Electron Bubbles in Liquid Helium

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1 Problem

When an electron (or positronium atom) is injected into liquid helium with nearly zero energy, a bubble quickly forms around it. This phenomenon (which also occurs in liquid hydrogen, liquid neon and possibly in solid helium) lowers the mobility of the electron to a value similar to that for a positive ion.

Estimate the radius of the bubble at zero pressure and temperature.

If the liquid is held in a state of negative pressure, the bubble will expand beyond the radius at zero pressure. Estimate the negative pressure such that a bubble once formed will grow without limit.

2 Solution

2.1 Bubble Radius at Zero Pressure

For a simple estimate, we follow the original paper by R.A. Ferrell [1]. We assume the bubble to be spherical with radius a.

Our estimate is based on an energy argument. The bubble is kept from collapsing by the pressure due to the collisions of the electron with the wall. At zero temperature, the motion of the electron inside the bubble is due to zero-point energy. We relate this to the zero-point momentum, which we estimate via the uncertainty principle,

$$\delta x\ \delta p_x \approx \frac{\hbar}{2}. \tag{1}$$

For an electron inside a bubble of radius a, the uncertainty in coordinate x is about $2a/3$, so we estimate that,

$$\left\langle p_x^2 \right\rangle^{1/2} \approx \delta p_x \approx \frac{3\hbar}{4a}. \tag{2}$$

The zero-point energy is therefore estimated as,[1]

$$U_{\rm zero-point} \approx \frac{\langle p^2 \rangle}{2m} \approx \frac{3\left\langle p_x^2 \right\rangle}{2m} \approx \frac{27\hbar^2}{32a^2m} \approx \frac{\hbar^2}{a^2m}. \tag{3}$$

The bubble tends to collapse to zero due to the force of surface tension. We characterize surface tension by a coefficient γ which is a force per unit length = energy per unit area. A bubble of radius a has a surface energy given by,

$$U_{\rm surface} = 4\pi a^2 \gamma. \tag{4}$$

[1]The formal result for the zero-point energy for a particle of mass m inside an infinite spherical potential well of radius a is $\pi^2\hbar^2/2a^2m$.

The physical radius of the bubble minimizes the total energy, so we solve,

$$0 = \frac{dU_{\rm total}}{da} = -\frac{2\hbar^2}{a^3 m} + 8\pi a\gamma, \tag{5}$$

which yields,

$$a_{\rm bubble} \approx \left(\frac{\hbar^2}{4\pi\gamma m}\right)^{1/4} = \left(\frac{(\hbar c)^2}{4\pi\gamma mc^2}\right)^{1/4}. \tag{6}$$

To estimate the surface tension coefficient γ for liquid helium, we recall that it liquifies at 4K where the kinetic energy per atom is about 1/4000 eV. So, we suppose that the binding of a helium atom to its neighbors on the surface is about 1/4000 eV, and that the distance between atoms is about one Ångstrom. Hence,[2]

$$\gamma \approx \frac{2.5 \times 10^{-4}\ \rm eV}{(10^{-8}\ \rm cm)^2} = 2.5 \times 10^{12}\ \rm eV/cm^2 = 4\ erg/cm^2. \tag{7}$$

We also recall that the electron rest energy is $mc^2 \approx 5 \times 10^5$ eV, and that $\hbar c \approx 200$ MeV-fermi $= 2 \times 10^{-5}$ eV-cm. Hence, we estimate that,

$$a_{\rm bubble} \approx \left(\frac{(2 \times 10^{-5})^2}{4\pi \cdot 2.5 \times 10^{12} \cdot 5 \times 10^5}\right)^{1/4} \ {\rm cm} \approx \left(\frac{10^{-28}}{4}\right)^{1/4} \ {\rm cm} \approx 7\ \text{Å}\,. \tag{8}$$

The experimental value for the bubble radius a is about 17 Å.[3]

2.2 Negative Pressure

When a bubble is formed in a liquid at positive pressure P, additional work PV must be done. We consider this work to be stored in the bubble in the form of an energy, and so the total energy of a bubble of radius a is,

$$U = U_{\rm zero-point} + U_{\rm surface} + PV \approx \frac{C\hbar^2}{a^2 m} + 4\pi a^2\gamma + \frac{4\pi a^3 P}{3}, \tag{9}$$

where $C = 1$ in our approximate model, and $C = \pi^2/2$ from a calculation based on a deep spherical potential well.

