The Electric Potential of Particle Detectors with Rectangular Cross-Section

Changguo Lu, Kirk T. McDonald
Joseph Henry Laboratories, Princeton University, Princeton, NJ 08544
(October 9, 2002)

1 Problem

This is a collection of related problems concerning calculation of the electric potential in structures whose unit cell is a 2-dimensional, hollow, rectangular prism with a collection wire at its center, as find application in gas-filled detectors of high-energy charged particles.

1.1 Basic Iarocci Tube

An Iarocci tube [1] is a low-cost descendent of a Geiger counter whose walls form a rectangular prism, with a wire along its center. In the basic configuration, the walls are conducting, and grounded, as shown in the left of Fig 1.

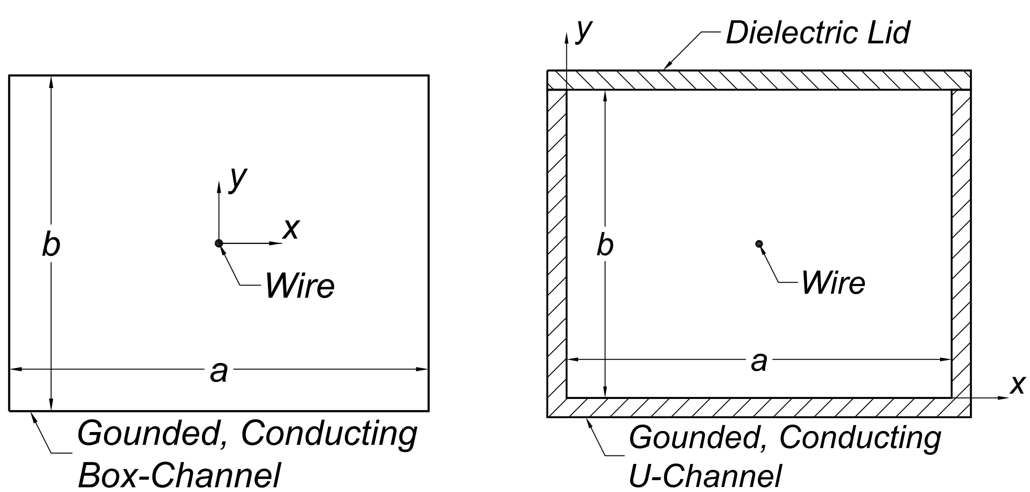

Figure 1: An Iarocci tube is a hollow rectangular prism of active area $a \times b$ with a wire running down the center. The wall of the tube may be a grounded, conducting box-channel (left), or a grounded, conducting U-channel with a dielectric lid (right).

Deduce the electrostatic potential $\phi(x, y)$ inside the Iarocci tube supposing the wire carries charge q per unit length.

The potential has a logarithmic divergence at the wire, so we specify the charge per unit length on the wire rather than its potential. Then, the presence of (nonsingular) surfaces at a specified potential permits a simple series expansion at points not on the wire. A physical device with this geometry will have a wire of nonzero radius r_0. Sum the series for points near the wire to show that potential of the wire is approximately $2q \ln(b/2\pi r_0)$.

1.2 Iarocci Tube with a Dielectric Lid

Another form of an Iarocci tube is based on a ground, conducting U-channel with a dielectric lid, as shown in the right of Fig. 1. This is not enough information to obtain a unique solution for the potential. We must know either the potential or its normal derivative everywhere on the boundary, according to a theorem of Green. During operation the dielectric surface becomes charge up by positive ions until the electric field just inside the lid is parallel to its surface, which implies that the same tangential electric field exists just outside the lid as well. Such an electric field configuration would hold in a structure consisting of the Iarocci tube as described plus its mirror image in the plane of the lid, *i.e.*, a grounded, conduction prism of size $a \times 2b$ with two wires.

1.3 Multiwire Proportional Chamber ("Electrodeless" Drift Chamber)

Closely related to an Iarocci tube is the so-called "electrodeless" drift chamber, shown in Fig. 2. Typically $a \gg b$, and both the upper and lower surfaces are dielectric. Find the of the potential assuming that the electric field is parallel to the dielectric surfaces, and that the wire carries charge q per unit length.

