"Hidden" Momentum in a Linked Pair of Gyrostats?

Kirk T. McDonald
Joseph Henry Laboratories, Princeton University, Princeton, NJ 08544
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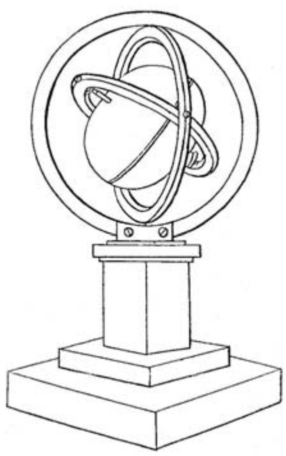
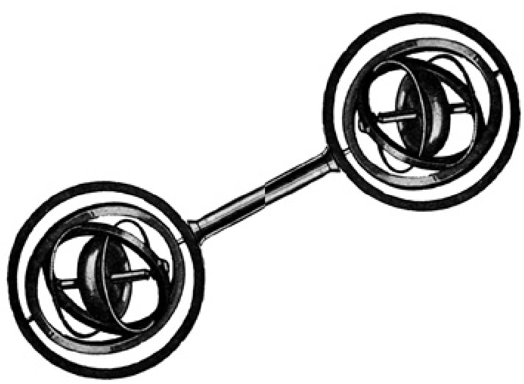
1 Problem

A gyrostat is a massive sphere (or disk) mounted such that the supporting apparatus exerts no torque about the center of the sphere. The earliest gyrostat was described by Bohnenberger [1] in 1817, as shown below. The name gyroscope was applied to this apparatus by Foucault [2] in 1852, although this term has come to imply a sphere or disk mounted with a fixed point on its axis. Lord Kelvin used the term gyrostat in 1874 [3], but in the more modern sense of gyroscope.



Consider here a pair of gyrostats joined by a link, as sketched below. Discuss the motion of the system when subject to no external forces, and when the "spin" angular momentum of the gyrostats is large compared to the orbital angular momentum of the system.



Show that if the spin angular velocity vectors of the two gyrostats are antiparallel, and both lie in the plane of their orbits, then the gyrostats "bob" (oscillate) perpendicular to the plane of the orbit, although the center of mass of the whole system is at rest

This problem is a mechanical analog of a pair of high-spin neutron stars [4].

2 Solution

We consider the problem in the lab frame where the center of mass of the system is at rest. Each sphere has rest mass M_0, radius a, and the separation between the centers of the spheres is $d = 2b$.

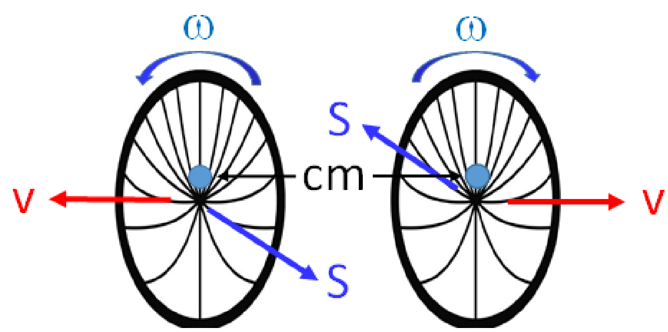
We approximate the link and mounting apparatus of the spheres as massless. Then, the link can only exert a tension along its length.

In the first approximation the spin angular momentum is unaffected by the orbital motion of the spheres (and the center of mass of each sphere is at it geometric center).[1] The force between the spheres, due to the link, is then a central force, so the system has constant orbital angular momentum, and the motion is simply uniform circular motion in a plane, with angular velocity vector $\mathbf{\Omega}$. Taking $\mathbf{b}$ to be the distance vector from the center of mass of the system to the center of the first sphere, its center has velocity $\mathbf{\Omega} \times \mathbf{b}$ (with $\mathbf{\Omega} \cdot \mathbf{b} = 0$), and the second sphere has the opposite velocity. We assume that $\Omega b \ll c$, where c is the speed of light in vacuum,

