Gravitational Acceleration of a Moving Object at the Earth's Surface

Kirk T. McDonald
Joseph Henry Laboratories, Princeton University, Princeton, NJ 08544
(April 4, 2016; updated July 5, 2019)

1 Problem

In Newton's theory of gravity the acceleration of a (small) mass at the surface of the Earth is g, independent of the velocity of the mass, where in the approximation of a spherical Earth of mass M_E and radius R_E, $g = GM_E/R_E^2$, with G being Newton's constant of gravitation. Deduce the velocity dependence of the acceleration in Einstein's theory of general relativity.

2 Solution

Einstein deduced (very briefly on p. 834 of [1], and in slightly more detail on pp. 820-822 of [2]) that the gravitational deflection of light according to general relativity is twice the "Newtonian" value.[1] Einstein did not deduce the acceleration of light (or that of other fast-moving particles), which has perhaps left the impression that the acceleration at the surface of the Earth is no different from that in Newton's theory (because curvature of space occurs only over large distances in "outer space").

Indeed, the topic of the acceleration due to gravity in Einstein's theory seems to be rarely discussed. It appears in somewhat disguised form in sec. VI.C, p. 2065 of [7] (titled "Laboratory experiments to test relativistic gravity"), where I interpret eq. (6.18) as implying,

$$a = g(1 + v^2/c^2), \tag{1}$$

[1]In [1, 2], Einstein did not refer to Newton, but to his earlier computations of the gravitational deflection of light [3, 4] based on the effect of gravity on the speed of light. In his book of 1920 [5], Einstein stated (p. 153 of the English translation):

> It may be added that, according to the theory, half of this deflection is produced by the Newtonian field of attraction of the sun, and the other half by the geometrical modification ("curvature") of space caused by the sun.

Newton may not have explicitly stated that light is subject to the acceleration due to gravity, but he came very close to this in Questions 29 and 31, pp. 345 and 350, of his *Opticks* [6]:

> Quest. 29. Are not the Rays of Light very small Bodies emitted from shining Substances?

> Quest. 31. Have not the small Particles of Bodies certain Powers, Virtues, or Forces, by which they act at a distance, not only upon the Rays of Light for reflecting, refracting, and inflecting them, but also upon one another for producing a great Part of the Phænomena of Nature? For it's well known, that Bodies act one upon another by the Attractions of Gravity, Magnetism, and Electricity; and these Instances shew the Tenor and Course of Nature, and make it not improbable but that there may be more attractive Powers than these.

for horizontal velocity v of a test mass at the Earth's surface, with c as the speed of light in vacuum (at the Earth's surface). The only other discussions that I have found is at the end of [8] (Hilbert, 1916)[2] which results are transcribed in sec. 6.7 of [9], where it is claimed that the acceleration for vertical motion at the surface of the Earth is,[3,4,5,6]

$$a = g(1 - 3v^2/c^2) \to \begin{cases} g & (v = 0), \\ -2g & (v = c). \end{cases} \tag{2}$$

However, as remarked on p. 191 of [64]:

> Whenever we obtain a prediction from general relativity the question always arises (or should arise) whether the result obtained really refers to an objective physical measurement or whether it has folded into it arbitrary subjective elements dependent on our choice of coordinate systems.

And, as somewhat sourly observed in the second paragraph of [65]:

> ... the fact that (Einstein's equations) retain their form under general coordinate transformation is an embarrassment rather than an advantage.

The standoffish attitude of the physics community to the issue of the gravitational acceleration of fast-moving particles is perhaps related to the comment of Born [66] on general relativity:

> Its connections with experience were slender. It appealed to me like a great work of art, to be enjoyed and admired from a distance.

As will be discussed below, corrections to the flat-space metric tensor at the surface of the Earth are of order $R_M/R_E \approx 10^{-9}$, where $R_M = 2GM/c^2$ is the Schwarzschild radius corresponding to mass M, which is about 1 cm for the Earth and 3 km for the Sun, and yet general-relativistic corrections to the Newtonian gravitational acceleration of a fast-moving mass at the surface of a spherical mass are of order unity. This is rather counterintuitive, but is little discussed in the literature.

Einstein deftly avoided this issue in his discussion [1, 2] of the gravitational deflection of light by only considering the deflection according to observers far from the surface of the deflecting mass.[7]

Einstein's original discussion [1] was based on an approximation to the metric of a static, spherical mass, but shortly thereafter an "exact" metric was found by Schwarzschild [68].

[2] April 21, 2020. This topic is also discussed, with some controversy, in [10]-[52].

[3] The gravitational deflection of high-speed particles was considered in [53] (1920) using what appears to be a spurious version of isotropic coordinates, with a claim of upwards acceleration for particles with $v > c/\sqrt{3}$, as in eq. (2). This was objected to in [54], followed by a more usual discussion of the gravitational deflection of light without mention of acceleration.

[4] It is claimed in [55] that the gravitational force of a spherical mass M on a particle of energy E, with effective mass $m = E/c^2$ is $\mathbf{F} = -GMm[(1 + \beta^2)\mathbf{r} - (\boldsymbol{\beta} \cdot \mathbf{r})\boldsymbol{\beta}]/r^3$.

[5] The present paper emphasizes the acceleration of "point" masses. For extended objects, internal motion can affect the acceleration of their "center of mass" in curved space [56]-[62].

[6] A discussion of rocket-assisted acceleration of particles in circular orbits near a black hole is given in [63].

[7] A typical example of this tradition is [67], which discusses the gravitational deflection of fast-moving particles, mentioning acceleration but only computing deflection angles.

In principle, the rotation of the Earth could add an effect of "magnetic gravity", but this is very small as first noted by de Sitter [69], and we will only consider the case of spherical symmetry here.[8]

2.1 Spherically Symmetric Metrics

This section is something of a historical digression. A "uniform" gravitational field is discussed in Appendix A below.

Schwarzschild [68] looked for a static, spherically symmetric solution to Einstein's equations for a point mass M at the origin in spherical coordinates $x^\mu = (t, r, \theta, \phi)$ such that the square of the invariant interval (line element) for small motion of a test mass would have the form, with $C' = dC/dr$,[9]

$$ds^2 = g_{\alpha\beta}\, dx^\alpha\, dx^\beta = A\, c^2\, dt^2 - B\, dr^2 - C^2\, d\Omega^2, \qquad d\Omega^2 = (d\theta^2 + \sin^2\theta\, d\phi^2), \tag{3}$$

where $A,\ B \overset{r\to\infty}{\longrightarrow} 1$, and $C \overset{r\to\infty}{\longrightarrow} r$, such that spacetime is asymptotically flat at large distances from the mass. The circumference of a circle of radius r is $2\pi C$ (not $2\pi r$),[10] which is a reminder that the physical interpretation of the coordinate r (and t) is not obvious.

A general result[11] is that A and B can be written in terms of $C(r)$ and the Schwarzschild radius R_M, such that the line element (3) takes the form,

$$ds^2 = \left(1 - \frac{R_M}{C}\right) c^2\, dt^2 - \frac{C'^2}{1 - R_M/C}\, dr^2 - C^2\,(d\theta^2 + \sin^2\theta\, d\phi^2), \tag{4}$$

Over the years, many variants of the Schwarzschild metric have been considered, with various forms of the function $C(r)$,

1. Schwarzschild [68] (Jan. 16, 1916): $C = R = (r^3 + R_M^3)^{1/3}$.

   Note that Schwarzschild's form is not that commonly associated with the "Schwarzschild" metric, which latter is actually the form of item 2.

2. Droste [10] (May 27, 1916);[12] Flamm [78] (Sept. 3, 1916); Hilbert [8] (Dec. 23, 1916); Weyl [79] (Aug. 8, 1917): $C = r$.

3. "Isotropic" Coordinates, Pauli [80] (1920):[13] $C = r(1 + R_M/4r)^2$.

[8]It has been shown in [70]-[74] that the only spherically symmetric solution to Einstein's equations with no source masses is for a static metric that can be obtained by coordinate transformations of eq. (4) with $C = r$. This result, known as Birkhoff's theorem, is sometimes misstated that eq. (4) with $C = r$ is the only possible form of a spherically symmetric metric.

Kerr [75] has found a static metric for a rotating spherical mass.

[9]Strictly, Schwarzschild wrote F for A, $G + Hr^2$ for B, and Gr^2 for C^2.

[10]Apparently, C is sometimes called the "curvature radius".

[11]This result is deduced in Appendix A of the paper [76], whose polemic against the notion of a "black hole" was carried further in [77].

[12]Droste's paper was discussed by de Sitter (Aug. 1916) [69].

[13]For isotropic coordinates, $B = C'^2/(1 - R_m/C) = C^2/r^2$, such that the spatial part of the line element (4) is $-(1 + R_M/4r)^4(dr^2 + r^2 d\theta^2 + r^2\sin^2\theta\, d\phi^2)$, which is isotropic. Hence, these coordinates may be the

4. Lanczos [81] (Oct. 1922), Fock, p. 194 of [82] (1959):[14] $\qquad C = r + R_M/2.$

5. Brillouin [83] (Jan. 1923): $\qquad C = r + R_M.$

Although it is not our main concern here, we note that for all of the above forms the metric is "peculiar" (if not "singular") at $r = R_M$, and the physical meaning of the solutions is unclear for smaller r. An eventual understanding emerged [84, 85] that a test mass cannot have constant r at values less than R_M, such that the notion of a static mass of radius $0 < r < R_M$ is not valid, and physically reasonable metrics in this regime must be time-dependent.

The first hint of this resolution to the (non)issue of a "singularity" at $r = R_M$ was given by Lemaître [86] (1933), when he found the (diagonal), time-dependent metric,

$$ds^2 = c^2\,dt^2 - \left(\frac{2R_M}{3(r-t)}\right)^{2/3} dr^2 - \left(\frac{3\sqrt{R_M}(r-t)}{2}\right)^{4/3} d\Omega^2. \tag{6}$$

The metric tensor is also diagonal for all of the Schwarzschild metrics. Time-dependent metrics with off-diagonal elements have also been considered in:

1. *Gullstrand [87] (May 25, 1921); Painlevé [88] (Oct. 1921),*

$$ds^2 = \left(1 - \frac{R_M}{r}\right) c^2\,dt^2 - 2\sqrt{\frac{r}{R_M}}\,c\,dt\,dr - dr^2 - r^2 d\Omega^2. \tag{7}$$

2. *Szekeres [89] (May 26, 1959); Kruskal [90] (Dec. 21, 1959); (extending earlier work of Synge [65]),*

$$ds^2 = \frac{4R_M}{r}\,e^{-r/R_M}(c^2\,d\mathsf{t}^2 - d\mathsf{r}^2) - r^2 d\Omega^2, \tag{8}$$

$$c^2\,\mathsf{t}^2 - \mathsf{r}^2 = \left(1 - \frac{r}{R_M}\right) e^{r/R_M}, \qquad \frac{2c\,\mathsf{r}\,\mathsf{t}}{\mathsf{r}^2 + c^2\,\mathsf{t}^2} = \tanh\frac{c\,t}{R_M}. \tag{9}$$

3. *Penrose [91] (1964) built on earlier work by Finkelstein [92] (1958) and a much earlier hint by Eddington [93] (1924) to provide yet another time-dependent, nondiagonal metric.*[15]

"closest" to those for flat space, if one emphasizes spatial geometry. These coordinates appear without attribution in eq. (421b) of [80] (Pauli, 1920). For completeness, $A = (1 - R_M/4r)^2/(1 + R_M/4r)^2$.