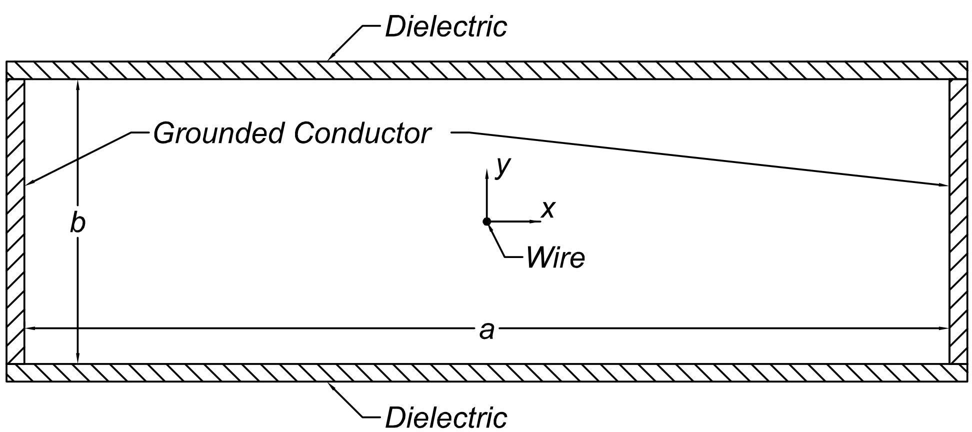

Figure 2: An "electrodeless" drift chamber is a hollow rectangular prism of active area $a \times b$ with two opposing faces that are conductors, two faces that are dielectric, and a wire running down the center. This is also a unit cell of a multiwire proportional chamber with planar cathodes.

If the unit cell shown in Fig. 2 is replicated so as to form a stack of cells in y with period b, we have in effect that case of a pair of grounded conducting planes distance a apart with a grid of wires of spacing b midway between. This is the geometry of a multiwire proportional chamber (MWPC).

1.4 MWPC with Alternating Anode and Cathode Wires

Crosstalk between transient signals of charge collected on the anode wires of a multiwire proportional chamber (the central wire in Fig. 2) can be suppressed by the addition of cathode wires between the anode wires, as shown in Fig. 3. Typically, these additional cathode wires would be grounded.

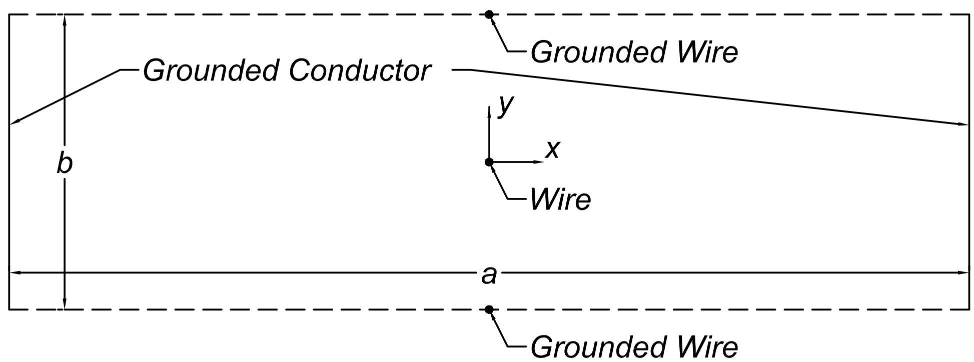

Figure 3: The unit cell of a multiwire proportional chamber with a plane of alternating anode and grounded cathode wires.

Recall that we cannot define the potential of a 1-dimensional wire, so we cannot analyze the problem where the anode wires are grounded. We can, however, make an analysis using 1-dimensional wires if the charges on the wires are given. Find the electric potential supposing the charge on each of the anode wires is $-\epsilon q$, where in practice ϵ would be determined by requiring the anode wire of finite radius to be at zero potential.

If $\epsilon \approx (2/\pi)\tan^{-1}(a/b)$, then the field pattern over most of the cell will resemble that of an Iarocci tube with grounded walls.

1.5 Array of Cells with One Anode and One Cathode Wire

Some particle detectors are in the form of an extended array of cells defined only by anode and cathode wires. A simple rectangular array is shown in Fig. 4, where the unit cell has size $2a \times 2b$ and contains one anode and one cathode wire. In the limit of an infinite array, no electric field lines would cross any cell boundary.

Give a series expansion for the potential $\phi(x, y)$ supposing that the anode and cathode wires carry charge $\pm q$ per unit length.

1.6 Array of Cells with 1 Anode and 3 Cathode Wires

To make the electric field in the cells of a rectangular array approach more closely that of a basic Iarocci tube, additional cathode wires can be added, as shown in Fig. 5. All field lines that leave an anode wire should terminate on cathode wires of the same cell, so the sum of the charges per unit length on the cathode wires must be equal and opposite to that on the anode wire. Roughly, if there are n cathode wires per unit cell, each will carry charge $-q/n$ per unit length, where q is the charge per unit length on the anode wire.