2.1 Perturbed Motion

In the next approximation, we note that the portion of a sphere whose spinning motion is in the same direction as the orbital motion of its center has larger relativistic mass/energy than the portion whose spinning motion is in the opposite direction. This results in a shift of the center of mass/energy, $\mathbf{x}_{\mathrm{cm},i}$ of sphere i away from its geometric center, illustrated in the figure below (adapted from [7]), with $\mathbf{x}_i$, given by [8],[2]

$$\mathbf{x}_{\mathrm{cm},i} \approx \mathbf{x}_i + \frac{\mathbf{v}_i \times a^2 k \boldsymbol{\omega}_i}{c^2} \approx \mathbf{x}_i + \frac{\mathbf{v}_i \times \mathbf{S}_i}{M_i c^2}\,, \tag{1}$$



where the spin angular momenta of the spheres are (to zeroth order,

$$\mathbf{S}_i = k M_i a^2 \boldsymbol{\omega}_i, \tag{2}$$

$k = 2/5$ for a uniform sphere, $k = 2/3$ for a spherical shell, $\boldsymbol{\omega}_i$ are the spin angular velocities of the spheres (with $\omega_i a \ll c$), whose centers have velocities,

$$\mathbf{v}_i \approx \pm \mathbf{\Omega} \times \mathbf{b}, \tag{3}$$

[1]If the mass distribution of the spheres were not spherically symmetric the motion would be quite complicated [5], being an extension of Poinsot's solution [6] for the free motion of a rigid body. When relativistic effects are included below, the mass/energy distribution is not spherically symmetric, but the spinning spheroids do not behave as rigid bodies, so the analysis of [5] does not apply.

[2]The spinning sphere can be regarded as a set of spinning hoops with a common axis.

the masses ($=$ energy$/c^2$) of the spheres in the lab frame are,

$$M_i \approx M_0 \left(1 + \frac{v_i^2}{2c^2} + \frac{k\omega_i^2 a^2}{2c^2}\right), \tag{4}$$

and the approximations in eqs. (1) and (4) are accurate to order $1/c^2$.

For simplicity we now assume that the two spin angular velocities have the same magnitude (but not necessarily the same direction). Then, $M_1 \approx M_2$, and since the center of mass/energy of the system is at the origin (by definition), we have from eqs. (1) and (3) that,

$$0 = 2\mathbf{x}_{\rm cm} \approx \mathbf{x}_{{\rm cm},1} + \mathbf{x}_{{\rm cm},2} \approx \mathbf{x}_1 + \mathbf{x}_2 + ka^2 \frac{[(\boldsymbol{\omega}_1 - \boldsymbol{\omega}_2)\cdot\boldsymbol{\Omega}]\mathbf{b} - [(\boldsymbol{\omega}_1 - \boldsymbol{\omega}_2)\cdot\mathbf{b}]\boldsymbol{\Omega}}{c^2}\,. \tag{5}$$

Writing the perturbed position of the center of the link as $\mathbf{x}_0 = \delta\,\hat{\mathbf{b}} + \epsilon\,\hat{\boldsymbol{\Omega}}$, the centers of the spheres are at $\mathbf{x}_i = \mathbf{x}_0 \pm \mathbf{b}$, such that, $\mathbf{x}_1 + \mathbf{x}_2 = 2\delta\,\hat{\mathbf{b}} + 2\epsilon\,\hat{\boldsymbol{\Omega}}$, and we find from eq. (5) that

$$\delta \approx -\frac{ka^2 b\,\Omega}{2c^2}(\boldsymbol{\omega}_1 - \boldsymbol{\omega}_2)\cdot\hat{\boldsymbol{\Omega}}, \qquad \epsilon \approx \frac{ka^2 b\,\Omega}{2c^2}(\boldsymbol{\omega}_1 - \boldsymbol{\omega}_2)\cdot\hat{\mathbf{b}}. \tag{6}$$