In [53], $A = 1 - R_M/r$ and $B - Cr^2 = 1 + R_M/r$, which are a kind of isotropic coordinates, but they do not correspond to a static mass M.

[14] Lanczos/Fock coordinates are the spherical-coordinate version of "harmonic" coordinates x^μ that are asymptotically Cartesian and obey Laplace's equation $\Box x^\mu = 0$ everywhere, eq. (5) of [81], p. 174 of [82]. On p. 188, Fock showed that this implies (in our notation, where $V^2 = c^2(1 - R_M/C)$, $F^2 = C'^2/(1 - R_M/C)$ and $\rho = C$),

$$2rC' = \frac{d}{dr}\left[\frac{C^2}{C'}\left(1 - \frac{R_M}{C}\right)\right] = 2C - R_M + \frac{C''C}{C'^2}(R_M - 2C), \tag{5}$$

which is satisfied by $C = r + R_M/2$. This form first appeared in eq. (16) of [81].

[15] Penrose gave a different vision of the metric in [94]. This is illustrated in Chap. 8 of [95]; Chaps. 6-7 illustrate Kruskal coordinates.

2.2 Equations of Motion

We now consider the equations of motion for a test particle (of small mass/energy compared to M) located on the x axis at $(r, \theta, \phi) = (r, \pi/2, 0)$ at time $t = 0$, and which moves in the "equatorial" plane with $\theta = \pi/2$ always.[16] We only consider metrics of the form (4), and compute the geodesic equations (eqs. (2) and (7) of [1]),

$$\frac{d^2 x^\lambda}{dp^2} = -\Gamma^\lambda_{\mu\nu} \frac{dx^\mu}{dp} \frac{dx^\nu}{dp}\,, \qquad \text{where} \qquad \Gamma^\lambda_{\mu\nu} = \frac{1}{2} g^{\lambda\alpha} \left(\frac{\partial g_{\mu\alpha}}{\partial x^\nu} + \frac{\partial g_{\nu\alpha}}{\partial x^\mu} - \frac{\partial g_{\mu\nu}}{\partial x^\alpha} \right), \tag{10}$$

in which the parameter p can be taken as the proper time $\tau = s/c$ for a particle with nonzero mass.

The generalized accelerations $d^2 x^\lambda / dp^2$ for geodesic motion in "curved" spacetime are the sums of terms that are products of generalized velocities, dx^μ/dp and dx^ν/dp with factors, $\Gamma^\lambda_{\mu\nu}$ = Christoffel symbols, that characterize the distortions of spacetime due to the presence of mass/energy.

In this view, it is less surprising that the acceleration due to "gravity" is velocity dependent, than that there is any effect independent of velocity. The latter occurs in that the "velocity" dt/dp for the time coordinate is essentially a constant (except in regions of extreme curvature). With this understanding, we should then expect that the accelerations have "corrections" of order unity for spatial velocities $d\mathbf{x}/dp$ that are of order unity (which in "relativistic" language means velocities of order the speed of light).

Thus, it is surprising (to this author) that 100 years of discussion of general relativity place such little emphasis on these "obvious" aspects of the geodesic equations (10).[17]

For the Christoffel symbols $\Gamma^\lambda_{\mu\nu}$ we can refer to the compilation in [96],[18] identifying $x^1 = r$, $x^2 = \theta$, $x^3 = \phi$ and $x^4 = t$. Then, in the notation of [96], the line element (4) tells us that,

$$A_{[96]} = \frac{C'^2(r)}{1 - R_M/C(r)}\,, \quad B_{[96]} = C^2(r), \quad C_{[96]} = C^2(r)\sin^2\theta, \quad D_{[96]} = c^2 \left(1 - \frac{R_M}{C(r)}\right). \tag{11}$$

The nonzero Christoffel symbols are then, from p. 560 of [96],

$$\Gamma^r_{rr} = \frac{A'_{[96]}}{2A_{[96]}} = \frac{C''}{C'} - \frac{R_M C'}{2C(C - R_M)}\,, \qquad \Gamma^r_{\theta\theta} = -\frac{B'_{[96]}}{2A_{[96]}} = \frac{R_M - C}{C'}\,, \tag{12}$$

$$\Gamma^r_{\phi\phi} = -\frac{C'_{[96]}}{2A_{[96]}} = \frac{R_M - C}{C'}\sin^2\theta, \qquad \Gamma^r_{tt} = \frac{D'_{[96]}}{2A_{[96]}} = \frac{c^2 R_M (C - R_M)}{2C'C^3}\,, \tag{13}$$

$$\Gamma^\theta_{r\theta} = \Gamma^\theta_{\theta r} = \Gamma^\phi_{r\phi} = \Gamma^\phi_{\phi r} = \frac{B'_{[96]}}{2B_{[96]}} = \frac{C'}{C}\,, \qquad \Gamma^\theta_{\phi\phi} = -\frac{1}{2B_{[96]}} \frac{\partial C_{[96]}}{\partial\theta} = -\sin\theta\cos\theta, \tag{14}$$

$$\Gamma^\phi_{\phi\theta} = \frac{1}{2C_{[96]}} \frac{\partial C_{[96]}}{\partial\theta} = \Gamma^\phi_{\theta\phi} = \cot\theta, \qquad \Gamma^t_{rt} = \Gamma^t_{tr} = \frac{D'_{[96]}}{2D_{[96]}} = \frac{R_M C'}{2C(C - R_M)}\,, \tag{15}$$

[16] This section follows sec. 3.8.4 of [64], which follows p. 68 ff of [8].

[17] While applications of the special relativity tend to emphasize physics associated with high velocities, applications of general relativity tend to emphasize rather low-velocity situations. An exception is the gravitational deflection of light, which is usually treated in a manner that makes no use of the notion of the gravitational acceleration of a fast-moving object.

[18] The author, Dingle, of [96] later became a vocal doubter of the theory of relativity. See the bibliography of [97].

where $A'_{[96]} = \partial A_{[96]}/\partial r$, *etc.* Using these, the four equations of motion (10) become,

$$\frac{d^2 r}{dp^2} = -\frac{A'_{[96]}}{2A_{[96]}}\left(\frac{dr}{dp}\right)^2 + \frac{B'_{[96]}}{2A_{[96]}}\left(\frac{d\theta}{dp}\right)^2 + \frac{C'_{[96]}}{2A_{[96]}}\left(\frac{d\phi}{dp}\right)^2 - \frac{D'_{[96]}}{2A_{[96]}}\left(\frac{dt}{dp}\right)^2, \quad (16)$$

$$\frac{d^2\theta}{dp^2} = -\frac{2C'}{C}\frac{dr}{dp}\frac{d\theta}{dp} + \sin\theta\cos\theta\left(\frac{d\phi}{dp}\right)^2, \quad (17)$$

$$\frac{d^2\phi}{dp^2} = -\frac{2C'}{C}\frac{dr}{dp}\frac{d\phi}{dp} - 2\cot\theta\frac{d\phi}{dp}\frac{d\theta}{dp} = -\frac{2}{C}\frac{dC}{dp}\frac{d\phi}{dp} - 2\cot\theta\frac{d\phi}{dp}\frac{d\theta}{dp}, \quad (18)$$

$$\frac{d^2 t}{dp^2} = -\frac{R_M C'}{C(C-R_M)}\frac{dr}{dp}\frac{dt}{dp} = -\frac{R_M}{C(C-R_M)}\frac{dC}{dp}\frac{dt}{dp}. \quad (19)$$

2.2.1 Constants of the Motion

As we consider only the case of motion in the plane $\theta = \pi/2$, the θ-equation (17) is automatically satisfied, and $d\theta/dp = 0$ in eqs. (16) and (18). Then, eq. (18) leads to,

$$\frac{d(d\phi/dp)/dp}{d\phi/dp} + 2\frac{dC/dp}{C} = 0, \qquad \Rightarrow \qquad \ln\frac{d\phi}{dp} + \ln C^2 = \text{const}, \qquad \Rightarrow \qquad C^2\frac{d\phi}{dp} = \frac{J}{c}, \quad (20)$$

where the constant J is related to the angular momentum (per unit rest mass) of the test particle.

Similarly, eq. (19) leads to, using Dwight 101.1 [98],

$$\frac{d(dt/dp)/dp}{dt/dp} + R_M\frac{dC/dp}{C(C-R_M)} = 0, \qquad \Rightarrow \qquad \ln\frac{dt}{dp} + \ln\left(1 - \frac{R_M}{C}\right) = \text{const}. \quad (21)$$

The choice of this constant turns out to be unimportant in the discussion of acceleration, secs. 2.2.3-3 below. On p. 186 of [64], it was recommended to take the constant as zero, which has the appeal that then parameter p goes to t for $r \gg R_M$. However, when discussing circular orbits, Appendix C below, it proves to be more convenient to take the constant to be $\ln(E/c^3)$, where E is a conserved "energy" (per unit rest mass). Then, the "velocity" dt/dp becomes,

$$\frac{dt}{dp} = \frac{E}{c^3}\frac{1}{1 - R_M/C}. \quad (22)$$

For large r, $dt/dp \to E/c^3$, and $p \to tc^3/E$ (which is just ct in case of a massive particle at rest).

Another constant of the motion can be found via a first integral of eq. (16), as discussed in Appendix C below.

2.2.2 "Antigravity" in the Radial Equation of Motion

We can now re-express derivatives with respect to parameter p as derivatives with respect to coordinate t,

$$\frac{dr}{dp} = \frac{dt}{dp}\frac{dr}{dt} = \frac{E}{c^3}\frac{C}{C-R_M}\frac{dr}{dt}, \qquad \frac{d\phi}{dp} = \frac{dt}{dp}\frac{d\phi}{dt} = \frac{E}{c^3}\frac{C}{C-R_M}\frac{d\phi}{dt}, \quad (23)$$

$$\frac{d^2 r}{dp^2} = \frac{d}{dp}\frac{dr}{dp} = \frac{E}{c^3}\frac{dt}{dp}\frac{d}{dt}\left[\frac{C}{C-R_M}\frac{dr}{dt}\right] = \frac{E^2}{c^6}\left(\frac{C}{C-R_M}\right)^2\left[\frac{d^2 r}{dt^2} - \frac{R_M C'}{C(C-R_M)}\left(\frac{dr}{dt}\right)^2\right], \quad (24)$$

Then, eq. (16) leads to, usings eqs. (12)-(13) with $\theta = \pi/2$,

$$\frac{d^2r}{dt^2} = -c^2 \frac{R_M(C - R_M)}{2C'C^3} - \left(\frac{C''}{C'} - \frac{3R_M C'}{2C(C - R_M)}\right)\left(\frac{dr}{dt}\right)^2 - \frac{R_M - C}{C'}\left(\frac{d\phi}{dt}\right)^2 . \tag{25}$$

If gravity is negligible, then $R_M = 0$, $C = r$, and eq. (25) simplifies to,

$$a_r = \frac{d^2r}{dt^2} - r\left(\frac{d\phi}{dt}\right)^2 = 0. \tag{26}$$

That is, the radial component of the acceleration is zero when there is no gravity (or any other force) acting on the test particle. Further, eq. (20) becomes $r^2\, d\phi/dt =$ constant, such that $a_\phi(\theta = \pi/2) = r\, d^2\phi/dt^2 + 2(dr/dt)(d\phi/dt) = 0$. The total acceleration is zero, spacetime is flat, and the particle's trajectory is a straight line (for nonzero velocity).