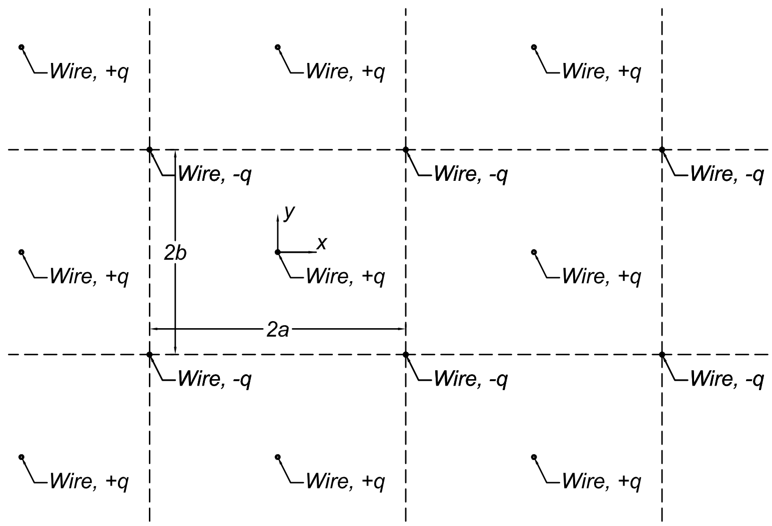

Figure 4: A region of a particle detector whose rectangular unit cell of size $2a \times 2b$ contains one anode wire and one cathode wire that carry charge $\pm q$ per unit length.

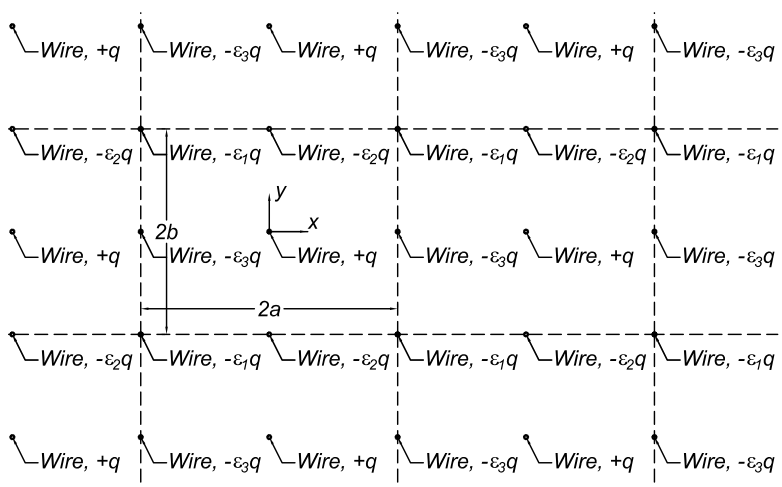

Figure 5: A region of a particle detector whose rectangular unit cell of size $2a \times 2b$ contains 1 anode wire and 3 cathode wires.

In practice, the cathode wires typically have the same potential, rather than the same charge. For a calculation using 1-dimensional wires we must, however, specify the charges rather than the potentials. Give a series expansion for the potential $\phi(x, y)$ supposing that the anode wires carry charge q per unit length and the three cathode wires carry charge $-\epsilon_1 q$, $-\epsilon_2 q$ and $-\epsilon_3 q$, where $\epsilon_1 + \epsilon_2 + \epsilon_3 = 1$.

2 Solution

Some additional aspects of this problem are considered in [2].

2.1 Basic Iarocci Tube

The case of a rectangular Iarocci tube is closely related to the very simple case of a structure with a wire along the axis of a grounded, conducting cylinder. In the latter case, the equipotentials are circles centered on the wire, and the electric field lines radiate from the wire. This pattern holds in the rectangular Iarocci tube for points close to the wire, but near the walls the field lines must bend until they are normally incident on the walls.

To use techniques for solving Laplace's equation, $\nabla^2\phi = 0$, for the potential ϕ, we subdivide the cells into rectangular regions that have no charge in their interior. We analyze the cells in a rectangular coordinate system with origin at the center of a cell (except in sec. 2.2).

2.1.1 Wire at Center of Cell

For the basic Iarocci tube shown on the left of Fig. 1, we solve separately in the regions $x < 0$ and $x > 0$, and match solutions at the "boundary" $x = 0$. In each region, we know the potential on three of the four bounding surfaces, and we know the charge distribution $\sigma \propto \partial\phi/\partial n$ on the fourth,

$$\phi(x, -b/2) = \phi(x, b/2) = \phi(-a/2, y) = \phi(a/2, y) = 0, \qquad \frac{\partial\phi(0^+, y)}{\partial x} = -2\pi q\delta(y), \tag{1}$$

in Gaussian units, where the symmetry of the potential and of E_x about $x = 0$ implies that $E_x(-\epsilon, y) = -E_x(\epsilon, y) = \partial\phi(\epsilon, y)/\partial x = -2\pi\sigma(0, y) = -2\pi q\delta(y)$.