The perturbations are maximal when $\boldsymbol{\omega} = \boldsymbol{\omega}_1 = -\boldsymbol{\omega}_2$, in which case,

$$\delta \approx -\frac{ka^2 b\,\Omega\omega}{c^2}\hat{\boldsymbol{\omega}}\cdot\hat{\boldsymbol{\Omega}} \equiv -\frac{ka^2 b\,\Omega\omega}{c^2}\cos\theta, \qquad \epsilon \approx \frac{ka^2 b\,\Omega\omega}{c^2}\hat{\boldsymbol{\omega}}\cdot\hat{\mathbf{b}} \equiv \frac{ka^2 b\,\Omega\omega}{c^2}\cos\theta\cos\Omega t, \tag{7}$$

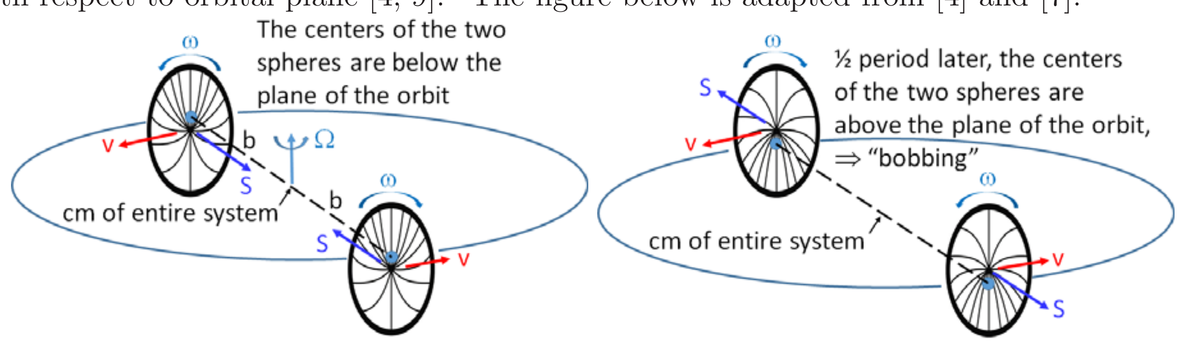
where θ is the angle between $\boldsymbol{\omega}$ and $\boldsymbol{\Omega}$, such that,

$$\hat{\boldsymbol{\omega}} = \cos\theta\,\hat{\boldsymbol{\Omega}} + \sin\theta\cos\Omega t\,\hat{\mathbf{b}} + \sin\theta\sin\Omega t\,\hat{\boldsymbol{\Omega}}\times\hat{\mathbf{b}}. \tag{8}$$

Noting that $\dot{\hat{\mathbf{b}}} = \boldsymbol{\Omega}\times\hat{\mathbf{b}}$, we have that the velocity of the center of the link is,

$$\dot{\mathbf{x}}_0 = \frac{\dot{\mathbf{x}}_1 + \dot{\mathbf{x}}_2}{2} = \delta\,\dot{\hat{\mathbf{b}}} + \dot{\epsilon}\,\hat{\boldsymbol{\Omega}} \approx -\frac{ka^2 b\,\Omega^2\omega}{c^2}(\cos\theta\,\hat{\boldsymbol{\Omega}}\times\hat{\mathbf{b}} + \sin\theta\sin\Omega t\,\hat{\boldsymbol{\Omega}}). \tag{9}$$

When $\boldsymbol{\omega}$ is parallel to $\boldsymbol{\Omega}$, the perturbation is in the plane of the orbits, whose geometric center $\mathbf{x}_0 = \delta\,\hat{\mathbf{b}}$ moves in a small circle with angular velocity $\boldsymbol{\Omega}$, which is hardly noticeable by a distant observer.[3] However, when the spin vector $\boldsymbol{\omega}$ is parallel to the plane of the orbit the perturbation is perpendicular to this plane and the center $\mathbf{x}_0 = \epsilon\,\hat{\boldsymbol{\Omega}}$ of the orbit oscillates sinusoidally with the orbital angular frequency $\boldsymbol{\Omega}$; the spheres appear to "bob" up and down with respect to orbital plane [4, 9].[4] The figure below is adapted from [4] and [7].



[3]This behavior is similar to that of a washing machine with an unbalanced load during its spin cycle [10]. The shaft of the drum moves in a small circle such that the center of mass of the unbalanced drum is at rest on the axis of rotation of a balanced drum.