In Einstein's theory, the existence of a mass M (other than the mass m of the test particle) "curves" the spacetime experienced by the test particle, which adds velocity-dependent terms of order R_M to the equations of motion, that represent the effect of gravity of mass M on the test particle.[19]

Note that in eq. (25), $C' = 1$ leads to a term $R_M(d\phi/dt)^2 = (c^2R_M/r^2)(v_\phi^2/c^2) = 2gv_\phi^2/c^2$, recalling that the Earth's surface gravitational acceleration is $g = GM/r^2 = c^2R_M/2r^2$. Hence, the result in case of horizontal motion depends on the form of C, *i.e.*, on the choice of coordinates!

For the "standard" Schwarzschild line element (4) with $C = r$, eq. (25) leads to,

$$\begin{aligned} a_r \equiv \frac{d^2r}{dt^2} - r\left(\frac{d\phi}{dt}\right)^2 &= -\frac{c^2R_M}{2r^2}\left[1 - \frac{R_M}{r} - \frac{3v_r^2}{c^2(1 - R_M/r)} + 2\frac{v_\phi^2}{c^2}\right] \qquad (C = r) \\ &\approx -g\left(1 - \frac{3v_r^2}{c^2} + 2\frac{v_\phi^2}{c^2}\right), \end{aligned} \tag{27}$$

setting $v_r = dr/dt$, $v_\phi = r\, d\phi/dt$, and the approximation holds for $r \gg R_M$.

This is suggestive,[20] but a bit odd in that the dependence of the radial acceleration a_r on velocity is not isotropic.

The result (27) holds for $C = r$, which is the "standard" parameterization of the Schwarzschild metric,[21] and also holds for Schwarzschild's original choice, $C = (r^3 + R_M^3)^{1/3} \approx r + R_M^3/3r^2$.

For Brillouin's coordinates, with $C = r + R_M$, the gravitational acceleration a_r is independent of v_ϕ (but not v_r),

$$a_r = \frac{d^2r}{dt^2} - r\left(\frac{d\phi}{dt}\right)^2 \approx -g\left(1 - \frac{3v_r^2}{c^2}\right) \qquad (C = r + R_M). \tag{28}$$

[19] As summarized by Wheeler, p. xi of [99]: "Spacetime grips mass, telling it how to move; mass grips spacetime telling it how to curve."

[20] We certainly should be impressed that the radial acceleration predicted for a test particle with $r \gg R_M$ and $v \ll c$ is the Newtonian value g. This should perhaps be called the zeroth experimental test of general relativity.

[21] Thus, we verify the claim of eq. (2), where "standard" Schwarzschild coordinates were used, with $v_\phi = 0$ and $v_r \equiv v$.

If we use isotropic coordinates, $C = r(1 + R_M/4r)^2 \approx r + R_M/2$, $C' \approx 1$, $C'' \approx 0$, and now the approximation to eq. (25) is,[22]

$$a_r = \frac{d^2 r}{dt^2} - r\left(\frac{d\phi}{dt}\right)^2 \approx -g\left(1 - \frac{3v_r^2}{c^2} + \frac{v_\phi^2}{c^2}\right) \qquad (C = r[1 + R_M/4r]^2). \tag{29}$$

Thus, we reproduce the claim of our eq. (1), following p. 2065 of [7], that objects with large horizontal velocity (and negligible v_r) fall with radial acceleration $a_r = -2g$ if we use isotropic coordinates, but not for "standard" Schwarzschild coordinates (for which $a_r = -3g$ when $v_r = 0$). It remains slightly disconcerting that switching from isotropic to "standard" coordinates at the surface of the Earth, where $R_M/r \ll 1$ makes a 50% difference in the acceleration due to gravity for fast, horizontally moving particles.

It is somewhat comforting that all static Schwarzschild metrics lead to the same expression for vertical acceleration ($a_r = -g(1 - 3v_r^2/c^2)$ when $v_\phi = 0$) at the surface of the Earth, but it is perhaps surprising that this acceleration is upwards for $|v_r| > c/\sqrt{3}$, as in eq. (2).[23] We could say that the gravitational force is repulsive, rather than attractive, in case of a high-speed particle moving vertically (either up or down); more dramatically, we could say that "antigravity" occurs in this case.

The prediction of "antigravity" for vertical motion of high-speed particles is one of the most dramatic consequences of Einstein's theory of general relativity, yet it is almost unknown (despite being deduced by Hilbert in 1916 [8]). This prediction has been confirmed experimentally, but the results are never discussed as providing this confirmation.

In particular, if a fast-moving particle falls from the Earth towards the Sun, then (once the effect of Earth's gravity can be ignored) its velocity decreases, rather than increases, according to eqs. (2), (27) and (29).[24] Hence, the fall time of the particle is longer than that expected in Newton's theory of gravity. If the particle is a photon, and is reflected back to the Earth, the time of the return trip is the same, longer time, according to time-reversal invariance.

This has been investigated in the radar-time-delay experiments of Shapiro *et al.* [101, 102, 103], in which signals sent from the Earth to Venus, and then reflected back to Earth, were observed to take longer than the Newtonian prediction, by amounts in agreement with Einstein's theory. However, the acceleration of the fast particles (photons) was never mentioned.[25]

[22] Equation (29) also holds for Lanczos/Fock coordinates, $C = r + R_M/2$.

[23] If a particle is shot upwards from the surface of the Earth with $v > c/\sqrt{3}$, which far exceeds the Earth's escape velocity, its velocity increases, rather than decreases, with time. However, the relative increase in velocity is small, since g decreases as $1/r^2$ for $r > r_E$.

[24] A simplified derivation of this is given in [100], which omits relating the change in velocity to an acceleration.

[25] The gravitational deflection of light and the time-delay for light during largely vertical travel could be considered as an examples of effects that combine general relativity and quantum physics, but apparently Einstein hesitated to make this connection, and instead emphasized an analysis via a Huygens' construction for wave fronts in [1, 2].

A quantum correction to the gravitational deflection of light has been considered in [104].

2.2.3 The Azimuthal Equation of Motion

Similarly to eq. (24), we have that,

$$\frac{d^2\phi}{dp^2} = \frac{d}{dp}\frac{d\phi}{dp} = \frac{dt}{dp}\frac{d}{dt}\left[\frac{C}{C-R_M}\frac{d\phi}{dt}\right] = \left(\frac{C}{C-R_M}\right)^2\left[\frac{d^2\phi}{dt^2} - \frac{R_M C'}{C(C-R_M)}\frac{dr}{dt}\frac{d\phi}{dt}\right]. \qquad (30)$$

Then, for constant θ as considered here, eq (18) can be written as,

$$a_\phi = C\frac{d^2\phi}{dt^2} + 2C'\frac{dr}{dt}\frac{d\phi}{dt} = -\frac{R_M C'}{C(C-R_M)}\frac{dr}{dt}\frac{d\phi}{dt}, \qquad (31)$$

noting that $2\pi C$ rather than $2\pi r$ is the circumference of a circle of radius r, For $r \gg R_M$, where $C \approx r$, this becomes,

$$a_\phi = r\frac{d^2\phi}{dt^2} + 2\frac{dr}{dt}\frac{d\phi}{dt} = 0, \qquad (32)$$

which vanishes as might be expected.

However, for motion near a black hole eq. (31) indicates that their exists a nonzero azimuthal acceleration, opposite to the direction of the azimuthal velocity $C\,d\phi/dt$ if dr/dt is negative. This azimuthal acceleration diverges as $r \to R_M$ in "standard" Schwarzschild coordinates, but not in the other Schwarzschild coordinate systems considered in sec. 2.1.

3 Comments

The result (29), that in isotropic coordinates the vertical acceleration due to gravity at the surface of the Earth is $2g$ downwards for horizontal velocity $v \approx c$ but $2g$ upwards for vertical motion, is hardly compatible with Galileo's hypothesis of universal gravitational acceleration (at g downwards).[26]

Galileo's law of universal acceleration is the basis of the equivalence principle, so the result (29), that the acceleration due to gravity depends on velocity, is in conflict with that principle (taking that principle to be that the acceleration is independent, not only of the mass, but of all other properties of a "body").[27] That is, an accelerated observer in flat spacetime would regard the motion of "free" bodies as having the same acceleration independent of their velocity. Thus, in principle one could determine whether or not a gravitational field is

[26]On p. 72 of the English translation of Galileo's *Discorsi* [105] Salviati says:

> ... in a medium totally devoid of resistance all would bodies fall with the same speed.

The meaning of "speed" is perhaps not clear here, so Galileo began a long discussion of "naturally accelerated motion" on p. 160, including on p. 164:

> Imagine a heavy stone held in the air at rest; the support is removed and the stone set free; then since it is heavier than the air it begins to fall, not with uniform motion but slowly at the beginning with a continuously accelerated motion.

[27]Jan. 25, 2022. A more careful statements of the equivalence principle is that the acceleration is independent of the mass for bodies of the same velocity. See, for example, [106, 107].

present (*i.e.*, that spacetime is curved) by performing experiments inside a small box as to the acceleration of high-speed particles moving in different directions.[28]

A speculation is that the "antigravity" between masses with large relative radial velocity might be related to the "dark energy" problem.

A Appendix: "Uniform" Gravitational Field

This Appendix was written on April 25, 2020.

For a "uniform" gravitational field, using coordinates $(x^0, x^1, x^2, x^3) = (t, x, y, z)$, the metric tensors g_{ij} and g^{ij} can be taken to have nonzero components,[29]

$$g_{00} = \frac{1}{g^{00}} = c^2 f^2(z), \qquad f(0) = 1, \qquad g_{11} = g_{22} = g_{33} = g^{11} = g^{22} = g^{33} = -1, \tag{33}$$

such that $g_{ik}\, g^{jk} = \delta_i^j$. The nonzero Christoffel symbols are,

$$c^2 f^2 \Gamma^0_{03} = c^2 f^2 \Gamma^0_{30} = \Gamma^3_{00} = c^2 f \frac{df}{dz} \equiv c^2 f f'. \tag{34}$$

The geodesic equations of motion (10) are,

$$\frac{d^2x}{dp^2} = \frac{d^2y}{dp^2} = 0, \tag{35}$$

$$\frac{d^2z}{dp^2} = -c^2 f f' \left(\frac{dt}{dp}\right)^2, \tag{36}$$

$$\frac{d^2t}{dp^2} = -\frac{2f'}{f}\frac{dz}{dp}\frac{dt}{dp}. \tag{37}$$

For a particle with mass, the parameter p can be taken as the proper time τ, which is also related by,

$$c^2 d\tau^2 = g_{\mu\nu} dx^\mu\, dx^\nu = (c^2 f^2 - v^2) dt^2, \tag{38}$$

$$d\tau = h\ dt, \quad h = \sqrt{f^2 - \frac{v^2}{c^2}}, \tag{39}$$

where $\mathbf{v} = d\mathbf{x}/dt$ is the 3-velocity of the particle. Hence,

$$0 = \frac{d^2x}{d\tau^2} = \frac{1}{h}\frac{d}{dt}\left(\frac{1}{h}\frac{dx}{dt}\right) = \frac{1}{h^2}\frac{d^2x}{dt^2} - \frac{dx}{dt}\frac{1}{h^3}\frac{dh}{dt}, \tag{40}$$

$$a_x = \frac{d^2x}{dt^2} = \frac{1}{h}\frac{dx}{dt}\frac{dh}{dt} = \frac{1}{h^2}\frac{dx}{dt}\left[f f' \frac{dz}{dt} - \frac{\mathbf{v}\cdot\mathbf{a}}{c^2}\right], \tag{41}$$

[28]These experiments would be very difficult to perform in a small space, but in the classical view that infinite precision of measurement is possible, they could be done. The one attempt [108, 109] to perform such an experiment at the surface of the Earth yielded a null result.

