Potentials of a Hertzian Dipole in the Gibbs Gauge

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1 Problem

In 1896, Lord Kelvin commented on the speed of propagation of electromagnetic fields when a spark discharges a pair of oppositely charge spheres [1], as in the famous studies of Hertz [2]. Kelvin noted that the speed in an incompressible solid is infinite, while that in an elastic solid is finite, and encouraged the reader to develop a theory of electromagnetism based on an elastic solid, apparently considering Maxwell's theory to be insufficiently equivalent to that of an elastic solid (æther).

Shortly thereafter, Gibbs remarked on solutions to Maxwell's equations via the vector potential $\mathbf{A}^{(\mathrm{G})}$, assuming that the scalar potential $V^{(\mathrm{G})}$ is set to zero [3].[1] Gibbs claimed that he followed Maxwell in this, although Maxwell worked in the Coulomb gauge,[2] where $V^{(\mathrm{C})}$ is the instantaneous electrostatic potential and $\boldsymbol{\nabla}\cdot\mathbf{A}^{(\mathrm{C})}=0$. Rather, it appears that Gibbs made the first known use of what has come to be called the Hamiltonian, or temporal, or Weyl gauge,[3] in which $V^{(\mathrm{G})}=0$ and the vector potential $\mathbf{A}^{(\mathrm{G})}$ is simply related to the electric field $\mathbf{E}$ by,

$$\mathbf{E}=-\frac{1}{c}\frac{\partial\mathbf{A}^{(\mathrm{G})}}{\partial t}\,, \tag{1}$$

where c is the speed of light in vacuum.

Give an expression for $\mathbf{A}^{(\mathrm{G})}$ in the Gibbs gauge (as well as for $\mathbf{E}$ and the magnetic field $\mathbf{B}$) for a time-dependent Hertzian electric dipole $\mathbf{p}(t)$.

2 Solution

As discussed in [8] (following [9]), the electric and magnetic fields of a "point" (Hertzian) electric dipole $\mathbf{p}$ at the origin can be written as,

$$\mathbf{E}(\mathbf{r},t) = \frac{([\ddot{\mathbf{p}}]\times\hat{\mathbf{r}})\times\hat{\mathbf{r}}}{c^2 r}+\frac{3([\dot{\mathbf{p}}]\cdot\hat{\mathbf{r}})\hat{\mathbf{r}}-[\dot{\mathbf{p}}]}{cr^2}+\frac{3([\mathbf{p}]\cdot\hat{\mathbf{r}})\hat{\mathbf{r}}-[\mathbf{p}]}{r^3}\,, \tag{2}$$

$$\mathbf{B}(\mathbf{r},t) = \frac{[\ddot{\mathbf{p}}]\times\hat{\mathbf{r}}}{c^2 r}+\frac{[\dot{\mathbf{p}}]\times\hat{\mathbf{r}}}{cr^2}\,, \tag{3}$$

[1] Gibbs, like Kelvin, was undoubtedly extrapolating from Hertz' famous discussion of the radiation of a small, oscillating electric dipole [2], in which Hertz employed so-called polarization potentials that are tacitly based on used of the Lorenz gauge [4] (as noted in sec. 6.13 of [5]. Thus, Gibbes demonstration of a vector potential $\mathbf{A}^{(\mathrm{G})}$ in his new gauge was perhaps the first example of a problem in which the electromagnetic fields were deduced from potentials in two different gauges.

[2] See, for example, sec. 617 of Maxwell's *Treatise* [6].

[3] See, for example, sec. VIII of [7]. The author has been unable to determine why these names are given to the "Gibbs" gauge, nor when those names were first used.