[4]This effect appears to have been first noticed in numerical simulations, fig.3 of [9], having been missed in analytic studies such as [11, 12, 13, 14].

In more detail, we note that the velocity of the center of sphere i is,

$$\mathbf{v}_i = \dot{\mathbf{x}}_i = \dot{\mathbf{x}}_0 \pm \dot{\mathbf{b}} \approx -\frac{ka^2 b\,\Omega^2\omega}{c^2}(\cos\theta\,\hat{\mathbf{\Omega}} \times \hat{\mathbf{b}} + \sin\theta \sin\Omega t\,\hat{\mathbf{\Omega}}) \pm \mathbf{\Omega} \times \mathbf{b}, \tag{10}$$

and that the acceleration of the center of sphere i is,

$$\mathbf{a}_i = \dot{\mathbf{v}}_i \approx \pm\ddot{\mathbf{b}} \approx \pm\mathbf{\Omega} \times (\mathbf{\Omega} \times \mathbf{b}) = \mp\Omega^2\,\mathbf{b}. \tag{11}$$

The velocity of the center of mass/energy of sphere i is, from eqs. (1) and (8),

$$\begin{aligned} \dot{\mathbf{x}}_{\mathrm{cm},i} &\approx \dot{\mathbf{x}}_i \mp \frac{ka^2 \boldsymbol{\omega} \times \mathbf{a}_i}{c^2} \\ &\approx -\frac{ka^2 b\,\Omega^2\omega}{c^2}(\cos\theta\,\hat{\mathbf{\Omega}} \times \hat{\mathbf{b}} + \sin\theta \sin\Omega t\,\hat{\mathbf{\Omega}}) \pm \mathbf{\Omega} \times \mathbf{b} + \frac{ka^2 b\,\Omega^2\omega}{c^2}\hat{\boldsymbol{\omega}} \times \hat{\mathbf{b}} \\ &= \pm\mathbf{\Omega} \times \mathbf{b}, \end{aligned} \tag{12}$$

which has no component perpendicular to the link, *i.e.*, no component along $\mathbf{\Omega}$. As expected from eq. (5), $\dot{\mathbf{x}}_{\mathrm{cm,total}} \approx \dot{\mathbf{x}}_{\mathrm{cm},1} + \dot{\mathbf{x}}_{\mathrm{cm},2} = 0$.

2.2 "Hidden" Momentum

Although the perturbations to the motion are visible to the careful observer, the system has been characterized in [4] as possessing "hidden" momentum. This term was introduced by Shockley [15] to describe certain electromechanical systems that contain nonzero electromagnetic field momentum while seeming to be at rest, with apparently zero total mechanical momentum.

The mechanical system of the present example does have its center of mass/energy at rest, and the total momentum of the system is always zero, while the center of the orbital motion (center of the link) oscillates with respect to the center of mass/energy. A naïve observer might assume that there is net momentum associated with the oscillation, in which case there might be an equal and opposite "hidden" momentum in the orbiting, spinning spheres.

Does this system actually contain "hidden" momentum in the sense introduced by Shockley?

To answer this we need a definition of "hidden" momentum that is applicable to all-mechanical systems as well as to electromechanical ones.

One possibility is to define the "relativistic" component of mechanical momentum to be "hidden", *i.e.*,

$$\mathbf{P}_{\mathrm{hidden}} \overset{?}{\equiv} (\gamma - 1)m_0\mathbf{v} = \mathbf{P} - m_0\mathbf{v} \approx \frac{v^2}{2c^2}m_0\mathbf{v}, \tag{13}$$

for a rest mass m_0 moving with velocity $\mathbf{v}$.[5] However, the momentum (13) is not really hidden in that if the momentum of the moving mass is measured the result is $\gamma m_0\mathbf{v}$ and not $m_0\mathbf{v}$.[6]

[5]The author contemplated the definition (13) in [16], but no longer favors it.