[29]See, for example, sec. 97 of [110]. Other metrics are possible, some of which are reviewed in [111].

where $\mathbf{a} = d\mathbf{v}/dt = d^2\mathbf{x}/dt^2$ is the 3-acceleration, and similarly for a_y. Only for $|v| \ll c$ are a_x and a_y close to zero.

For motion in z, we restrict our attention to the case that $x = y = 0$. Then, $\mathbf{v} = \hat{\mathbf{z}}\, dz/dt$, and recalling the manipulations in eqs. (40)-(41), eq. (36) leads to,

$$\frac{d^2z}{d\tau^2} = \frac{1}{h}\frac{d}{dt}\left(\frac{1}{h}\frac{dz}{dt}\right) = \frac{1}{h^2}\frac{d^2z}{dt^2} - \frac{1}{h^4}\frac{dz}{dt}\frac{dh}{dt} = \frac{1}{h^2}\frac{d^2z}{dt^2} - \frac{1}{h^4}\frac{dz}{dt}\left[ff'\frac{dz}{dt} - \frac{1}{c^2}\frac{dz}{dt}\frac{d^2z}{dt^2}\right] = -\frac{c^2 ff'}{h^2}, \tag{42}$$

$$\frac{1}{h^2}\frac{d^2z}{dt^2}\left[1 + \frac{1}{c^2h^2}\left(\frac{dz}{dt}\right)^2\right] = \frac{f^2}{h^4}\frac{d^2z}{dt^2} = -\frac{c^2 ff'}{h^2}\left[1 - \frac{1}{c^2h^2}\left(\frac{dz}{dt}\right)^2\right] = -\frac{c^2 ff'}{h^4}\left[f^2 - \frac{2}{c^2}\left(\frac{dz}{dt}\right)^2\right], \tag{43}$$

$$a_z = \frac{d^2z}{dt^2} = -c^2 f'\left[f - \frac{2}{c^2 f}\left(\frac{dz}{dt}\right)^2\right]. \tag{44}$$

In general, the "vertical" acceleration a_z depends on the velocity $v_z = dz/dt$.

A.1 A Common Choice for f

A common choice for the function f of eq. (33) is that considered in sec. 97 of [110],[30]

$$f = 1 + \frac{gz}{c^2}, \tag{45}$$

which is only physically meaningful in the region $z > -c^2/g$. Then, the "vertical" acceleration of a free, massive particle is,

$$a_z = \frac{d^2z}{dt^2} = -g\left[\left(1 + \frac{gz}{c^2}\right) - \frac{2}{c^2(1 + gz/c^2)}\left(\frac{dz}{dt}\right)^2\right]. \tag{46}$$

Here, the "vertical" acceleration a_z approaches $-g$ only for $|z| \ll c^2/g$ and $|v| \ll c$. The acceleration is upwards ("antigravity") for $|v_z| > cf/\sqrt{2} = \sqrt{g_{00}/2} = c(1 + gz/c^2)/\sqrt{2}$.[31,32]

However, the "antigravity" effect does not persist indefinitely, since eq. (46) indicates that $a_z < 0$ for large z, such that any initial "upward" motion of a free particle eventually becomes "downward".

The nonlinear differential equation (46) (amazingly) has the analytic solution,

$$z = \frac{c^2}{g}\left[\left(1 + \frac{gz_0}{c^2}\right)\frac{1}{\cosh g(t - t_0)/c} - 1\right], \tag{47}$$

where z_0 is the z-coordinate at time t_0 when the velocity v_z is zero. That is,

$$v_z = \frac{dz}{dt} = -c\left(1 + \frac{gz_0}{c^2}\right)\frac{\sinh g(t - t_0)/c}{\cosh^2 g(t - t_0)/c}, \tag{48}$$

[30]This is based on a uniformly accelerated frame in flat spacetime (special relativity), in which the acceleration is constant with respect to the accelerated frame, as first discussed by Born (1909) [112].

[31]The result (44) was deduced in sec. 97 of [110], without comment as to the "antigravity".

[32]For the unphysical region, $z < -c^2/g$, the acceleration would be upwards for any velocity.

whose maximum is,

$$v_{z,\max} = \frac{c}{2}\left(1 + \frac{gz_0}{c^2}\right), \tag{49}$$

when,

$$\sinh\frac{g(t-t_0)}{c} = -1, \qquad \cosh\frac{g(t-t_0)}{c} = \sqrt{2}, \qquad z = \frac{c^2}{g}\left[\frac{1}{\sqrt{2}}\left(1 + \frac{gz_0}{c^2}\right) - 1\right]. \tag{50}$$

Note also that the speed of a massless particle propagating in the z-direction is not c, the speed of light in vacuum (according to inertial observers in flat spacetime), but for metrics of the form (33), where for massless particles, $ds^2 = 0 = g_{00}dt^2 - d\mathbf{x}^2$,

$$v_{\rm massless} = \sqrt{g_{00}} = cf = c\left(1 + \frac{gz}{c^2}\right), \tag{51}$$

where the last form holds for f of eq. (45). Thus, the speed of light at the location of the maximum (49) of the velocity of a massive particle is, using eq. (50),

$$v_{\rm light} = \frac{c}{\sqrt{2}}\left(1 + \frac{gz_0}{c^2}\right) > v_{z,\max}. \tag{52}$$

Although $v_{z,\max}$ can be greater than c, it is always less than $v_{\rm light}$.

A.2 An Alternative f

The choice (46) leads to a_z that depends on z even for zero velocity. It might be preferable to have $a_z = -g$ for zero velocity, independent of z, which requires $c^2 ff' = g$ in eq. (44), *i.e.*,[33]

$$f = \sqrt{\frac{1 + 2gz}{c^2}}, \qquad a_z = -g\left[1 - \frac{2}{c^2(1 + 2gz/c^2)}\left(\frac{dz}{dt}\right)^2\right]. \tag{53}$$

In this case, a_z is upwards ("antigravity") for $|v_z| > c\sqrt{1 + 2gz/c^2}/\sqrt{2}$, which is upwards for any velocity if $z < -c^2/2g$.

In any case, a "uniform" gravitational field does not satisfy the equivalence principle except for particles with $|v| \ll c$ (unless one takes the view that the equivalence principle is "local", and observation of the velocity dependence of the acceleration can not be made "locally").

[33] In Einstein's approximate theory of 1914, sec. 17 of [113]), he favored the f of eq. (47) when he took $g_{00} = c^2(1 + 2\Phi/c^2)$, with Φ being the gravitational potential, gz for a uniform field.

B Appendix: Proper, Coordinate and Geometric Acceleration

This appendix was written on July 15, 2017, following a suggestion from Ricardo Vieira.

When considering a test particle with nonzero mass, one often speaks of its proper 4-velocity,

$$u^\alpha = \frac{dx^\alpha}{d\tau} = c\frac{dx^\alpha}{ds}\,, \tag{54}$$

for $x^\alpha = (x^0, \mathbf{x}) = (x^0, x^1, x^2, x^3)$. In coordinate systems where $x^0 = ct$ with t as the "time", one speaks of the ordinary (or coordinate) 3-velocity, $\mathbf{v} = d\mathbf{x}/dt$. Introducing $\gamma = 1/\sqrt{1 - v^2/c^2}$, one can identity the proper-time interval as $d\tau = dt/\gamma$, and the proper 4-velocity can be written as,

$$u^\alpha = \gamma(c, \mathbf{v}). \tag{55}$$

While one might then suppose that the 4-vector $du^\alpha/d\tau$ would be called the proper 4-acceleration, this seems not to be the convention in general relativity, where it is (sometimes) called the coordinate 4-acceleration. Instead, the geodesic equation (10) is sometimes written as,

$$A^\alpha = \frac{d^2x^\alpha}{d\tau^2} + \Gamma^\alpha_{\mu\nu}\frac{dx^\mu}{dp}\frac{dx^\nu}{d\tau} = \frac{du^\alpha}{d\tau} + \Gamma^\alpha_{\mu\nu}u^\mu u^\nu = 0, \tag{56}$$

and the 4-vector $A^\lambda = (0, 0, 0, 0)$ is called the proper 4-acceleration. In this view, the proper 4-acceleration is zero for a particle with no other interaction than "gravity".

The geodesic equation,

$$\frac{du^\alpha}{d\tau} = -\Gamma^\alpha_{\mu\nu}u^\mu u^\nu, \tag{57}$$

can then be interpreted as saying that the coordinate 4-acceleration $du^\lambda/d\tau$ is equal and opposite to the geometric 4-acceleration $\Gamma^\alpha_{\mu\nu}u^\mu u^\nu$ associated with the curvature of spacetime.

We can also use eq. (55) to write,

$$\frac{du^\alpha}{d\tau} = \gamma\frac{du^\alpha}{dt} = \left(\frac{\gamma^4}{c}\mathbf{v}\cdot\mathbf{a},\ \gamma^2\mathbf{a} + \frac{\gamma^4}{c^2}(\mathbf{v}\cdot\mathbf{a})\mathbf{v}\right), \qquad \text{where} \qquad \mathbf{a} = \frac{d\mathbf{v}}{dt}\,. \tag{58}$$

Comparing this with eq. (57), we see that,

$$\frac{\gamma^4}{c}\mathbf{v}\cdot\mathbf{a} = -\Gamma^0_{\mu\nu}u^\mu u^\nu, \qquad \gamma^2 a^k = \Gamma^0_{\mu\nu}u^\mu u^\nu\frac{v^k}{c} - \Gamma^k_{\mu\nu}u^\mu u^\nu \qquad (k = 1, 2, 3). \tag{59}$$

This provides an alternative approach to the accelerations a_r and a_ϕ as found in secs. 2.2.2-3 above.

C Appendix: Circular Orbits

This Appendix added April 16, 2020.