where quantities within brackets are evaluated at the retarded time $t' = t - r/c$. Integrating eq. (1), we find,

$$\begin{aligned} \mathbf{A}^{(G)}(\mathbf{r},t) &= -c\int_{t_0}^{t} \mathbf{E}(\mathbf{r},t')\,dt' + \mathbf{A}_0^{(G)}(\mathbf{r}) \\ &= -\frac{([\dot{\mathbf{p}}]\times\hat{\mathbf{r}})\times\hat{\mathbf{r}}}{cr} - \frac{3([\mathbf{p}]\cdot\hat{\mathbf{r}})\hat{\mathbf{r}} - [\mathbf{p}]}{r^2} - c\int_{t_0}^{t} \frac{3(\mathbf{p}(t'-r/c)\cdot\hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{p}(t'-r/c)}{r^3}\,dt', \end{aligned} \tag{4}$$

choosing $\mathbf{A}_0^{(G)}(\mathbf{r})$ to cancel the terms from the first 2 integrals at time t_0. The Gibbs-gauge vector potential (4) includes both retarded terms, and "instantaneous" terms, as was noted by Gibbs [3].

If the dipole moment is constant, $\mathbf{p} = \mathbf{p}_0$, the vector potential reduces to,[4]

$$\mathbf{A}^{(G)}(\mathbf{r},t) = -ct\mathbf{E}_0(\mathbf{r}), \tag{5}$$

ignoring terms independent of t (whose curls are zero), where,

$$\mathbf{E}_0(\mathbf{r}) = \frac{3(\mathbf{p}_0\cdot\hat{\mathbf{r}})\,\hat{\mathbf{r}} - \mathbf{p}_0}{r^3}\,. \tag{6}$$

If the dipole moment oscillates according to $\mathbf{p} = \mathbf{p}_0\,e^{-i\omega t}$, the vector potential is,

$$\mathbf{A}^{(G)}(\mathbf{r},t) = ik(\mathbf{p}_0\times\hat{\mathbf{r}})\times\hat{\mathbf{r}}\,\frac{e^{i(kr-\omega t)}}{r} - \left(r + \frac{i}{k}\right)\mathbf{E}_0(\mathbf{r})\,e^{i(kr-\omega t)}, \tag{7}$$

where $k = \omega/c$, after absorbing a term in t_0 into $\mathbf{A}_0^{(G)}(\mathbf{r})$.

Alternative forms for the vector potential in the Gibbs gauge can be given using eqs. (8.5) and (8.7) of [7], but these do not seem to have any crisper interpretations.

A Appendix: Potentials in the Lorenz Gauge

As noted in eqs. (9)-(10) of [9], the potentials in the Lorenz gauge, where $\boldsymbol{\nabla}\cdot\mathbf{A}^{(L)} = -\partial V^{(L)}/\partial ct$, of a Hertzian (point) dipole with arbitrary time dependence are,

$$\mathbf{A}^{(L)}(\mathbf{r},t) = \frac{[\dot{\mathbf{p}}]}{cr}\,, \qquad V^{(L)}(\mathbf{r},t) = \frac{[\mathbf{p}]\cdot\hat{\mathbf{r}}}{r^2} + \frac{[\dot{\mathbf{p}}]\cdot\hat{\mathbf{r}}}{cr}\,. \tag{8}$$

If the dipole moment oscillates according to $\mathbf{p} = \mathbf{p}_0\,e^{-i\omega t}$, the potentials are,

$$\mathbf{A}^{(L)}(\mathbf{r},t) = -ik\mathbf{p}_0\,\frac{e^{i(kr-\omega t)}}{r}\,, \qquad V^{(L)}(\mathbf{r},t) = \mathbf{p}_0\cdot\hat{\mathbf{r}}\,\frac{e^{i(kr-\omega t)}}{r^2} - ik\mathbf{p}_0\cdot\hat{\mathbf{r}}\,\frac{e^{i(kr-\omega t)}}{r}\,. \tag{9}$$

The potentials in both the Gibbs and Lorenz gauges propagate at speed c, but they have different forms.

[4]That the vector potential (5) is time-dependent in a static situation reinforces the general perception that the vector potential does not have direct physical significance. This conventional wisdom is not contradicted by the Aharonov-Bohm effect [10], which depends on differences in the vector potential (as do $\mathbf{E}$ and $\mathbf{B}$) and not on its absolute value.

