Heat Flow from a Point Source at the End of a Bar

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1 Problem

As a simple example of 3-dimesional heat flow, deduce the steady-state temperature distribution inside a semi-infinte square bar with a point source of heat somewhere on its square face, assuming no heat flow across other surfaces (except the square face at infinity).[1]

2 Solution

The heat flux vector $\mathbf{J}$ obeys,

$$\mathbf{J} = -\kappa \boldsymbol{\nabla} T, \tag{1}$$

where κ is the thermal conductivity and T is the temperature distribution. Energy is conserved in the interior of the bar, so in a steady state $\boldsymbol{\nabla} \cdot \mathbf{J} = 0$ there, and hence $\nabla^2 T = 0$.

We consider a separation-of-variable solution in a rectangular coordinate system, taking the heat source Q to be at $(x_0, y_0, 0)$, with the bar extending over the $z \geq 0$ with square cross section $|x|, |y| \leq a/2$. The normal derivative of the temperature is zero at the surfaces across which no heat flows, so the boundary conditions are,[2]

$$\frac{\partial T(x, y, 0)}{\partial z} = -\frac{Q}{\kappa} \delta(x - x_0, y - y_0), \tag{2}$$

$$\frac{\partial T(0, y, z)}{\partial x} = \frac{\partial T(a, y, z)}{\partial x} = 0 = \frac{\partial T(x, 0, z)}{\partial y} = \frac{\partial T(x, a, z)}{\partial y}. \tag{3}$$

A separated form that obeys $\nabla^2 T = 0$ and satisfies condition (3) is,[3]

$$T = \sum_{m,n=0}^{\infty} C_{mn} \cos\frac{2m\pi x}{a} \cos\frac{2n\pi y}{a}\, e^{-2\sqrt{m^2+n^2}\pi z/a} - Az. \tag{4}$$

Condition (2) is then,

$$A + \frac{2\pi}{a} \sum_{m,n=0}^{\infty} \sqrt{m^2+n^2}\, C_{mn} \cos\frac{2m\pi x}{a} \cos\frac{2n\pi y}{a} = \frac{Q}{\kappa} \delta(x - x_0, y - y_0). \tag{5}$$

[1] Equivalently, consider a square bar of infinite length with a point source somewhere inside.
[2] It seems not possible to obtain an analytic solution for a bar of finite length with a point source on one end and the other end at fixed temperature, if no heat flows across its other surfaces.
[3] This type of solution may have been first given by Fourier, sec. 321 of [1].

On multiplying by $\cos\frac{2k\pi x}{a}\cos\frac{2l\pi y}{a}$ and integrating over the area of the square cross section of the bar, we find that,

$$A = \frac{Q}{a^2\kappa}, \qquad C_{kl} = \frac{2Q}{\pi a\kappa}\begin{cases} \text{undefined} & (k = l = 0), \\ \frac{1}{l}\cos\frac{l\pi y_0}{a} & (k = 0,\ l \geq 1), \\ \frac{1}{k}\cos\frac{k\pi x_0}{a} & (l = 0, k \geq 1), \\ \frac{2}{\sqrt{k^2+l^2}}\cos\frac{k\pi x_0}{a}\cos\frac{l\pi y_0}{a} & (k, l \geq 1). \end{cases} \tag{6}$$

Hence, on redefinint the undetermined constant C_{00} as T_0,

$$\begin{aligned} T &= T_0 - \frac{Qz}{a^2\kappa} + \frac{2Q}{\pi a\kappa}\left(\sum_{m=1}^{\infty}\cos\frac{2m\pi x}{a}\cos\frac{2m\pi x_0}{a}\frac{e^{-2m\pi z/a}}{m} + \sum_{n=1}^{\infty}\cos\frac{2n\pi y}{a}\cos\frac{2n\pi y_0}{a}\frac{e^{-2n\pi z/a}}{n}\right. \\ &\quad \left. +2\sum_{m,n=1}^{\infty}\cos\frac{2m\pi x}{a}\cos\frac{2m\pi x_0}{a}\cos\frac{2n\pi y}{a}\cos\frac{2n\pi y_0}{a}\frac{e^{-2\sqrt{m^2+n^2}\pi z/a}}{\sqrt{m^2+n^2}}\right). \end{aligned} \tag{7}$$