A suitable form of the solution to Laplace's equation for a potential that vanishes on the outer boundaries and is symmetric in both x and y is,

$$\phi(x, y) = \sum_n A_n \sinh k_n(a/2 - |x|) \cos k_n y. \tag{2}$$

The boundary condition at $y = \pm b/2$ requires that $\cos k_n b/2 = 0$, and hence that $k_n = (2n+1)\pi/b$. The boundary condition at $x = 0$ can now be written as,

$$-2\pi q\delta(y) = \frac{\partial\phi(0^+, y)}{\partial x} = -\sum_n \frac{(2n+1)\pi}{b} A_n \cosh\frac{(2n+1)\pi a}{2b} \cos\frac{(2n+1)\pi y}{b}. \tag{3}$$

On multiplying eq. (3) by $\sin n\pi y/b$ and integrating from 0 to b we find that,

$$A_n = \frac{4q}{(2n+1)\cosh\frac{(2n+1)\pi a}{2b}}. \tag{4}$$

Hence, the potential for a basic Iarocci tube with a wire at its center can be written as,

$$\phi(x, y) = 4q \sum_n \frac{\sinh\frac{(2n+1)\pi(a/2-|x|)}{b}}{(2n+1)\cosh\frac{(2n+1)\pi a}{2b}} \cos\frac{(2n+1)\pi y}{b} \qquad \text{(wire, origin at center)}$$

$$= \quad 4q \sum_n \frac{\tanh\frac{(2n+1)\pi a}{2b} \cosh\frac{(2n+1)\pi|x|}{b} - \sinh\frac{(2n+1)\pi|x|}{b}}{(2n+1)} \cos\frac{(2n+1)\pi y}{b}\,. \tag{5}$$

The potential at the origin diverges. But, of course, a physical realization of an Iarocci tube involves a wire of finite radius r. We can estimate the potential at the surface of the wire at position (x, y), where $x^2 + y^2 = r^2 \ll a,\ b$, using the second form of eq. (5),

$$\begin{aligned}
\phi_{\rm wire} &= \phi(x,y) \approx 4qRe \sum_{n=0} \frac{\cosh\frac{(2n+1)\pi x}{b} - \sinh\frac{(2n+1)\pi x}{b}}{2n+1} e^{(2n+1)\pi i y/b} \\
&= 4qRe \sum_{n=0} \frac{e^{-(2n+1)\pi x/b} e^{(2n+1)\pi i y/b}}{2n+1} = 4qRe \sum_{n=1} \frac{\left[e^{\pi(-|x|+iy)/b}\right]^{2n+1}}{2n+1} \\
&= 2qRe \ln\frac{1 + e^{\pi(-|x|+iy)/b}}{1 - e^{\pi(-|x|+iy)/b}} = 2qRe \ln\frac{\sinh\frac{\pi|x|}{b} + i\sin\frac{\pi y}{b}}{\cosh\frac{\pi|x|}{b} - \cos\frac{\pi y}{b}}\,.
\end{aligned} \tag{6}$$

Note that the approximation in line 1 of eq. (6) is "exact" in the limit that $a \gg b$, *i.e.*, in the case of a wire halfway between a pair of grounded conducting planes.

Then, writing $\ln\left[\left(\sinh\frac{\pi|x|}{b} + i\sin\frac{\pi y}{b}\right) / \left(\cosh\frac{\pi|x|}{b} - \cos\frac{\pi y}{b}\right)\right] = u + iv$ we have,

$$e^{u+iv} \quad = \quad e^u \cos v + i e^u \sin v = 1 - e^{-2\pi(x-iy)/a} = \frac{\sinh\frac{\pi|x|}{b} + i\sin\frac{\pi y}{b}}{\cosh\frac{\pi|x|}{b} - \cos\frac{\pi y}{b}} \tag{7}$$

$$\begin{aligned}
e^{2u} &= \frac{\sinh^2\frac{\pi|x|}{b} + \sin^2\frac{\pi y}{b}}{\left(\cosh\frac{\pi|x|}{b} - \cos\frac{\pi y}{b}\right)^2} = \frac{\cosh\frac{\pi|x|}{b} + \cos\frac{\pi y}{b}}{\cosh\frac{\pi|x|}{b} - \cos\frac{\pi y}{b}} \\
&\approx \frac{2}{\frac{1}{2}[(\frac{\pi x}{b})^2 + (\frac{\pi y}{b})^2]} = \left(\frac{2b}{\pi r_0}\right)^2,
\end{aligned} \tag{8}$$

$$u \approx \ln\frac{2b}{\pi r_0}\,, \tag{9}$$

and we finally have,

$$V_{\rm wire} \approx 2q \ln\frac{2b}{\pi r_0} = 2q \ln\frac{0.64b}{r_0}\,. \tag{10}$$

This is consistent with the general argument that if a wire of radius r carries charge q per unit length, the electric field near the wire will have strength $E \approx 2q/r$. So if the wire is surrounded by a grounded conductor of characteristic size s, the potential on the wire will be $\phi_{\rm wire} \approx \int_r^s E dr \approx 2q\ln(s/r)$.