If the bubble is to grow indefinitely we must have $\partial U/\partial a < 0$ for all a. The zero-point energy term decreases monotonically with radius a, while the surface energy term increases with radius. Together, the first two terms have a single minimum at the radius found in sec. 2.1.

For negative pressures, the pressure term decreases with radius more rapidly than the surface term increases with radius. So, at large radii the negative pressure term dominates, and $\partial U/\partial a$ becomes negative. For small negative pressures there is a maximum of U at a radius larger than that of the minimum of U, and between these two extrema the slope is positive. As the magnitude of the negative pressure increases, the two extrema approach one another, until at the desired critical pressure they coalesce, and there is a radius for which both $\partial U/\partial a = 0$ and $\partial^2 U/\partial a^2 = 0$, as illustrated in Fig. 1

[2] Apparently, this estimate is about a factor of ten high.

[3] Noting that our estimate for $U_{\rm zero-point}$ is a factor of five low and that for γ is a factor of ten high, the estimate (8) should be multiplied by $\sqrt[4]{50}$ to yield a prediction that $a_{\rm bubble} \approx 19$ Å.

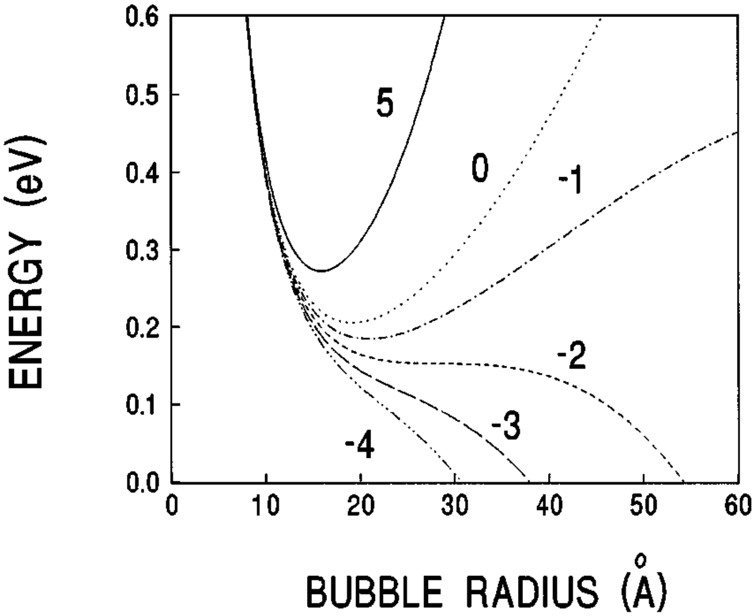

Figure 1: The energy (9) of an electron bubble in liquid helium at zero temperature as a function of the radius s. The curves are labeled by the pressure in bars [2].

From eq. (9), the critical pressure and radius are related by,

$$\frac{\partial U}{\partial a} = 0 = -\frac{2C\hbar^2}{a^3 m} + 8\pi a\gamma + 4\pi a^2 P, \qquad (10)$$

$$\frac{\partial^2 U}{\partial a^2} = 0 = \frac{6C\hbar^2}{a^4 m} + 8\pi\gamma + 8\pi a P. \qquad (11)$$

We quickly find that,

$$a = \left(\frac{5C\hbar^2}{4\pi\gamma m}\right)^{1/4} = 5^{1/4} a_{\rm bubble}(P = 0) \approx 10 C^{1/4}\ \text{Å}. \qquad (12)$$

and,

$$P = -\frac{8\gamma}{5a}\,. \qquad (13)$$

Using our estimate (7) that $\gamma \approx 0.004$ N/m and taking $C = 1$, we find the critical pressure to be $P = -64$ bar. This is about 40 times the reported value of -1.6 bars [2]. Since the result (13) varies as $\gamma^{5/4}$, it is more sensitive to the value of surface tension than is the bubble radius. If we use $\gamma = 0.0004$ N/m and $C = \pi^2/2$, then our estimate for the critical negative pressure would be -2.4 bars.

References

[1] R.A. Ferrell, *Long Lifetime of Positronium in Liquid Helium*, Phys. Rev. **108**, 167 (1957), http://kirkmcd.princeton.edu/examples/fluids/ferrell_pr_108_167_57.pdf

[2] J. Classen, C.-K. Su and H.J. Maris, *Observation of Exploding Electron Bubbles in Liquid Helium*, Phys. Rev. Lett. **77**, 2006 (1996),
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