and the heat-flow vector (1) has components,

$$\begin{aligned} J_x &= \frac{2Q}{a^2}\left(\sum_{m=1}^{\infty}\sin\frac{2m\pi x}{a}\cos\frac{2m\pi x_0}{a}e^{-2m\pi z/a}\right. \\ &\quad \left. +2\sum_{m,n=1}^{\infty}m\sin\frac{2m\pi x}{a}\cos\frac{2m\pi x_0}{a}\cos\frac{2n\pi y}{a}\cos\frac{2n\pi y_0}{a}\frac{e^{-2\sqrt{m^2+n^2}\pi z/a}}{\sqrt{m^2+n^2}}\right), \end{aligned} \tag{8}$$

$$\begin{aligned} J_y &= \frac{2Q}{a^2}\left(\sum_{n=1}^{\infty}\sin\frac{2n\pi y}{a}\cos\frac{2n\pi y_0}{a}e^{-2n\pi z/a}\right. \\ &\quad \left. +2\sum_{m,n=1}^{\infty}n\cos\frac{2m\pi x}{a}\cos\frac{2m\pi x_0}{a}\sin\frac{2n\pi y}{a}\cos\frac{2n\pi y_0}{a}\frac{e^{-2\sqrt{m^2+n^2}\pi z/a}}{\sqrt{m^2+n^2}}\right), \end{aligned} \tag{9}$$

$$\begin{aligned} J_z &= \frac{Q}{a^2}\left(1 + 2\sum_{m=1}^{\infty}\cos\frac{2m\pi x}{a}\cos\frac{2m\pi x_0}{a}e^{-2m\pi z/a} + 2\sum_{n=1}^{\infty}\cos\frac{2n\pi y}{a}\cos\frac{2n\pi y_0}{a}e^{-2n\pi z/a}\right. \\ &\quad \left. +4\sum_{m,n=1}^{\infty}\cos\frac{2m\pi x}{a}\cos\frac{2m\pi x_0}{a}\cos\frac{2n\pi y}{a}\cos\frac{2n\pi y_0}{a}e^{-2\sqrt{m^2+n^2}\pi z/a}\right). \end{aligned} \tag{10}$$

For the particular case that the point source is at the center of the end face, $x_0 = y_0 = 0$,

$$\begin{aligned} T &= T_0 - \frac{Qz}{a^2\kappa} + \frac{Q}{\pi a\kappa}\left(\sum_{m=1}^{\infty}\cos\frac{2m\pi x}{a}\frac{e^{-2m\pi z/a}}{m} + \sum_{n=1}^{\infty}\cos\frac{2n\pi y}{a}\frac{e^{-2n\pi z/a}}{n}\right. \\ &\quad \left. +2\sum_{m,n=1}^{\infty}\cos\frac{2m\pi x}{a}\cos\frac{2n\pi y}{a}\frac{e^{-2\sqrt{m^2+n^2}\pi z/a}}{\sqrt{m^2+n^2}}\right). \end{aligned} \tag{11}$$

and the heat-flow vector (1) has components,

$$J_x = \frac{2Q}{a^2}\left(\sum_{m=1}^{\infty}\sin\frac{2m\pi x}{a}\, e^{-2m\pi z/a} + 2\sum_{m,n=1}^{\infty} m \sin\frac{2m\pi x}{a}\cos\frac{2n\pi y}{a}\,\frac{e^{-2\sqrt{m^2+n^2}\pi z/a}}{\sqrt{m^2+n^2}}\right), \tag{12}$$

$$J_y = \frac{2Q}{a^2}\left(\sum_{n=1}^{\infty}\sin\frac{2n\pi y}{a}\, e^{-2n\pi z/a} + 2\sum_{m,n=1}^{\infty} n \cos\frac{2m\pi x}{a}\sin\frac{2n\pi y}{a}\,\frac{e^{-2\sqrt{m^2+n^2}\pi z/a}}{\sqrt{m^2+n^2}}\right), \tag{13}$$

$$J_z = \frac{Q}{a^2}\left(1 + 2\sum_{m=1}^{\infty}\cos\frac{2m\pi x}{a}\, e^{-2m\pi z/a} + 2\sum_{n=1}^{\infty}\cos\frac{2n\pi y}{a}\, e^{-2n\pi z/a} + 4\sum_{m,n=1}^{\infty}\cos\frac{2m\pi x}{a}\cos\frac{2n\pi y}{a}\, e^{-2\sqrt{m^2+n^2}\pi z/a}\right). \tag{14}$$