The image method can be used to generate another solution to this problem. A doubly infinite set of charges $(-1)^{m+n}q$ at positions (ma, nb), where m and n are any integer (positive or negative), is consistent with all four bounding planes of the box-channel being at ground potential. Hence, we can write,

$$\begin{aligned}
\phi(x,y) &= 2q \sum_m \sum_n (-1)^{m+n} \ln\frac{1}{\sqrt{(x-ma)^2 + (y-nb)^2}} + C \\
&= -q \sum_m \sum_n (-1)^{m+n} \ln\left[(x-ma)^2 + (y-nb)^2\right] + C.
\end{aligned} \tag{11}$$

We require that the potential be zero on the boundary, which leads to an infinite set of representations of constant C. For example, forcing $V(a/2, b/2) = 0$, we can write,

$$V(x,y) = -q \sum_m \sum_n (-1)^{m+n} \ln \frac{(m - x/a)^2 + (n - y/b)^2}{(m - 1/2)^2 + (n - 1/2)^2}\,. \tag{12}$$

For (x, y) near the origin, we can suppose that the series is dominated by the term with $m = n = 0$, which implies that $\phi \approx -q \ln[2(x^2/a^2 + y^2/b^2)]$. In this approximation, the equipotentials are ellipsoidal cylinders, rather than circular cylinders as is to be expected. When $a = b$, we obtain $\phi \approx 2q \ln(b/\sqrt{2}r) = 2q \ln(0.71b/r)$, which is very close to eq. (10).

2.1.2 Wire at Arbitrary Position

If the wire were not at the center of the Iarocci tube, but at an aribtrary position (x_0, y_0), then the matching condition at the plane $x = x_0$ would be,

$$\frac{\partial\phi(x_{0,-}, y)}{\partial x} - \frac{\partial\phi(x_{0,+}, y)}{\partial x} = 4\pi q\delta(y - y_0). \tag{13}$$

A suitable expansion for the potential that satisfies the other boundary conditions of (1) and which is continuous at the plane $x = x_0$ is,

$$\phi(x < x_0, y) = \sum_n A_n \sinh \frac{(2n+1)\pi(a/2 + x)}{b} \cos \frac{(2n+1)\pi y}{b}\,, \tag{14}$$

$$\phi(x > x_0, y) = \sum_n A_n \frac{\sinh \frac{(2n+1)\pi(a/2+x_0)}{b}}{\sinh \frac{(2n+1)\pi(a/2-x_0)}{b}} \sinh \frac{(2n+1)\pi(a/2 - x)}{b} \cos \frac{(2n+1)\pi y}{b}\,. \tag{15}$$

The Fourier coefficients A_n are readily re-evaluated using eq. (13), and the series expansion for the potential is,

$$\phi(x < x_0, y) = 8q \sum_n \frac{\sinh \frac{(2n+1)\pi(a/2-x_0)}{b} \cos \frac{(2n+1)\pi y_0}{b}}{\sinh \frac{(2n+1)\pi a}{b}} \sinh \frac{(2n+1)\pi(a/2 + x)}{b} \cos \frac{(2n+1)\pi y}{b}\,. \tag{16}$$

2.2 Iarocci Tube with Electric Field Parallel to a Dielectric Lid

This configuration does not have symmetry in the y direction, so it proves simpler to use a coordinate system with its origin in, say, the lower left corner of the cell, as shown on the right of Fig. 1. The boundary conditions, assuming that no field lines cross the surface $y = b$, are,

$$\phi(0, y) = \phi(a, y) = \phi(x, 0) = 0, \qquad \frac{\partial\phi(x, b)}{\partial y} = 0, \qquad \frac{\partial\phi(a/2, y)}{\partial x} = 2\pi q\delta(y - b/2), \tag{17}$$

in Gaussian units, where the symmetry of the potential and of E_x about $x = a/2$ implies that $E_x(a/2 - \epsilon, y) = -E_x(a/2 + \epsilon, y) = -2\pi\sigma(a/2, y) = -2\pi q\delta(y - b/2)$.