The main theme of this note has been that even at $r \gg R_M$ from a mass there are notable effects of general relativity for fast moving objects. We now add a few remarks about circular orbits with $r \approx R_M$.[34]

We can use eqs. (20) and (22) in eq. (16) for d^2r/dp^2 with $\theta = \pi/2$, and multiply it by $2A_{[96]}\, dr/dp$ to find, noting that $C_{[96]} = C^2$ for $\theta = \pi/2$,

$$2A_{[96]}\frac{dr}{dp}\frac{d^2r}{dp^2} + \frac{dA_{[96]}}{dp}\left(\frac{dr}{dp}\right)^2 - \frac{J^2}{c^2C^4}\frac{dC^2}{dp} + \frac{c^4}{D_{[96]}^2}\frac{dD_{[96]}}{dp} = 0, \tag{60}$$

$$A_{[96]}\left(\frac{dr}{dp}\right)^2 + \frac{J^2}{c^2C^2} - \frac{c^4}{D_{[96]}} = -\kappa = \text{const.} \tag{61}$$

$$\frac{C'^2}{1 - R_M/C}\left(\frac{dr}{dp}\right)^2 + \frac{J^2}{c^2C^2} - \frac{1}{1 - R_M/C} = -\kappa, \tag{62}$$

where κ is another constant of the motion.

The line element (4) can now be written (for $\theta = \pi/2$) in terms of parameter p as,

$$\begin{aligned} ds^2 &= dp^2\left[\left(1 - \frac{R_M}{C}\right)\left(\frac{c\,dt}{dp}\right)^2 - \frac{C'^2}{1 - R_M/C}\left(\frac{dr}{dp}\right)^2 - C^2\left(\frac{d\phi}{dp}\right)^2\right] \\ &= dp^2\left[-\frac{C'^2}{1 - R_M/C}\left(\frac{dr}{dp}\right)^2 - \frac{J^2}{c^2C^2} + \frac{E^2}{c^4(1 - R_M/C)}\right] = \kappa\, dp^2. \end{aligned} \tag{63}$$

For a massless particle, the constant κ is zero, as first noted by Einstein in eq. (5) of [1], while for a particle with nonzero rest mass, $\kappa = 1$ when parameter p is $c\tau$ and τ is the proper time, as perhaps first noted by Einstein in eq. (20a) and following of [2].

It is useful to rewrite the last line of eq. (63) as,

$$\frac{E^2}{c^4C'2} = \left(\frac{dr}{dp}\right)^2 + \frac{1}{C'^2}\left(1 - \frac{R_M}{C}\right)\left(\kappa + \frac{J^2}{c^2C^2}\right) \equiv \left(\frac{dr}{dp}\right)^2 + V(r), \tag{64}$$

where,

$$V(r) = \frac{1}{C'^2}\left(1 - \frac{R_M}{C}\right)\left(\kappa c^2 + \frac{J^2}{c^2C^2}\right), \tag{65}$$

can be considered as an effective potential (per unit rest mass when $\kappa = 1$ and the test particle has mass).

The interpretation of eq. (64) is most straightforward for the "standard" Schwarzschild metric, with $C = r$. Then,

$$\frac{E^2}{c^4} = \left(\frac{dr}{dp}\right)^2 + \left(1 - \frac{R_M}{r}\right)\left(\kappa + \frac{J^2}{c^2r^2}\right), \qquad V(r) = \left(1 - \frac{R_M}{r}\right)\left(\kappa + \frac{J^2}{c^2r^2}\right). \tag{66}$$

[34]A succinct treatment of this theme via a Lagrangian method is given at https://hepweb.ucsd.edu/ph110b/110b_notes/node79.html

Circular orbits exist for r_0 such that $dV(r_0)/dr = 0$, and these are stable if $d^2V(r_0)/dr^2 > 0$,

$$\frac{dV(r)}{dr} = \frac{R_M}{r^2}\left(\kappa + \frac{J^2}{c^2r^2}\right) - \left(1 - \frac{R_M}{r}\right)\frac{2J^2}{c^2r^3} = \frac{\kappa R_M}{r^2} - \frac{2J^2}{c^2r^3} + \frac{3R_MJ^2}{c^2r^4}\,, \tag{67}$$

$$\frac{d^2V(r)}{dr^2} = -\frac{2\kappa R_M}{r^3} + \frac{6J^2}{c^2r^4} - \frac{12R_MJ^2}{c^2r^5}\,. \tag{68}$$

For $\kappa = 0$, $r_0 = 3R_M/2$ and $d^2V(r_0)/dr^2 = -2J^2/c^2r_0^4 < 0$, so photons have no stable circular orbits, and have a possible unstable circular orbit at $r_0 = 3R_M/2$ (sometimes called the photon sphere).

For $\kappa = 1$, circular orbits exist at the roots of,

$$r_0^2 - 2\frac{J^2}{c^2R_M}r_0 + \frac{3J^2}{c^2} = 0, \qquad \Rightarrow \qquad r_0 = \frac{J^2}{c^2R_M}\left(1 \pm \sqrt{1 - \frac{3c^2R_M^2}{J^2}}\right). \tag{69}$$

Such orbits exist only $J^2/c^2 \geq 3R_M^2$. Orbits with minimal J have $r_0(J_{\rm min}) = 3R_M$, for which $d^2V(r_0)/dr^2 = 0$, which implies this radius is the borderline for stability. Since (Newtonian) stable orbits exist for large r_0, orbits with $r_0 < 3R_M$ are unstable.

Orbits with very large angular momentum (per unit rest mass), $J = \gamma\, r v_\phi = r v_\phi/\sqrt{1 - v_\phi^2/c^2}$, exist for $r_0 = r_{0,\rm min} = 3R_M/2$, but these are unstable.

In sum, unstable circular orbits for massive particles exist for $3R_M/2 < r_0 < 3R_M$ in standard Schwarzschild coordinates. The innermost stable circular orbit is at $r_{\rm ISCO} = 3R_M$, for which $J = \sqrt{3}cR_M = 3\gamma\, vR_M$, *i.e.*, $v_{\rm ISCO} = c/2$. The only circular orbit possible for a photon is at $r_0 = 3R_M/2$, which is unstable.[35] Since the minimum radius of an (unstable) orbit of a massive particle is that same as that for a photon, we can say that in both cases the orbital speed is the speed of light.[36]

C.1 Historical Note

The earliest discussion of circular orbits in general relativity was by Droste (1917) [10], who was the first to use the now-standard Schwarzschild metric with $C = r$. On p. 204 he noted that circular orbits exist only for $r > 3R_M/2$, but he did not comment on their stability.

This result was also obtained in 1917 by Hilbert, p. 74 of [8], who also did not comment on the stability of the orbit.

An extensive discussion of circular, elliptical and hyperbolic orbits was given in 1931 by Hagihara [114] using Hamilton-Jacobi methods, including characterization of their stability, for both massive particles and photons. He did not anticipate the possibility of black holes, saying instead on p. 174: *Hence the statement that a very massive star can entirely absorb the light emitted from its surface and never be seen from outside, is quite fallacious.* Hagihara's work went largely unnoticed for over 30 years.

[35]In Fock coordinates where the radial coordinate is $R = r - R_M/2$, $R_{\rm ISCO} = 5R_M/2$, and the photon sphere has radius $R_0 = R_M$.

[36]For a massive particle with velocity approaching the speed of light, the momentum and angular momentum diverge, such that $J \to \infty$ as $r_0 \to r_{0,\rm min}$ from above.

In 1939, Einstein [115] discussed "stationary" orbits, using isotropic coordinates for the Schwarzschild metric, and deduced that such circular orbits exist only for $r_{\rm iso} > (2 + \sqrt{3})R_M/4$.[37] His concluding remarks have been interpreted as arguing that black holes could not exist:[38] *The essential result of this investigation is a clear understanding as to why the "Schwarzschild singularities" do not exist in physical reality. Although the theory given here treats only clusters whose particles move along circular paths it does not seem to be subject to reasonable doubt that more general cases will have analogous results. The "Schwarzschild singularity" does not appear for the reason that matter cannot be concentrated arbitrarily. And this is due to the fact that otherwise the constituting particles would reach the velocity of light.*

In 1956, McCrea [117] ended with the comments: *It may not have been remarked before that ... $3R_M/2$ is a critical distance for the Schwarzschild space-time. It is the distance for which a circular orbit demands an orbital speed equal to the speed of light.*

In 1958, Darwin [118] gave a discussion of the stability of circular orbits, while claiming to be unaware of any previous such effort. He extended the discussion to elliptic and hyperbolic orbits in [119].

In 1961, Goldhammer [120] published a brief note on the stability of circular orbits.

In 1963, Metzner [121] extended the analysis of Darwin somewhat.

The first textbook discussion of the stability of circular orbits in general relativity may be that in sec. 25 of [122] (1973).

C.2 Gravitational Radiation in Binary Black-Hole Mergers

That the innermost stable circular orbit (ISCO) is at $3R_M$ in standard Schwarzschild coordinates is relevant to the recent observations of gravitational waves from mergers of black holes initially in quasicircular orbits [123].[39] The peak intensity of the radiation occurs at this distance between centers of the black holes, after which they "plunge" towards one another with lesser intensity of the radiation.[40] The frequency f of the peak (quadrupole) radiation is at twice the ISCO orbital frequency for, say, a pair of black holes of $30M_\odot$, for which the orbital velocity is $v_{\rm ISCO} = c/2$. Hence, $f = 2v_{\rm ISCO}/2\pi r_{\rm ISCO} = c/6\pi R_M(30M_\odot) \approx c/600R_M(M_\odot) \approx 150$ Hz, noting that the Schwarzschild radius of the Sun is $R_M(M_\odot) \approx 3$

[37]As noted by Pauli in eq. (422) of [80], r in standard Schwarzschild coordinates is related by $r = r_{\rm iso}(1 + R_M/4r_{\rm iso})^2$, so Einstein's result corresponds to $r = 3R_M/2$ in standard coordinates, which is the minimum radius for an unstable circular orbit, rather than for a stable one.

Since $r_{\rm iso} = [(r^2 - R_M\, r)^{1/2} + r - R_M/2]/2$ (p. 327 of [110]), the innermost stable circular orbit (ISCO) ($r_{\rm ISCO} = 3R_M$) is at $r_{\rm iso,ISCO} = (5 + 4\sqrt{2})R_M/4 = 3.55R_M$ in isotropic coordinates.

[38]The story of the slow theoretical acceptance of black holes is recounted in chaps. 3-6 of [116].

[39]The story of the slow acceptance of gravitational waves is recounted in [124], and more briefly in [125].

[40]Gravitational radiation emitted at radius r outside a black hole with Schwarzschild radius R_M arrives at an observer at distance $r_{\rm obs} \gg R_M$ after time delay,

$$\Delta t = \frac{1}{c}\int_r^{r_{\rm obs}} \frac{dr}{1 - R_m/r} = \frac{r_{\rm obs} - r}{c} + \frac{R_M}{c}\ln\frac{r_{\rm obs} - R_M}{r - R_M}\,, \qquad (70)$$

in standard Schwarzschild coordinates. While the last term diverges as $r \to R_M$, it is negligible for $r = 3R_M$, so the peak intensity of gravitational radiation from black-hole mergers is little diluted or delayed by the strong gravitational field of the black holes.

km.[41]

Acknowledgment

The author thanks Malcolm Perry for discussions of this problem — 30 years ago, and Frans Pretorius for recent discussions of black holes.