[6]It may be that arguing that the "mass" of a moving particle is just its rest mass m_0 [17] leads some people to consider that the "momentum" of a moving particle is just $m_0\mathbf{v}$. Then, $\gamma m_0\mathbf{v}$ is not the "momentum", and so $(\gamma - 1)m_0\mathbf{v}$ might be called the "hidden momentum".

The term "hidden" momentum appears in the text of [4] after their eq. (35) where it seems to be said that,

$$\mathbf{P}_{\rm hidden} \stackrel{?}{\equiv} = \mathbf{P} - m_0 \mathbf{v}_{\rm geometric}, \tag{14}$$

with the velocity $\mathbf{v}_{\rm geometric}$ of sphere i taken to be $\mathbf{v}_i$ of eq. (10) for the geometric center of the sphere, rather than the velocity $\mathbf{v}_{{\rm cm},i}$ of its center of mass/energy. This peculiar definition leads to the identification of "hidden" momentum as the quantity $kM_0 a^2 \boldsymbol{\omega}_i \times \mathbf{a}_i/c^2 = \mathbf{S}_i \times \mathbf{F}_i/M_0 c^2$ (which would appear in eq. (12) when multiplied by M_0 or M).

The author favors a definition of "hidden" momentum inspired by discussions with Daniel Vanzella [18],

$$\mathbf{P}_{\rm hidden} \equiv \mathbf{P} - M\mathbf{v}_{\rm cm} - \oint_{\rm boundary} (\mathbf{x} - \mathbf{x}_{\rm cm})\,(\mathbf{p} - \rho \mathbf{v}_b) \cdot d\mathbf{Area} = -\int \frac{f^0}{c} (\mathbf{x} - \mathbf{x}_{\rm cm})\, d{\rm Vol}, \tag{15}$$

where $\mathbf{P}$ is the total momentum of the subsystem, $M = U/c^2$ is its total "mass", U is its total energy, $\mathbf{x}_{\rm cm}$ is its center of mass/energy, $\mathbf{v}_{\rm cm} = d\mathbf{x}_{\rm cm}/dt$, $\mathbf{p}$ is its momentum density, $\rho = u/c^2$ is its "mass" density, u is its energy density, $\mathbf{v}_b$ is the velocity (field) of its boundary, and,

$$f^\mu = \frac{\partial T^{\mu\nu}}{\partial x^\nu}, \tag{16}$$

is the 4-force density exerted on the subsystem by the rest of the system, with $T^{\mu\nu}$ being the stress-energy-momentum 4-tensor of the subsystem. The rationale for this definition, and links to several illustrative examples, are given in [19].[7,8]

A basic consequence of the definition (15) is that an isolated system, such as the present example, has zero total "hidden" momentum. An additional consequence is that spatially disjoint subsystems cannot contain "hidden" momentum. So, if we consider each gyrostat (plus half of the link) to be a separate subsystem, there will be no "hidden" momentum in either gyrostat. As discussed in [8], $\mathbf{P} = M\mathbf{v}_{\rm cm}$ for each spinning, translating sphere. Further, each of the integrals in eq. (15) vanishes for each subsystem, so there is no "hidden" momentum in either subsystem by this definition.

A pair of spinning neutron stars can contain "hidden" momentum (according to definition (15)) in its subsystems of matter and gravitational field. Likewise, an electromechanical analog of the neutrons stars as a pair of charge magnetic moments can contain "hidden" momentum in its subsystems of matter and electromagnetic fields. However, this "hidden" momentum has negligible effect on the motion of the overall geometric center of the matter subsystem, as acknowledged in [4].

[7]Steps towards the definition (15) appear in sec. I of [4], but the authors omit the boundary integral in their eq. (3) when integrating their eq. (2) by parts. The boundary integral does appear in eq. (2.16) of [20], which would be the same as eq. (15) if the terms labeled "kinetic" and "SSC" were replaced by the $M\mathbf{v}_{\rm cm}$ as suggested by eq. (2.14).

[8]As discussed in sec. 3 of [19], we do not advocate replacing $\mathbf{v}_{\rm cm}$ by $\mathbf{v}_{\rm centroid}$ in definition (15), where the centroid of sphere i is the position $\mathbf{x}_i$ of its geometric center.