The figures below[4] shows the lines of the heat-flow vector $\mathbf{J}$ in the midplane $y = 0$,

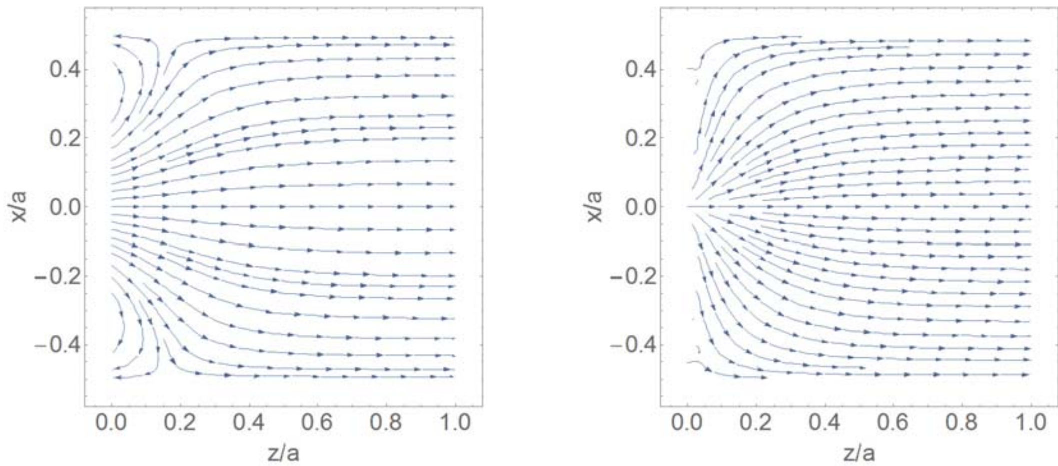
$$J_x(x,0,z) = \frac{2Q}{a^2}\left(\sum_{m=1}^{\infty}\sin\frac{2m\pi x}{a}\, e^{-2m\pi z/a} + 2\sum_{m,n=1}^{\infty} m \sin\frac{2m\pi x}{a}\,\frac{e^{-2\sqrt{m^2+n^2}\pi z/a}}{\sqrt{m^2+n^2}}\right), \tag{15}$$

$$J_y(x,0,z) = 0, \tag{16}$$

$$J_z(x,0,z) = \frac{Q}{a^2}\left(1 + 2\sum_{m=1}^{\infty}\cos\frac{2m\pi x}{a}\, e^{-2m\pi z/a} + 2\sum_{n=1}^{\infty} e^{-2n\pi z/a} + 4\sum_{m,n=1}^{\infty}\cos\frac{2m\pi x}{a}\, e^{-2\sqrt{m^2+n^2}\pi z/a}\right). \tag{17}$$

Indices m and n were evaluated up to 1 in the left figure and to 20 in the right; indices higher than 1 mainly affect the region close to $z = 0$ where the delta-function boundary condition (5) is being approximated.

[4]The figures were generated via the Mathematica notebook http://kirkmcd.princeton.edu/examples/heatflow.nb.



The figure indicates that the heat flow is essentially parallel to the z-axis for $z \gtrsim a/2$ from the point source, which is agreeable with naïve expectations.[5]

References

[1] J. Fourier, *Theorie Analytique de la Chaleur* (Firmin Didot, 1822),
http://kirkmcd.princeton.edu/examples/statmech/fourier_22.pdf
http://kirkmcd.princeton.edu/examples/statmech/fourier_22_english.pdf

[2] H.S. Carslaw and J.C. Jaeger, *Conduction of Heat in Solids*, $2^{\rm nd}$ ed. (Clarendon Press, 1959), http://kirkmcd.princeton.edu/examples/statmech/carslaw_59.pdf

[5]This example does not appear in the great compendium [2] of lore on heat conduction, although the ingredients of the solution are, of course, well represented there. For example, sec. 14.3-III gives a 2-dimensional version of the present problem.