B Appendix: Potentials in the Coulomb Gauge

In the Coulomb gauge, where $\boldsymbol{\nabla}\cdot\mathbf{A}^{(\mathrm{G})} = 0$, the scalar potential is the instantaneous Coulomb potential, which for a Hertzian dipole with arbitrary time dependence is,

$$V^{(\mathrm{C})}(\mathbf{r},t) = \frac{\mathbf{p}(\mathbf{r},t)\cdot\hat{\mathbf{r}}}{r^2}\,. \tag{10}$$

If the dipole moment oscillates according to $\mathbf{p} = \mathbf{p}_0\, e^{-i\omega t}$, the Coulomb-gauge scalar potential is,

$$V^{(\mathrm{C})}(\mathbf{r},t) = \mathbf{p_0}\cdot\hat{\mathbf{r}}\,\frac{e^{-i\omega t}}{r^2}\,. \tag{11}$$

To compute the vector potential in the Coulomb gauge, we relate it to the vector potential in the Gibbs gauge according to the prescription in eq. (23) of [11],

$$\mathbf{A}^{(\mathrm{C})}(\mathbf{r},t) = \mathbf{A}^{(\mathrm{G})}(\mathbf{r},t) - c\boldsymbol{\nabla}\int_{-\infty}^{t} V^{(\mathrm{C})}(\mathbf{r},t')\,dt'. \tag{12}$$

If the dipole moment oscillates according to $\mathbf{p} = \mathbf{p}_0\, e^{-i\omega t}$,

$$c\int_{-\infty}^{t} V^{(\mathrm{C})}(\mathbf{r},t')\,dt' = c\frac{(\mathbf{p_0}\cdot\hat{\mathbf{r}})}{r^2}\int_{-\infty}^{t} e^{-i\omega t'}\,dt' = \frac{i}{k}\frac{(\mathbf{p_0}\cdot\hat{\mathbf{r}})}{r^2}\,e^{-i\omega t} = \frac{i}{k}V^{(\mathrm{C})}(\mathbf{r},t), \tag{13}$$

taking the scalar potential at $t = -\infty$ to be its average value of zero at large negative times, such that the Coulomb-gauge vector potential is, recalling eq. (7),

$$\begin{aligned}\mathbf{A}^{(\mathrm{C})}(\mathbf{r},t) &= ik(\mathbf{p}_0\times\hat{\mathbf{r}})\times\hat{\mathbf{r}}\frac{e^{i(kr-\omega t)}}{r} - \left(r + \frac{i}{k}\right)\mathbf{E}_0\,e^{i(kr-\omega t)} - \frac{i}{k}\boldsymbol{\nabla}V^{(\mathrm{C})}(\mathbf{r},t) \\ &= ik(\mathbf{p}_0\times\hat{\mathbf{r}})\times\hat{\mathbf{r}}\frac{e^{i(kr-\omega t)}}{r} - \left(r + \frac{i}{k}\right)\mathbf{E}_0\,e^{i(kr-\omega t)} + \frac{i\,\mathbf{E}_0}{k}\,e^{-i\omega t}. \end{aligned} \tag{14}$$

The first two terms of eq. (14) propagate at speed $c = \omega/k$, while the third term is instantaneous.

Since $\boldsymbol{\nabla}\times\mathbf{E}_0 = 0$, the magnetic field $\mathbf{B} = \boldsymbol{\nabla}\times\mathbf{A}^{(\mathrm{C})}$ propagates at speed c. Likewise, the electric field $\mathbf{E} = -\boldsymbol{\nabla}V^{(\mathrm{C})} - \partial\mathbf{A}^{(\mathrm{C})}/\partial ct = -\boldsymbol{\nabla}V^{(\mathrm{C})} + ik\mathbf{A}^{(\mathrm{C})}$ propagates at speed c, as the term $-\boldsymbol{\nabla}V^{(\mathrm{C})}$ is canceled by the third term in $ik\mathbf{A}^{(\mathrm{C})}$ according to the form (14).

That is, the Coulomb-gauge vector potential $\mathbf{A}^{(\mathrm{C})}$ always contains an instantaneous term whose time derivative cancels the instantaneous term $-\boldsymbol{\nabla}V^{(\mathrm{C})}$, such that the electric field $\mathbf{E}$ propagates at speed c.

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