2.3 Electromechanical Analog

An electromechanical analog of a pair of orbiting, spinning neutron stars is a pair of spheres with opposite electrical charges $\pm Q$ and equal rest masses M_0, which spheres are rotating with spin angular velocities $\boldsymbol{\omega}_i$ as well as orbiting about one another.

As noted in [21] the analysis is somewhat simpler if the charged spheres are taken to be spherical shells (in their rest frames), of radius a and $k = 2/3$. The mechanical angular momenta of the spinning shells, if their centers are at rest, are,

$$\mathbf{S}_{\mathrm{mech},i} = k M_0 a^2 \boldsymbol{\omega}_i, \tag{17}$$

and their electromagnetic-field spin angular momenta, relative to their geometric centers, are,

$$\begin{aligned}
\mathbf{S}_{\mathrm{EM},i} &= \int \mathbf{r}_i \times \frac{\mathbf{E}_i \times \mathbf{B}_i}{4\pi c}\, d\mathrm{Vol}_i = \frac{1}{4\pi c} \int \mathbf{r}_i \times \left(\frac{Q_i\, \hat{\mathbf{r}}_i}{r_i^2} \times \frac{3(\boldsymbol{\mu}_i \cdot \hat{\mathbf{r}}_i)\hat{\mathbf{r}}_i - \boldsymbol{\mu}_i}{r_i^3} \right) d\mathrm{Vol}_i \\
&= \frac{Q_i \mu_i}{4\pi c} \int \mathbf{r}_i \times \left(\frac{\hat{\mathbf{r}}_i}{r_i^2} \times \frac{2\cos\theta_i \hat{\mathbf{r}}_i + \sin\theta_i\, \hat{\boldsymbol{\theta}}_i}{r_i^3} \right) d\mathrm{Vol}_i = \frac{Q_i \boldsymbol{\mu}_i}{2c} \int \sin^3\theta_i\, d\theta_i \int_a^\infty \frac{dr_i}{r_i^2} \\
&= \frac{2 Q_i \boldsymbol{\mu}_i}{3ac} = \frac{2aQ^2 \boldsymbol{\omega}_i}{9c^2},
\end{aligned} \tag{18}$$

noting that the magnetic moments of the spinning charged shells have magnitudes,

$$\mu = \int \frac{\pi a^2 \sin^2\theta\, dI}{c} = \frac{\pi a^2}{c} \int \sin^2\theta\, Q \frac{2\pi a^2 \sin\theta\, d\theta}{4\pi a^2} \frac{\omega}{2\pi} = \frac{a^2 Q \omega}{3c}\,. \tag{19}$$

The ratio of the magnitudes of the spin angular momenta is,

$$\frac{S_{\mathrm{EM}}}{S_{\mathrm{mech}}} = \frac{Q^2}{3aM_0 c} = \frac{a_0}{3a}\,, \tag{20}$$

where $a_0 = Q^2/M_0c^2$ is the classical charge radius of the spherical shell, which is small compared to the electron's classical charge radius, 3×10^{-13} cm. Hence, the electromagnetic field angular momentum is very small compared to the mechanical angular momentum for any macroscopic (non-quantum) system of charged, spinning spherical shells, and the motion of the electromechanical system is identical to that of the all-mechanical system of sec. 2.1 to a very good approximation. See sec. V of [4] for greater detail.

However, the electromechanical system does possess "hidden" momentum in its electromagnetic and mechanical subsystems, according to definition (15).

We first note that a magnetic moment $\boldsymbol{\mu}$ which is at rest in an external electric field $\mathbf{E}^\star$ (due to, say, a distant electric charge also at rest in the $\star$ frame) has nonzero electromagnetic field momentum [22].[9]

$$\mathbf{P}^\star_{\mathrm{EM}} = \frac{\mathbf{E}^\star \times \boldsymbol{\mu}}{c}\,, \tag{21}$$

[9]Only an Ampèrian magnetic moment (due to electrical currents) has "hidden" momentum in an external electric field. A Gilbertian moment (due to a pair of opposite magnetic charges) would have none [23].