References

[1] A. Einstein, *Erklärung der Perihelbewegung des Merkur aus der allgemeinen Relativitätstheorie*, S.K.P. Akad. Wiss. 831 (1915),
http://kirkmcd.princeton.edu/examples/GR/einstein_skpaw_831_15.pdf
Explanation of the Perihelion Motion of Mercury from the General Theory of Relativity,
http://kirkmcd.princeton.edu/examples/GR/einstein_skpaw_831_15_english.pdf

[2] A. Einstein, *Die Grundlage der allgemeinem Relativitätstheorie*, Ann. d. Phys. **49**, 769 (1916), http://kirkmcd.princeton.edu/examples/GR/einstein_ap_49_769_16.pdf
The Foundation of the General Theory of Relativity,
http://kirkmcd.princeton.edu/examples/GR/einstein_ap_49_769_16_english.pdf

[3] A. Einstein, *Über das Relativitätsprinzip und die aus demselben gezogene Folgerungen*, Jahrb. Radioakt. Elektron. **4**, 411 (1908),
http://kirkmcd.princeton.edu/examples/GR/einstein_jre_4_411_08.pdf
On the Relativity Principle and Conclusions Drawn from It,
http://kirkmcd.princeton.edu/examples/GR/einstein_jre_4_411_08_english.pdf
http://kirkmcd.princeton.edu/examples/GR/einstein_jre_4_411_08_english2.pdf

[4] A. Einstein, *Über den Einfluß der Schwerkraft auf die Ausbreitung des Lichtes*, Ann. d. Phys. **35**, 898 (1911), http://kirkmcd.princeton.edu/examples/GR/einstein_ap_35_898_11.pdf
On the Influence of Gravitation on the Propagation of Light,
http://kirkmcd.princeton.edu/examples/GR/einstein_ap_35_898_11_english2.pdf

[5] A. Einstein, *Über die spezielle und die allgemeine Relativitätstheorie*, (Vieweg & Sohn, 1920), http://kirkmcd.princeton.edu/examples/GR/einstein_relativititatstheorie_20.pdf
Relativity: The Special and General Theory (Holt, 1920),
http://kirkmcd.princeton.edu/examples/GR/einstein_relativity_20_english.pdf

[6] I. Newton, *Optics*, 4th ed. (1730),
http://kirkmcd.princeton.edu/examples/optics/newton_opticks.pdf

[7] V.B. Braginsky, C.M. Caves and K.S. Thorne, *Laboratory experiments to test relativistic gravity*, Phys. Rev. D **15**, 2047 (1977),
http://kirkmcd.princeton.edu/examples/GR/braginsky_prd_15_2047_77.pdf

[41] Since $v \approx c/2$ during the peak of the gravitational radiation, low-velocity (post-Newtonian) approximations to general relativity are not adequate for a good analysis thereof. Rather, numerical analysis is required, as in [126] (which used a variant of harmonic coordinates).

[8] D. Hilbert, *Die Grundlagen der Physik*, Nachr. Gesell. Wiss. Göttingen, 53 (1917), http://kirkmcd.princeton.edu/examples/GR/hilbert_ngwg_53_17.pdf

[9] K. Brown, *Reflections on Relativity* (2016), http://www.mathpages.com/rr/

[10] J. Droste, *The field of a single centre in EINSTEIN'S theory of gravitation, and the motion of a particle in that field*, Proc. Roy. Neth. Acad. Arts. Sci. **19**, 197 (1917), http://kirkmcd.princeton.edu/examples/GR/droste_pknaw_19_197_17.pdf

[11] H. Bauer, *Mathematische Einführung in die Gravitationstheorie Einsteins* (Leipzig, 1922). http://kirkmcd.princeton.edu/examples/GR/bauer_22.pdf

[12] D. Hilbert, *Die Grundlagen der Physik*, Math. Ann. **92**, 1 (1924), http://kirkmcd.princeton.edu/examples/GR/hilbert_ma_92_1_24.pdf

[13] G.C. McVittie, *General Relativity and Cosmology* (Chapman and Hall, 1956; 2nd ed. 1965), http://kirkmcd.princeton.edu/examples/GR/mcvittie_64.pdf

[14] J. Jaffe and I.I. Shapiro, *Lightlike Behavior of Particles in a Schwarzschild Field*, Phys. Rev. D **6**, 405 (1972), http://kirkmcd.princeton.edu/examples/GR/jaffe_prd_6_405_72.pdf

[15] M. Carmelli, *Behaviour of Fast Particles in the Schwarzschild Field*, Lett. Nuovo Cim. **6**, 5 (1973), http://kirkmcd.princeton.edu/examples/GR/carmelli_lnc_3_379_72.pdf

[16] F.L. Markley, *Motion in the Schwarzschild Metric*, Am. J. Phys. **41**, 45 (1973), http://kirkmcd.princeton.edu/examples/GR/markley_ajp_41_45_73.pdf

[17] G. Cavalleri and G. Spinelli, *Motion of Particles Entering a Schwarzschild Field*, Lett. Nuovo Cim. **6**, 5 (1973), http://kirkmcd.princeton.edu/examples/GR/cavalleri_lnc_6_5_73.pdf

[18] A.I. Janis, *Note on Motion in the Schwarzschild Field*, Phys. Rev. D **18**, 2360 (1973), http://kirkmcd.princeton.edu/examples/GR/janis_prd_8_2360_73.pdf

[19] H. Treder and K. Fritze, *Die Fallbeschleunigung in der Schwarzschild-Metrik*, Astron. Nachr. **296**, 109 (1975), http://kirkmcd.princeton.edu/examples/GR/treder_an_296_109_75.pdf

[20] G. Cavalleri and G. Spinelli, *Note on Motion in the Schwarzschild Field*, Phys. Rev. D **15**, 3065 (1977), http://kirkmcd.princeton.edu/examples/GR/cavelleri_prd_15_3065_77.pdf

[21] A.I. Janis, *Motion in the Schwarzschild field: A reply*, Phys. Rev. D **15**, 3068 (1977), http://kirkmcd.princeton.edu/examples/GR/janis_prd_15_3068_77.pdf

[22] M. Enosh and A. Kovetz, *Is Active Gravitational Mass Equal to Inertial Mass?* Int. J. Theor. Phys. **17**, 549 (1978), http://kirkmcd.princeton.edu/examples/GR/enosh_ijtp_17_549_78.pdf

[23] M.P. Silverman, *Gravitational deflection of relativistic massive particles*, Am. J. Phys. **48**, 72 (1980), http://kirkmcd.princeton.edu/examples/GR/silverman_ajp_48_72_80.pdf

[24] C.H. McGruder III, *Gravitational Repulsion in the Schwarzschild field*, Phys. Rev. D **25**, 3195 (1982), http://kirkmcd.princeton.edu/examples/GR/mcgruder_prd_25_3191_82.pdf

[25] K.D. Krori, J.C. Sarmah and D. Goswami, *Gravitational repulsion in the Einstein-zero-mass scalar theory*, Can. J. Phys. **62**, 629 (1984), http://kirkmcd.princeton.edu/examples/GR/krori_cjp_62_629_84.pdf

[26] M. Barua, *Gravitational repulsion by a Reissner-Nordström Black Hole*, Indian J. Pure Appl. Phys. **23**, 634 (1985).

[27] K.D. Krori and M. Barua, *Gravitational repulsion by Kerr and Kerr-Newman black holes*, Phys. Rev. D **31**, 3135 (1985), http://kirkmcd.princeton.edu/examples/GR/krori_prd_31_3135_85.pdf

[28] J. Ponce de Leon, *Gravitational repulsion in sources of the Reissner-Nordström field*, J. Math. Phys. **29**, 197 (1988), http://kirkmcd.princeton.edu/examples/GR/poncedeleon_jmp_29_197_88.pdf

[29] I.P. Denisova, B.V. Mehta and A.A. Zubrilo, *The Investigation of the Model of Gravitational Repulsion in Einstein's General Theory of Relativity*, Gen. Rel. Grav. **31**, 821 (1999), http://kirkmcd.princeton.edu/examples/GR/denisova_grg_31_821_99.pdf

[30] L. Herrera, *Geodesics in a Quasi-Spherical Spacetime: A Case of Gravitational Repulsion*, Found. Phys. **18**, 21 (2005), http://kirkmcd.princeton.edu/examples/GR/herrera_fp_18_21_05.pdf

[31] J.B. Broekaert, *A "Lorentz-Poincaré"–Type Interpretation of Relativistic Gravitation*, Proc. EPS-13, 23 (2005), http://kirkmcd.princeton.edu/examples/GR/broekaert_esasp_63_23_05.pdf

[32] B. Mashoon, *Beyond Gravitoelectromagnetism: Critical Speed in Gravitational Motion*, Int. J. Mod. Phys. D **14**, 2025 (2005), http://kirkmcd.princeton.edu/examples/GR/mashhoon_ijmpd_14_2025_05.pdf

[33] C. Chicone and B. Mashoon, *Explicit Fermi coordinates and tidal dynamics in de Sitter and Gödel spacetimes*, Phys. Rev. D **74**, 064019 (2006), http://kirkmcd.princeton.edu/examples/GR/chicone_prd_74_064019_06.pdf

[34] L. Herrera and N.O. Santos, *A general relativistic effect in quasi-spherical objects as the possible origin of relativistic jets*, Astro. Space Sci. **310**, 251 (2007), http://kirkmcd.princeton.edu/examples/GR/herrera_ass_310_251_07.pdf

[35] J.B. Broekaert, *A Spatially-VSL Gravity Model with 1-PN Limit of GRT*, Found. Phys. **39**, 409 (2008), http://kirkmcd.princeton.edu/examples/GR/broekaert_fp_38_409_08.pdf

[36] F.S. Felber, *Exact 'antigravity-field' solutions of Einstein's equation*, (Oct. 28, 2008), https://arxiv.org/abs/0803.2864

[37] A. Loinger and T. Marsico, *On Hilbert's gravitational repulsion (A historical Note)*, (Apr. 9, 2009), https://arxiv.org/pdf/0904.1578.pdf

[38] A. Loinger and T. Marsico, *Protons do not exert any Hilbertian gravitational repulsion*, (Dec. 7, 2009), https://arxiv.org/abs/0912.1323

[39] M. Kutschera and W. Zajiczek, *Shapiro effect for relativistic particles — testing General Relativity in a new window*, Acta Phys. Polon. B **41**, 1237 (2010), http://kirkmcd.princeton.edu/examples/GR/kutschera_appb_41_1237_10.pdf

[40] A. Spallicci, *Free fall and self-force: an historical perspective*, (May 4, 2010), https://arxiv.org/abs/1005.0611

[41] F. Felber, *Test of Relativistic Gravity for Propulsion at the Large Hadron Collider*, AIP Conf. Proc. **1208**, 247 (2010), http://kirkmcd.princeton.edu/examples/GR/felber_aipcp_1208_247_10.pdf

[42] H.C. Ohanian, *Reversed Gravitational Acceleration for High-speed Particles* (Nov. 11, 2011), https://arxiv.org/abs/1102.2870

[43] F.G. Winkler, *Diagonal Metrics of Static, Spherically Symmetric Fields: The Geodesic Equations and the Mass-Energy Relation from the Coordinate Perspective*, Int. J. Theor. Phys. **52**, 3045 (2013), http://kirkmcd.princeton.edu/examples/GR/winkler_ijtp_52_3045_13.pdf