A suitable form of the solution to Laplace's equation for a potential that vanishes at $x = 0$ and $y = 0$ is,

$$\phi(x, y) = \sum_n A_n \sinh k_n x \sin k_n y. \tag{18}$$

The boundary condition at $y = b$ requires that $\cos k_n b = 0$, and hence that $k_n = (2n+1)\pi/2b$. The boundary condition at $x = a/2$ can now be written as,

$$2\pi q \delta(y - b/2) = \frac{\partial \phi(a/2, y)}{\partial x} = \sum_n \frac{(2n+1)\pi}{2b} A_n \cosh \frac{(2n+1)\pi a}{4b} \sin \frac{(2n+1)\pi y}{2b}. \tag{19}$$

On multiplying eq. (19) by $\sin(2n + 1)\pi y/2b$ and integrating from 0 to b, we find that,

$$A_n = \frac{4q \sin \frac{(2n+1)\pi}{4}}{(2n+1) \cosh \frac{(2n+1)\pi a}{4b}}. \tag{20}$$

The potential is therefore given by,

$$\phi(x, y) = 4q \sum_n \frac{\sin \frac{(2n+1)\pi}{4}}{2n+1} \frac{\sinh \frac{(2n+1)\pi x}{2b}}{\cosh \frac{(2n+1)\pi a}{4b}} \sin \frac{(2n+1)\pi y}{2b}. \tag{21}$$

Equipotentials and field lines based on eq. (21) are shown in Fig. 6.

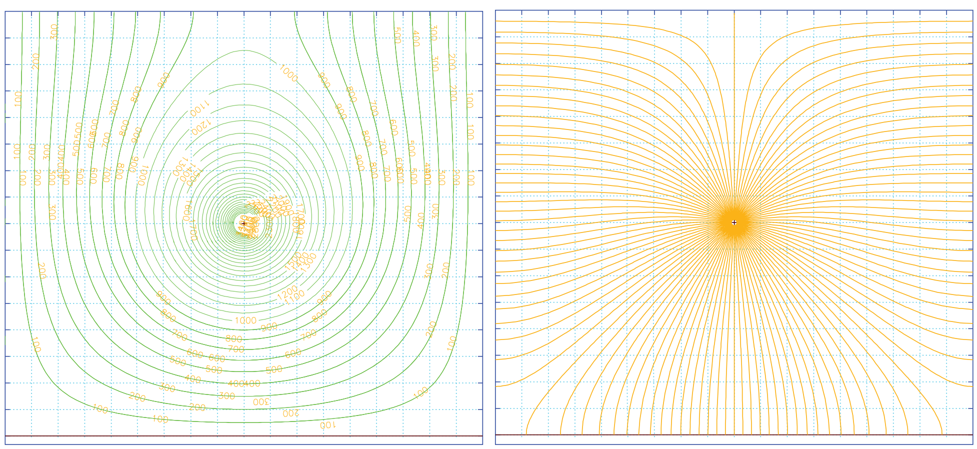

Figure 6: The electric potential (left) and field lines (right) as calculated for the geometry of an Iarocci tube using the computer program GARFIELD [3].

2.3 "Electrodeless" Drift Chamber (MWPC)

2.3.1 Wire at the Center of a Cell

We analyze the cell, shown in Fig. 2, in a coordinate system with its origin on the wire. The boundary conditions on the outer surfaces are,

$$\phi(-a/2, y) = \phi(a/2, y) = 0, \qquad \frac{\partial \phi(x, -b/2)}{\partial y} = \frac{\partial \phi(x, b/2)}{\partial y} = 0, \tag{22}$$

assuming that no electric field lines cross the surfaces $y = \pm b/2$. We consider the cell to be divided into two regions, $x < 0$ and $x > 0$. The matching condition on the plane $x = 0$ is,

$$\frac{\partial \phi(0^+, y)}{\partial x} = -2\pi q \delta(y), \tag{23}$$

The boundary conditions (22) on the outer surfaces of the cell indicate that a suitable series expansion is,

$$\phi(x, y) = A_0(a/2 - |x|) + \sum_{n=1}^{\infty} A_n \sinh \frac{2n\pi(a/2 - |x|)}{b} \cos \frac{2n\pi y}{b}. \tag{24}$$

Using the matching condition (23) at $x = 0$ we find the coefficient A_n to be,

$$A_0 = \frac{2\pi q}{b}, \qquad A_n = \frac{2q}{n \cosh \frac{n\pi a}{b}}, \tag{25}$$

Hence, the potential can be written as,

$$\phi(x, y) = \frac{\pi q a}{b} - \frac{2\pi q\,|x|}{b} + 2q \sum_{n=1}^{\infty} \frac{1}{n} \frac{\sinh \frac{2n\pi(a/2-|x|)}{b}}{\cosh \frac{n\pi a}{b}} \cos \frac{2n\pi y}{b} \qquad \text{(origin at the wire)}. \tag{26}$$

For $a \gg b$, eq. (26) simplifies to,

$$\phi(x, y) \approx -\frac{2\pi q\,|x|}{b} + 2q \sum_{n=1}^{\infty} \frac{1}{n} e^{-2n\pi|x|/b} \cos \frac{2n\pi y}{b} \qquad \text{(origin at the wire)}, \tag{27}$$

neglecting the constant term $\pi q a/b$, which then agrees with the known solution [4].