The center of energy of the electromagnetic field is at rest in the * frame, so we say that the electromagnetic field momentum (21) is "hidden" momentum according to definition (15).[10]

The mechanical subsystem of the spinning shell (and the distant electrical charge at rest) also has "hidden" momentum, which is equal and opposite to that of eq. (21). If the charged shell consists of charges embedded in a rigid nonconductor, the charges on the side of the spinning shell (taken for simplicity to have $\boldsymbol{\mu} \perp \mathbf{E}^\star$) closer to the distant (positive) charge are in a higher electrical potential and so have a higher electrical energy. The total energy of the charges is the same around a current loop on the shell (which spins like a rigid body), so the mechanical mass is lower on the side closer to the distance charge. Hence, the mechanical momentum of the moving charges is lower than for those on the opposite side of spinning shell, so there is a net mechanical momentum in the direction $\boldsymbol{\omega} \times \mathbf{E}^\star$, which is opposite to the vector $\mathbf{P}^\star_{\rm EM}$.[11]

The above argument is essentially unchanged on transforming to the lab frame, and to a good approximation the "hidden" momentum in the electromagnetic fields of the pair of charged spinning shells is, combining eqs. (19) and (21),

$$\mathbf{P}_{\rm hidden,EM} = \frac{a^2 Q^2}{12 b^2 c^2} \hat{\mathbf{b}} \times (\boldsymbol{\omega}_2 - \boldsymbol{\omega}_1), \tag{22}$$

and the mechanical "hidden" momentum is the negative of this. However, as noted above, and in [4], this "hidden" momentum has negligible effect on the motion of the orbiting, charge, spinning shells, which motion is related to the "ordinary" (though subtle at order $1/c^2$) mechanical properties of the system as in sec. 2.1.[12]

References

[1] V. Bohnenberger, *Beschreibung einer Machine zur Erläuterung der Gesetze der Umdrehung der Erde um ihre Axe, und der Veränderung der Lage der letzteren*, Tübinger Blätter für Naturwissenschaften und Arzneikunde, **3**, 72 (1817),
http://kirkmcd.princeton.edu/examples/mechanics/bohnenberger_tbna_3_72_17.pdf

[2] L. Foucault, *Sur les phénomènes d'orientation des corps tournants entraînés par un axe fixe à la surface de la terre – Nouveaux signes sensibles du mouvement diurne*, Compt. Rend. Acad. Sci. **35**, 424 (1852),
http://kirkmcd.princeton.edu/examples/mechanics/foucault_cras_35_424_52.pdf

[3] W. Thomson, *Vibrations and Waves in a stretched uniform Chain of Symmetrical Gyrostats*, Proc. London Math. Sci. **6**, 190 (1874),
http://kirkmcd.princeton.edu/examples/mechanics/thomson_plms_6_190_74.pdf

[10] The boundary of the electromagnetic field subsystem can be taken at infinity, so the boundary integral in eq. (15) is negligible.

[11] This argument was first given in footnote 9 of [24].

[12] See [25] for discussion of "paradoxes" in the behavior of a magnetic moment plus distant charge that both move with respect to the lab frame.

[4] S.E. Gralla, A.I. Harte, and R.M. Wald, *Bobbing and kicks in electromagnetism and gravity*, Phys. Rev. D **81**, 104012 (2010),
http://kirkmcd.princeton.edu/examples/GR/gralla_prd_81_104012_10.pdf

[5] Č. Masaitis, *Motion of Two Linked Bodies*, Ann. Rat. Mech. Anal. **8**, 23 (1961),
http://kirkmcd.princeton.edu/examples/mechanics/masaitis_arma_8_23_61.pdf

[6] L. Poinsot, *Théorie Nouvelle de la Rotation des Corps*, (1834, 1852),
http://kirkmcd.princeton.edu/examples/mechanics/poinsot_rotation_52.pdf

[7] V.N. Matvejev, O.V. Matvejev, and Ø. Grøn, *A relativistic trolley paradox*, Am. J. Phys. **84**, 419 (2016), http://kirkmcd.princeton.edu/examples/GR/matvejev_ajp_84_419_16.pdf

