24}$$

Hence,

$$\mathbf{E}_{\rm near} \cdot \mathbf{B}_{\rm near} \propto 3[\sin\theta\sin\theta_m \cos(\phi - \phi_m) + \cos\theta\cos\theta_m]\cos\theta + \cos\theta_m, \tag{25}$$

which cannot vanish for all θ *and* ϕ *for any choice of* θ_m *and* ϕ_m*.*

Similarly, the transverse parts of the near electric and magnetic fields cannot be at right angles to one another everywhere.

We close by considering radiation from a combination of a small electric and small magnetic antenna with common centers, taken to be the origin. The radiation fields have the same form for any $r \gtrsim 2a$, which region includes most of the near zone and all of the far zone,

$$\mathbf{E}_{\rm rad}(r \gtrsim 2a) = k^2[(\hat{\mathbf{r}} \times \mathbf{p}) \times \hat{\mathbf{r}} - \hat{\mathbf{r}} \times \mathbf{m}]\frac{e^{i(kr-\omega t)}}{r} = k^2[\mathbf{p} - (\hat{\mathbf{r}} \cdot \mathbf{p})\hat{\mathbf{r}} - \hat{\mathbf{r}} \times \mathbf{m}]\frac{e^{i(kr-\omega t)}}{r} \tag{26}$$

$$\mathbf{B}_{\rm rad}(r \gtrsim 2a) = k^2[(\hat{\mathbf{r}} \times \mathbf{m}) \times \hat{\mathbf{r}} + \hat{\mathbf{r}} \times \mathbf{p}]\frac{e^{i(kr-\omega t)}}{r} = k^2[\mathbf{m} - (\hat{\mathbf{r}} \cdot \mathbf{m})\hat{\mathbf{r}} - \hat{\mathbf{r}} \times \mathbf{p}]\frac{e^{i(kr-\omega t)}}{r} \tag{27}$$

The time-average radiated power has the angular distribution[9]

$$\left\langle \frac{dP(\hat{\mathbf{r}})}{d\Omega} \right\rangle = \frac{cr^2}{8\pi}\hat{\mathbf{r}} \cdot Re(\mathbf{E} \times \mathbf{B}^\star) = \frac{ck^4}{8\pi}\left(|p|^2 \sin^2\theta_p + |m|^2 \sin^2\theta_m\right), \tag{28}$$

where θ_p is the angle between $\hat{\mathbf{r}}$ and $\mathbf{p}$, and θ_m is the angle between $\hat{\mathbf{r}}$ and $\mathbf{m}$. A possibly surprising result is that there is no interference between the radiation from the electric dipole

[8]To have equality of electric and magnetic fields in the near zone we must have both electric and magnetic antennas. The use of two electric antennas with moments $\mathbf{p}_1$ and $\mathbf{p}_2$, as advocated in one design of a "crossed-field" antenna [13], merely leads to an electric antenna of total moment $\mathbf{p} = \mathbf{p}_1 + \mathbf{p}_2$, for which the near electric field is always larger than the near magnetic field.

[9]$\hat{\mathbf{r}} \cdot [\mathbf{p} - (\hat{\mathbf{r}} \cdot \mathbf{p})\hat{\mathbf{r}}] \times [\mathbf{m}^\star - (\hat{\mathbf{r}} \cdot \mathbf{m}^\star)\hat{\mathbf{r}}] = \hat{\mathbf{r}} \cdot \mathbf{p} \times \mathbf{m}^\star$, while, $\hat{\mathbf{r}} \cdot (\hat{\mathbf{r}} \times \mathbf{m}) \times (\hat{\mathbf{r}} \times \mathbf{p}^\star) = -(\hat{\mathbf{r}} \times \mathbf{m}) \cdot \hat{\mathbf{r}} \times (\hat{\mathbf{r}} \times \mathbf{p}^\star) = -(\hat{\mathbf{r}} \times \mathbf{m}) \cdot [(\hat{\mathbf{r}} \cdot \mathbf{p}^\star)\hat{\mathbf{r}} - \mathbf{p}^\star] = -\hat{\mathbf{r}} \cdot \mathbf{p}^\star \times \mathbf{m}$, so the sum of these two terms has no real part.

p and the magnetic dipole **m**, no matter what are their directions and relative phases. The total time-average radiated power follows from integration of eq. (28),

$$\langle P \rangle = \frac{ck^4}{3} \left(|p|^2 + |m|^2 \right) = P_E + P_M, \tag{29}$$

where P_E and P_M are the time-average powers radiated by the small electric and magnetic antennas if operated separately. Thus, there is no advantage (in terms of radiated power) to a combination of a small electric dipole and a small magnetic dipole antenna compared to either of these two separately.[10]

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[10]Suitably phased combinations of small electric and magnetic antennas can lead to interesting forms of the polarization of the radiation. See, for example, sec. 2.2.2 of [14].

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