Except near $x = 0$, the exponential terms in eqs. (26) and (27) are small. So, over most of the cell the potential varies linearly with x, and the electric field is parallel to the x axis. The field strength is $\pm 2\pi q/b$, as if the charge q were uniformly distributed over the plane $(x = 0, |y| < b/2)$. For $x \approx 0$, $y \approx 0$ the equipotentials become cylinders around the wire, as shown in Fig. 7. This is, of course, the desirable field configuration for a drift chamber.

For a point (x, y) on the wire of radius $r = \sqrt{x^2 + y^2} \ll a,\ b$, an argument as given in eqs. (6)-(10) indicates that the sum in eq. (26) is approximately $2q \ln(b/2\pi r)$. Hence, the potential on the wire is,

$$\phi_{\text{wire}} \approx \frac{\pi q a}{b} + 2q \ln \frac{b}{2\pi r}. \tag{28}$$

The first term of this is just the "uniform" electric field $E_x = 2\pi q/b$ multiplied by the distance $a/2$ between the wire and a ground plane. The second term is the same as the potential for a wire on an isolated grid [4].

Equation (27) can also be written in closed form,

$$\phi(x, y) = -q \ln \left[2 \left(\cosh \frac{2\pi x}{a} - \cos \frac{2\pi y}{a} \right) \right], \tag{29}$$

as can be deduced by the use of functions of a complex variable [5].

Apparently, the potential (26) can be expressed in "closed form" using Jacobian elliptic functions [6, 7], although it is not clear this has much practical advantage.

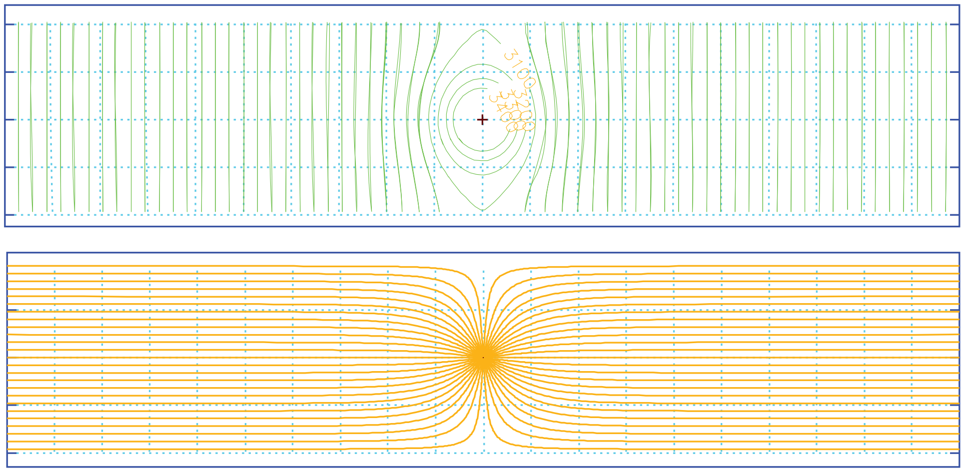

Figure 7: The electric potential (top) and field lines (bottom)as calculated for the geometry of an "electrodeless" drift chamber using the computer program GARFIELD [3].

2.3.2 Wires at Arbitrary Positions

The wires in a multiwire proportional chamber may not all be at the centers of their cells, due to "errors" in fabrication. The method used to find the potential (26) is not appropriate for this case, in that the boundary conditions (23) are satisfied only if the "error" in the placement of one wire is replicated in the placement of the other wires as well.

We recall that solutions to Laplace's equation in rectangular coordinates in two dimensions can have oscillatory functions in one coordinate and exponential functions in the one. Let us now consider solutions that are exponential is x and oscillatory in y.

We first analyze the case of a single wire of charge λ_0 (per unit length) at position (x_0, y_0). Then, the potential from an arbitray set of wires can be built up from this result, supposing that we know the charges, rather than the potentials on the wires.