[8] K.T. McDonald, *Center of Mass of a Relativistic Rolling Hoop* (Sept. 7, 2012),
http://kirkmcd.princeton.edu/examples/hoop.pdf

[9] M. Campanelli *et al.*, *Maximum Gravitational Recoil*, Phys. Rev. Lett. **98**, 231102 (2007), http://kirkmcd.princeton.edu/examples/GR/campanelli_prl_98_231102_07.pdf

[10] K.T. McDonald, *Physics in the Laundromat* (Apr. 5, 1997),
http://kirkmcd.princeton.edu/examples/washer.pdf

[11] H.C. Corben, *Classical and Quantum Theories of Spinning Particles* (Holden-Day, San Francisco, 1968), Chap. II, http://kirkmcd.princeton.edu/examples/QM/corben_68_chap2.pdf

[12] B.M. Barker and R.F. O'Connell, *On the completion of the post-Newtonian gravitational two-body problem with spin*, J. Math. Phys. **28**, 661 (1987),
http://kirkmcd.princeton.edu/examples/GR/barker_jmp_28_661_87.pdf

[13] O.B. Karpov, *Gyroscope Deviation from Geodesic Motion: Quasiresonant Oscillations on a Circular Orbit*, J. Expt. Theor. Phys. **96**, 581 (2003),
http://kirkmcd.princeton.edu/examples/GR/karpov_jetp_96_581_03.pdf

[14] O.B. Karpov, *The Papapetrou equations and supplementary conditions*, arXiv:gr-qc/0406002 (June 2, 2004), http://arxiv.org/abs/gr-qc/0406002

[15] W. Shockley and R.P. James, *"Try Simplest Cases" Discovery of "Hidden Momentum" Forces on "Magnetic Currents,"* Phys. Rev. Lett. **18**, 876 (1967),
http://kirkmcd.princeton.edu/examples/EM/shockley_prl_18_876_67.pdf

[16] K.T. McDonald, *"Hidden" Momentum of a Steady Current Distribution in a System at "Rest"* (Apr. 21, 2009), http://kirkmcd.princeton.edu/examples/current.pdf

[17] L.B. Okun, *Mass versus relativistic and rest masses*, Am. J. Phys. **77**, 430 (2009),
http://kirkmcd.princeton.edu/examples/mechanics/okun_ajp_77_430_09.pdf

[18] D. Vanzella, Private communication, (June 29, 2012).

[19] K.T. McDonald, *On the Definition of "Hidden" Momentum* (July 9, 2012),
http://kirkmcd.princeton.edu/examples/hiddendef.pdf

[20] D. Keppel *et al.*, *Momentum flow in black-hole binaries. I. Post-Newtonian analysis of the inspiral and spin-induced bobbing*, Phys. Rev. D **80**, 124015 (2009), http://kirkmcd.princeton.edu/examples/GR/keppel_prd_80_124015_09.pdf

[21] N.L. Sharma, *Field versus action-at-a-distance in a static situation*, Am. J. Phys. **56**, 420 (1988), http://kirkmcd.princeton.edu/examples/EM/sharma_ajp_56_420_88.pdf

[22] W.H. Furry, *Examples of Momentum Distributions in the Electromagnetic Field and in Matter*, Am. J. Phys. **37**, 621 (1969), http://kirkmcd.princeton.edu/examples/EM/furry_ajp_37_621_69.pdf

[23] D.J. Griffiths, *Dipoles at rest*, Am. J. Phys. **60**, 979 (1992), http://kirkmcd.princeton.edu/examples/EM/griffiths_ajp_60_979_92.pdf

[24] S. Coleman and J.H. Van Vleck, *Origin of "Hidden Momentum" Forces on Magnets*, Phys. Rev. **171**, 1370 (1968), http://kirkmcd.princeton.edu/examples/EM/coleman_pr_171_1370_68.pdf

[25] K.T. McDonald, *Mansuripur's Paradox* (May 2, 2012), http://kirkmcd.princeton.edu/examples/mansuripur.pdf