[44] M.O. Katanaev, *Point massive particle in General Relativity*, Gen. Rel. Grav. **45**, 1861 (2013), http://kirkmcd.princeton.edu/examples/GR/katanaev_grg_45_1861_13.pdf

[45] A. Spallicci and P. Ritter, *A fully relativistic radial fall*, Int. J. Geom. Meth. Mod. Phys. **11**, 1450090 (2014), http://kirkmcd.princeton.edu/examples/GR/spallicci_ijgmmp_11_1450090_14.pdf

[46] J. Gariel, N.O. Santos and J. Silk, *Unbound geodesics from the ergosphere and potential observability of debris from ultrahigh energy particle collisions*, Phys. Rev. D **90**, 06305 (2014), http://kirkmcd.princeton.edu/examples/GR/gariel_prd_90_063505_14.pdf

[47] J. Gariel, N.O. Santos and A. Wang, *Kerr geodesics following the axis of symmetry*, Gen. Rel. Grav. **48**, 66 (2016), http://kirkmcd.princeton.edu/examples/GR/gariel_grg_48_66_16.pdf

[48] A. Spallicci and M. van Putten, *Gauge dependence and self-force in Galilean to Einsteinian free fall, compact stars falling into black holes, Hawking radiation and the Pisa tower at the general relativity centennial*, Int. J. Geom. Meth. Mod. Phys. **13**, 1630014 (2016), http://kirkmcd.princeton.edu/examples/GR/spallicci_ijgmmp_13_16300014_16.pdf

[49] N. Gorkavyi and A. Vasilkov, *A repulsive force in the Einstein theory*, MNRAS **461**, 2929 (2016), http://kirkmcd.princeton.edu/examples/GR/gorkavyi_mnras_461_2929_16.pdf

[50] C.H. McGruder III, *Acceleration of particles to high energy via gravitational repulsion in the Schwarzschild field* (Sept. 17, 2018), https://arxiv.org/abs/1802.08546

[51] M.-N. Célérier, N.O. Santos and V.H. Satheeshkumar, *Can gravity be repulsive?* (Nov. 6, 2018), https://arxiv.org/abs/1606.08300

[52] D.L. Berkahn, J.M. Chappell and D. Abbott, *Hilbert's forgotten equation, the equivalence principle and velocity dependence of free fall*, Eur. J. Phys. **41**, 035604 (2020), http://kirkmcd.princeton.edu/examples/GR/berkahn_ejp_41_035604_20.pdf

[53] L. Page, *Gravitational Deflection of High-speed Particles*, Nature **104**, 692 (1920), http://kirkmcd.princeton.edu/examples/GR/page_nature_104_692_20.pdf
Nature **105**, 233 (1920), http://kirkmcd.princeton.edu/examples/GR/page_nature_105_233_20.pdf

[54] A.S. Eddington, *Gravitational Deflection of High-speed Particles*, Nature **105**, 37 (1920), http://kirkmcd.princeton.edu/examples/GR/page_nature_105_37_20.pdf

[55] L.B. Okun, *The Concept of Mass*, Phys. Today **42** (6), 31 (1989), http://kirkmcd.princeton.edu/examples/GR/okun_pt_42(6)_31_89.pdf

[56] J. Wisdom, *Swimming in Spacetime: Motion by Cyclic Changes in Body Shape*, Science **299**, 1865 (2003), http://kirkmcd.princeton.edu/examples/GR/wisdom_science_299_1865_03.pdf

[57] E. Gueŕon, C.A.S. Maia and G.E.A. Matsas, *Swimming versus swinging effects in spacetime*, Phys. Rev. D **73**, 024020 (2006), http://kirkmcd.princeton.edu/examples/GR/gueron_prd_73_024020_06.pdf

[58] J.E. Avron and O. Kenneth, *Swimming in curved space or the Baron and the cat*, New J. Phys. **8**, 69 (2006), http://kirkmcd.princeton.edu/examples/GR/avron_njp_8_69_06.pdf

[59] E. Gueŕon and R.A. Mosna, *Relativistic glider*, Phys. Rev. D **75**, 081501 (2007), http://kirkmcd.princeton.edu/examples/GR/gueron_prd_75_081501_07.pdf

[60] L. Bergamin, P. Delva and A. Hees, *Vibrating systems in Schwarzschild spacetime: toward new experiments in gravitation?* Class. Quant. Grav. **26**, 195006 (2009), http://kirkmcd.princeton.edu/examples/GR/bergamin_cqg_26_185006_09.pdf

[61] S. Li *et al.*, *Robotic swimming in curved space via geometric phase*, Proc. Nat. Acad. Sci. **119**, 2200924119 (2022), http://kirkmcd.princeton.edu/examples/mechanics/li_pnas_119_2200924119.pdf

[62] K.T. McDonald, *The Falling Cat and All That: A Short Story of the "Geometric Phase"* (Nov. 10, 2023), http://kirkmcd.princeton.edu/examples/cat.pdf

[63] M.A. Abramowicz and J.P. Lacosta, *On traveling round without feeling it and uncurving curves*, Am. J. Phys. **54**, 936 (1986), http://kirkmcd.princeton.edu/examples/GR/abramowicz_ajp_54_936_86.pdf

[64] S. Weinberg, *Gravitation and Cosmology* (Wiley, 1972), http://kirkmcd.princeton.edu/examples/GR/weinberg_72.pdf

[65] J.L. Synge, *The Gravitational Field of a Particle*, Proc. Roy. Irish Acad. A **53**, 83 (1950), http://kirkmcd.princeton.edu/examples/GR/synge_pria_a53_83_50.pdf

[66] M. Born, *Physics and Relativity*, Helv. Phys. Acta **29**(Suppl. 4), 244 (1956), p. 253, http://kirkmcd.princeton.edu/examples/GR/born_hpa_29-s4_244_56.pdf

[67] M.P. Silverman, *Gravitational deflection of relativistic massive particle*, Am. J. Phys. **48**, 72 (1980), http://kirkmcd.princeton.edu/examples/GR/silverman_ajp_48_72_80.pdf

[68] K. Schwarzschild, *Über das Gravitationsfeld eines Massenpunktes nach der EINSTEINschen Theorie*, S.K.P. Akad. Wiss. 189 (1916),
http://kirkmcd.princeton.edu/examples/GR/schwarzschild_skpaw_189_16.pdf
On the Gravitational Field of a Point-Mass, According to Einstein's Theory,
http://kirkmcd.princeton.edu/examples/GR/schwarzschild_skpaw_189_16_english.pdf

[69] W. de Sitter, *Einstein's Theory of Gravitation, and its Astronomical Consequences*, Mon. Not. Roy. Astron. Soc. **76**, 699 (1916),
http://kirkmcd.princeton.edu/examples/GR/desitter_mnras_76_699_16.pdf
Mon. Not. Roy. Astron. Soc. **77**, 155 (1916),
http://kirkmcd.princeton.edu/examples/GR/desitter_mnras_77_155_16.pdf
Mon. Not. Roy. Astron. Soc. **78**, 3 (1917),
http://kirkmcd.princeton.edu/examples/GR/desitter_mnras_78_3_17.pdf

[70] J.T. Jebsen, *Über die allgemeinen kugelsymmetrischen Lösungen der Einsteinschen Gravitationsgleichungen im Vakuum*, Ark. Math. Astr. Fys. **15**(18), 1 (1921),
On the general spherically symmetric solutions of Einstein's gravitational equations in vacuo, http://kirkmcd.princeton.edu/examples/GR/jebsen_amaf_15(18)_1_21_english.pdf

[71] J. Eiesland, *The Group of Motions of an Einstein Space*, Trans. Am. Math. Soc. **27**, 213 (1925), http://kirkmcd.princeton.edu/examples/GR/eiesland_tams_27_213_25.pdf

[72] W. Alexandrow, *Über den kugelsymmetroschen Vakuurmvorgang in der Einteinschen Gravitationstheorie*, Ann. d. Phys. **72**, 141 (1923),
http://kirkmcd.princeton.edu/examples/GR/alexandrow_ap_72_141_23.pdf

[73] G.D. Birkhoff, *Relativity and Modern Physics* (Harvard U.P., 1923), p. 253,
http://kirkmcd.princeton.edu/examples/GR/birkhoff_23.pdf

[74] N.J. Johansen and F. Ravndal, *On the discovery of Birkhoff's theorem*, Gen. Rel. Grav. **38**, 213 (2006), http://kirkmcd.princeton.edu/examples/GR/johansen_grg_38_537_06.pdf

[75] R.P. Kerr, *Gravitational Field of a Spinning Mass as an Example of Algebraically Special Metrics*, Phys. Rev. Lett. **11**, 237 (1963),
http://kirkmcd.princeton.edu/examples/GR/kerr_prl_11_237_63.pdf
Discovering the Kerr and Kerr-Schild metrics (Jan. 14, 2008), http://arxiv.org/abs/0706.1109

[76] L.S. Abrams, *Black holes: the legacy of Hilbert's error*, Can. J. Phys. **67**, 919 (1989),
http://kirkmcd.princeton.edu/examples/GR/abrams_cjp_67_919_89.pdf

[77] S.J. Crothers, *On the General Solution to Einsteins Vacuum Field and Its Implications for Relativistic Degeneracy*, Prog. Phys. **1**(1), 68 (2005),
http://kirkmcd.princeton.edu/examples/GR/crothers_pp_1(1)_68_05.pdf
http://www.sjcrothers.plasmaresources.com/

[78] L. Flamm, *Beiträge zur Einsteinschen Gravitationstheorie* Phys. Z. **19**, 197 (1917),
http://kirkmcd.princeton.edu/examples/GR/flamm_pz_17_448_16.pdf
Contributions to Einsteins theory of gravitation,
http://kirkmcd.princeton.edu/examples/GR/flamm_pz_17_448_16_english.pdf

[79] H. Weyl, *Zur Gravitationstheorie*, Ann. d. Phys. **54**, 117 (1917),
http://kirkmcd.princeton.edu/examples/GR/weyl_ap_54_117_17.pdf
On the theory of gravitation, http://kirkmcd.princeton.edu/examples/GR/weyl_ap_54_117_17_english.pdf

[80] W. Pauli, *Relativitätstheorie*, Ency. Math. Phys. **5**(2), 539 (1921),
http://kirkmcd.princeton.edu/examples/GR/pauli_emp_5_2_539_21.pdf
Theory of Relativity (Pergamon, 1958).

