Relativistic Harmonic Oscillator

Kirk T. McDonald
Joseph Henry Laboratories, Princeton University, Princeton, NJ 08544
(April 28, 2013)

1 Problem

Estimate the period τ of a "simple" harmonic oscillator consisting of a zero-rest-length massless spring of constant k that is connected to a rest mass m_0 (with the other end of the spring fixed to the origin), taking in account the relativistic mass.

2 Solution

2.1 Quick Estimates

Ignoring relativistic effects, the angular frequency ω_0 and the period τ_0 of the oscillator are,

$$\omega_0 = \sqrt{\frac{k}{m_0}}\,, \qquad \tau_0 = \frac{2\pi}{\omega_0} = 2\pi\sqrt{\frac{m_0}{k}}\,. \tag{1}$$

In this approximation, the oscillating mass has position and velocity,

$$x = A\cos\omega_0 t, \qquad v = -A\omega_0\sin\omega_0 t. \tag{2}$$

In general, the oscillating mass has (time-dependent) relativistic mass,

$$m = \frac{m_0}{\sqrt{1 - v^2/c^2}} \approx m_0\left(1 + \frac{v^2}{2c^2}\right), \tag{3}$$

where c is the speed of light in vacuum. We expect that the period τ of oscillation of the relativistic mass can be approximated as,

$$\tau \approx \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{\langle m\rangle}{k}} > \tau_0\,, \tag{4}$$

where $\langle m\rangle > m_0$ is an appropriate average of the relativistic mass. This might be the time average,

$$\begin{aligned}\langle m\rangle_t &= \frac{1}{\tau}\int_0^\tau m(t)\,dt \approx \frac{m_0}{\tau}\int_0^\tau \left(1 + \frac{v^2}{2c^2}\right)dt \approx m_0\left(1 + \frac{1}{2\tau_0 c^2}\int_0^{\tau_0} A^2\omega_0^2\cos^2\omega_0 t\,dt\right) \\ &= m_0\left(1 + \frac{A^2\omega_0^2}{4c^2}\right) = m_0\left(1 + \frac{kA^2}{4m_0c^2}\right),\end{aligned} \tag{5}$$

in which case,

$$\tau \approx \tau_0\sqrt{1 + \frac{kA^2}{4m_0c^2}} \approx \tau_0\left(1 + \frac{kA^2}{8m_0c^2}\right), \qquad \langle m\rangle = \langle m\rangle_t\,. \tag{6}$$

However, it could be that the spatial average is more appropriate,

$$\begin{aligned}
\langle m\rangle_x &= \frac{1}{A}\int_0^A m(x)\,dx \approx \frac{m_0}{A}\int_0^A\left(1+\frac{v^2}{2c^2}\right)dx \approx m_0\left[1+\frac{1}{2Ac^2}\int_0^A A^2\omega_0^2\left(1-\frac{x^2}{A^2}\right)dx\right] \\
&= m_0\left(1+\frac{A^2\omega_0^2}{4c^2}\right) = m_0\left(1+\frac{kA^2}{3m_0c^2}\right), \qquad (7)
\end{aligned}$$

noting that $\sin\omega t = \sqrt{1-\cos^2\omega t} \approx \sqrt{1-x^2/A^2}$, in the approximation that oscillating mass has x-coordinate $x = A\cos\omega t$. In this case,

$$\tau \approx \tau_0\sqrt{1+\frac{kA^2}{3m_0c^2}} \approx \tau_0\left(1+\frac{kA^2}{6m_0c^2}\right), \qquad \langle m\rangle = \langle m\rangle_x\,. \qquad (8)$$

As many other averages of the relativistic mass can be imagined, we seek a method that clarifies what type of approximation is best.

2.2 A Better Estimate

A different approach is to note that the motion is periodic with spatial amplitude A, and so the period τ can be computed as,

$$\tau = 4\int_0^A \frac{dt}{dx}\,dx = 4\int_0^A \frac{dx}{v}\,. \qquad (9)$$

Total energy E is conserved in this example,

$$E = mc^2 + \frac{kx^2}{2} = \frac{m_0c^2}{\sqrt{1-v^2/c^2}} + \frac{kx^2}{2} = m_0c^2 + \frac{kA^2}{2}, \qquad (10)$$

where the potential energy of the system is $kx^2/2$, such that,[1]

$$\frac{1}{v} = \frac{\tau_0}{2\pi}\frac{1+k(A^2-x^2)/2m_0c^2}{\sqrt{A^2-x^2}\sqrt{1+k(A^2-x^2)/4m_0c^2}} \approx \frac{\tau_0}{2\pi}\left(\frac{1}{\sqrt{A^2-x^2}}+\frac{3k\sqrt{A^2-x^2}}{8m_0c^2}\right). \qquad (11)$$

Hence,

$$\tau \approx \frac{2\tau_0}{\pi}\left(\int_0^A \frac{dx}{\sqrt{A^2-x^2}} + \frac{3k}{8m_0c^2}\int_0^A \sqrt{A^2-x^2}\,dx\right) = \tau_0\left(1+\frac{3kA^2}{16m_0c^2}\right). \qquad (12)$$

The correction term in this result is 2% larger than that in the estimate (8) based on the spatial average of the relativistic mass.

The "exact" periord of a relativistic harmonic oscillator can be given as an elliptic integral. A series approximation to this integral is given in [2].

[1]There is a sign error in the correction term of eq. (7-150), p. 325 of [1], which corresponds to eq. (11) of the present note. Thanks to Bill Jones for pointing this out.

References

[1] H. Goldstein, *Classical Mechanics*, $2^{\rm nd}$ ed. (Addison-Wesley, 1980),
http://kirkmcd.princeton.edu/examples/mechanics/goldstein_3ed.pdf

[2] L.A. MacColl, *Theory of the Relativistic Harmonic Oscillator*, Am. J. Phys. **25**, 535 (1957), http://kirkmcd.princeton.edu/examples/mechanics/maccoll_ajp_25_535_57.pdf