[81] K. Lanzcos, *Ein vereinfachendes Koordinatensystem für die Einsteinschen Gravitationsgleichungen*, Phys. Z. **23**, 537 (1923),
http://kirkmcd.princeton.edu/examples/GR/lanczos_pz_23_537_23.pdf

[82] V. Fock, *The Theory of Space Time and Gravitation* (Pergamon, 1959),
http://kirkmcd.princeton.edu/examples/GR/fock_spacteime_59.pdf

[83] M. Brillouin, *Les Points Singuliers de l'Univers d'Einstein*, J. Phys. Rad. **4**, 43 (1923),
http://kirkmcd.princeton.edu/examples/GR/brillouin_jpr_4_43_23.pdf
The singular points of Einstein's universe,
http://kirkmcd.princeton.edu/examples/GR/brillouin_jpr_4_43_23_english.pdf

[84] E. Serini, *Euclideità dello spazio completamente vuoto nella relatività generale di Einstein*, Rend. Reale Acad. Lincei **27**, 235 (1918),
http://kirkmcd.princeton.edu/examples/GR/serini_rral_27_235_13.pdf

[85] A. Einstein and W. Pauli, *On the Non-Existence of Regular Stationary Solutions of Relativistic Field Equations*, Ann. Math. **44**, 131 (1943),
http://kirkmcd.princeton.edu/examples/GR/einstein_am_44_131_43.pdf

[86] G. Lemaître, *L'Univers en expansion*, Ann. Soc. Sci. Bruxelles A **53**, 51 (1933),
http://kirkmcd.princeton.edu/examples/GR/lemaitre_assb_a53_51_33.pdf
The Expanding Universe,
http://kirkmcd.princeton.edu/examples/GR/lemaitre_assb_a53_51_33_english.pdf

[87] A. Gullstrand, *Allgemeine Lösung des statischen Einkörperproblems in der Einsteinschen Gravitationstheorie*, Ark. Math. Astr. Fys. **16**(8), 1 (1922),
http://kirkmcd.princeton.edu/examples/GR/gullstrand_amaf_16(8)_1_22.pdf

[88] P. Painlevé, *La Mécanique classique et la théorie de la relativité*, Comp. Rend. Acad. Sci. **173**, 677 (1921), http://kirkmcd.princeton.edu/examples/GR/painleve_cras_173_677_21.pdf
La gravitation dans la Mécanique de Newton et dans la Mécanique d'Einstein, Comp. Rend. Acad. Sci. **173**, 873 (1921),
http://kirkmcd.princeton.edu/examples/GR/painleve_cras_173_873_21.pdf

[89] G. Szekeres, *On the singularities of a Riemannian manifold*, Pub. Math. Debrecen **7**, 285 (1960), http://kirkmcd.princeton.edu/examples/GR/szekeres_pmd_7_285_60.pdf

[90] M.D. Kruskal, *Maximal Extension of Schwarzschild Metric*, Phys. Rev. **119**, 1743 (1960), http://kirkmcd.princeton.edu/examples/GR/kruskal_pr_119_1743_60.pdf

[91] R. Penrose, *Gravitational Collapse and Space-Time Singularities*, Phys. Rev. Lett. **14**, 57 (1965), http://kirkmcd.princeton.edu/examples/GR/penrose_prl_14_57_65.pdf

[92] D. Finkelstein, *Past-Future Asymmetry of the Gravitational Field of a Point Particle*, Phys. Rev. **110**, 965 (1958), http://kirkmcd.princeton.edu/examples/GR/finkelstein_pr_110_965_58.pdf

[93] A.S. Eddington, *A Comparison of Whitehead's and Einstein's Formulæ*, Nature **113**, 192 (1924), http://kirkmcd.princeton.edu/examples/GR/eddington_nature_113_192_24.pdf

[94] R. Penrose, *Asymptotic Properties of Fields and Space-Times*, Phys. Rev. Lett. **10**, 66 (1963), http://kirkmcd.princeton.edu/examples/GR/penrose_prl_10_66_63.pdf

[95] L. Susskind and A. Cabannes, *General Relativity* (Basic Books, 2022). https://cosmolearning.org/courses/theoretical-minimum-iv-general-relativity/

[96] H. Dingle, *Values of T_μ^ν, and the Christoffel Symbols for a Line Element of Considerable Generality*, Proc. Nat. Acad. Sci. **19**, 959 (1933), http://kirkmcd.princeton.edu/examples/GR/dingle_pnas_19_959_33.pdf

[97] K.T. McDonald, *The Clock Paradox and Accelerometers* (Jan. 11, 2020), http://kirkmcd.princeton.edu/examples/clock.pdf

[98] H.B. Dwight, *Tables of Integrals and Other Mathematical Data*, $4^{\rm th}$ ed. (Macmillan, 1961), http://kirkmcd.princeton.edu/examples/EM/dwight_57.pdf

[99] J.A. Wheeler, *A Journey Into Gravity and Spacetime* (W.H. Freeman, 1990), http://kirkmcd.princeton.edu/examples/GR/wheeler_90.pdf

[100] M. Pössel, *The Shapiro time delay and the equivalence principle* (Dec. 31, 2019), https://arxiv.org/abs/2001.00229

[101] I.I. Shapiro, *Fourth Test of General Relativity*, Phys. Rev. Lett. **13**, 789 (1964), http://kirkmcd.princeton.edu/examples/GR/shapiro_prl_13_789_64.pdf

[102] I.I. Shapiro *et al.*, *Fourth Test of General Relativity: Preliminary Results*, Phys. Rev. Lett. **20**, 265 (1968), http://kirkmcd.princeton.edu/examples/GR/shapiro_prl_20_1265_68.pdf

[103] I.I. Shapiro *et al.*, *Fourth Test of General Relativity: New Radar Result*, Phys. Rev. Lett. **26**, 1132 (1971), http://kirkmcd.princeton.edu/examples/GR/shapiro_prl_26_1132_71.pdf

[104] N.E.J. Bjerrum-Bohr *et al.*, *Bending of Light in Quantum Gravity*, Phys. Rev. Lett. **114**, 061301 (2015), http://kirkmcd.princeton.edu/examples/GR/bjerrum-bohr_prl_114_061301_15.pdf

[105] G. Galilei, *Discorsi e Dimonstazioni Matematiche intorno a due nuove scienze attenenti alla mecanica & i movimenti locali* (1638),
http://kirkmcd.princeton.edu/examples/mechanics/galileo_discorsi_38.pdf
Dialogues concerning Two New Sciences (Macmillan, 1914),
http://kirkmcd.princeton.edu/examples/mechanics/galileo_two_new_sciences.pdf

[106] R.H. Dicke, in *Relativity, Groups, and Topology*, C. DeWitt and B. DeWitt, eds. (Gordon & Breach, 1964), p. 168. http://kirkmcd.princeton.edu/examples/GR/dicke_rgt_168_63.pdf

[107] H.C. Ohanian, *What is the Equivalence Principle?* Am. J. Phys. **45**, 903 (1977),
http://kirkmcd.princeton.edu/examples/GR/ohanian_ajp_45_903_77.pdf

[108] A.C. Melissinos, *Experimental Tests of Newtonian Gravity at Relativistic Velocities*, Nuovo Cim. B **62**, 190 (1981),
http://kirkmcd.princeton.edu/examples/GR/meslissinos_nc_62b_190_81.pdf

[109] P. Reiner *et al.*, *Search for Long Range Interactions at Highly Relativistic Velocities*, Phys. Lett. B **176**, 233 (1986),
http://kirkmcd.princeton.edu/examples/GR/reiner_pl_176b_233_86.pdf

[110] C. Møller, *The Theory of Relativity* (Oxford U. Press, 1952),
http://kirkmcd.princeton.edu/examples/GR/moller_relativity_52.pdf

[111] C.G. Adler and R.W. Brehme, *Relativistic solutions to the falling body in a uniform gravitation field*, Am. J. Phys. **59**, 209 (1991),
http://kirkmcd.princeton.edu/examples/GR/adler_ajp_59_209_91.pdf

[112] M. Born, *Die Theorie des starren Elektrons in der Kinematik des Relativitätsprinzips*, Ann. d. Phys. **30**, 1 (1909), http://kirkmcd.princeton.edu/examples/EM/born_ap_30_1_09.pdf
The Theory of the Rigid Electron in the Kinematics of the Principle of Relativity,
http://kirkmcd.princeton.edu/examples/GR/born_ap_30_1_09_english.pdf

[113] A. Einstein, *Die formale Grundlage der allgemeinen Relativitätstheorie*, Sitzber. König. Preuss. Akad. Wiss., 1030 (1914),
http://kirkmcd.princeton.edu/examples/GR/einstein_skpaw_1030_14.pdf
translated as *The Formal Foundations of the General Theory of Relativity*,
http://kirkmcd.princeton.edu/examples/GR/einstein_skpaw_1030_14_english.pdf

[114] Y. Hagihara, *Theory of the Relativistic Trajectories in a Gravitational Field of Schwarzschild*, Japan. J. Astron. Geophys. **8**, 67 (1931),
http://kirkmcd.princeton.edu/examples/GR/hagihara_jjag_8_67_31.pdf

[115] A. Einstein, *On a Stationary System With Spherical Symmetry Consisting of Many Gravitating Masses*, Ann. Math. **40**, 922 (1939),
http://kirkmcd.princeton.edu/examples/GR/einstein_am_40_922_39.pdf

[116] K.S. Thorne, *Time Warps and Black Holes* (W.W. Norton, 1994),
http://kirkmcd.princeton.edu/examples/GR/thorne_94.pdf

[117] W.H. McCrea, *A Time-keeping Problem Connected with the Gravitational Red-shift*, Helv. Phys. Acta. Suppl. **4**, 121 (1956),
http://kirkmcd.princeton.edu/examples/GR/mccrea_hva_4s_121_56.pdf

[118] C. Darwin, *The gravity field of a particle*, Proc. Roy. Soc. London A **246**, 180 (1959),
http://kirkmcd.princeton.edu/examples/GR/darwin_prsla_246_180_59.pdf

[119] C. Darwin, *The gravity field of a particle. II*, Proc. Roy. Soc. London A **263**, 39 (1961),
http://kirkmcd.princeton.edu/examples/GR/darwin_prsla_263_39_61.pdf

[120] P. Goldhammer, *A Note on Circular Gravitational Orbits*, Lett. Nuovo Cim. **20**, 1205 (1961), http://kirkmcd.princeton.edu/examples/GR/goldhammer_nc_1205_61.pdf

[121] A.W.K. Metzner, *Observable Properties of Large Relativistic Masses*, J. Math. Phys. **4**, 1194 (1963), http://kirkmcd.princeton.edu/examples/GR/metzner_jmp_4_1194_63.pdf

[122] C.W. Misner, K.S. Thorne and J.A. Wheeler, *Gravitation* (W.H. Freeman, 1973),
http://kirkmcd.princeton.edu/examples/GR/mtw_73.pdf

[123] B.P. Abbott *et al.*, *GWTC-1: A Gravitational-Wave Transient Catalog of Compact Binary Mergers Observed by LIGO and Virgo during the First and Second Observing Runs*, Phys. Rev. X **9**, 031040 (2019),
http://kirkmcd.princeton.edu/examples/GR/abbott_prx_9_031040_19.pdf

[124] D. Kennefick, *Travelling at the Speed of Thought: Einstein and the Quest for Gravitational Waves* (Princeton U. Press, 2007),
http://kirkmcd.princeton.edu/examples/GR/kennefick_06.pdf

[125] D. Kennefick, *Controversies in the History of the Radiation Reaction problem in General Relativity* (Apr. 1, 1997), https://arxiv.org/abs/gr-qc/9704002

[126] F. Pretorius, *Evolution of Binary Black-Hole Spacetimes*, Phys. Rev. Lett. **95**, 121101 (2005), http://kirkmcd.princeton.edu/examples/GR/pretorius_prl_95_121101_05.pdf