We begin the analysis in a coordinate system with the grounded planes at $x = 0,\ a$, and build up solutions in the two regions $y < y_0$ and $y > y_0$. The boundary conditions are now,

$$\phi(0, y) = \phi(a, y) = \phi(x, \pm\infty) = 0, \qquad \frac{\partial \phi(x, y_{0,-})}{\partial y} - \frac{\partial \phi(x, y_{0,+})}{\partial y} = 4\pi q_0 \delta(x - x_0). \tag{30}$$

An expansion of the potential that satisfies the first three conditions of (30), and is continuous at $y = y_0$, is given by,

$$\phi(x, y) = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi x}{a} e^{-n\pi|y-y_0|/a}. \tag{31}$$

Using the fourth condition of of (30) we find $A_n = 4(q_0/n) \sin n\pi x_0/a$, so the potential due to a single wire is,

$$\phi(x, y) \quad = \quad 4q_0 \sum_{n=1}^{\infty} \frac{1}{n} \sin \frac{n\pi x_0}{a} \sin \frac{n\pi x}{a} e^{-n\pi|y-y_0|/a}$$

$$
\begin{aligned}
&= 2q_0 \sum_{n=1}^{\infty} \frac{1}{n} \left[\cos \frac{n\pi(x-x_0)}{a} - \cos \frac{n\pi(x+x_0)}{a} \right] e^{-n\pi|y-y_0|/a} \\
&= 2q_0 \; Re \sum_{n=1}^{\infty} \frac{1}{n} \left[e^{in\pi(x-x_0)/a} - e^{in\pi(x+x_0)/a} \right] e^{-n\pi|y-y_0|/a} \\
&= q_0 \left[\frac{e^{\pi|y-y_0|/a} - \cos \frac{\pi(x-x_0)}{a}}{\cosh \frac{\pi|y-y_0|}{a} - \cos \frac{\pi(x-x_0)}{a}} - \frac{e^{\pi|y-y_0|/a} - \cos \frac{\pi(x+x_0)}{a}}{\cosh \frac{\pi|y-y_0|}{a} - \cos \frac{\pi(x+x_0)}{a}} \right] \\
&= 2q_0 \frac{\sin \frac{\pi x_0}{a} \sin \frac{\pi x}{a} \sinh \frac{\pi|y-y_0|}{a}}{\cosh^2 \frac{\pi|y-y_0|}{a} - 2\cos \frac{\pi x_0}{a} \cos \frac{\pi x}{a} \cosh \frac{\pi|y-y_0|}{a} + \cos^2 \frac{\pi x_0}{a} + \cos^2 \frac{\pi x}{a} - 1} . \quad (32)
\end{aligned}
$$

We can now shift the origin in x from being on the left grounded plate to being on the midplane of the detector by replacing x by $x + a/2$ (and x_0 by $x_0 + a/2$) in eq. (32). This implies simply that $\sin \to \cos$ and $\cos \to -\sin$,

$$
\phi(x, y) = 2q_0 \frac{\cos \frac{\pi x_0}{a} \cos \frac{\pi x}{a} \sinh \frac{\pi|y-y_0|}{a}}{\cosh^2 \frac{\pi|y-y_0|}{a} - 2\sin \frac{\pi x_0}{a} \sin \frac{\pi x}{a} \cosh \frac{\pi|y-y_0|}{a} + \sin^2 \frac{\pi x_0}{a} + \sin^2 \frac{\pi x}{a} - 1} . \quad (33)
$$

The potential due to a set of wires at (x_i, y_i) carrying charges λ_i can therefore be written,

$$
V(x, y) = 2\sum_i q_i \frac{\cos \frac{\pi x_0}{a} \cos \frac{\pi x}{a} \sinh \frac{\pi|y-y_i|}{a}}{\cosh^2 \frac{\pi|y-y_i|}{a} - 2\sin \frac{\pi x_i}{a} \sin \frac{\pi x}{a} \cosh \frac{\pi|y-y_i|}{a} + \sin^2 \frac{\pi x_i}{a} + \sin^2 \frac{\pi x}{a} - 1} . \quad (34)
$$

For a single wire midway between the two grounded planes, $(x_0, y_0) = (0, y_0)$, the potential (32) becomes,

$$
\phi(x, y) = 2q_0 \frac{\cos \frac{\pi x}{a} \sinh \frac{\pi|y-y_0|}{a}}{\cosh^2 \frac{\pi|y-y_0|}{a} - \cos^2 \frac{\pi x}{a}} . \quad (35)
$$

Hence, the potential due to a regular grid of wires at $x = a/2$, $y = nb$, each carrying charge q, is

$$
\phi(x, y) = 2q \sum_{n=-\infty}^{\infty} \frac{\cos \frac{\pi x}{a} \sinh \frac{\pi|y-nb|}{a}}{\cosh^2 \frac{\pi|y-nb|}{a} - \cos^2 \frac{\pi x}{a}} \qquad \text{(origin at a wire).} \quad (36)
$$

This provides an alternative expansion to eq. (26) for the case of an ideal multiwire proportional chamber.

2.4 MWPC with Alternating Anode and Cathode Wires

This problem is the same as that of the previous section, with the addition of cathode wires at $(0, \pm b/2)$, as shown in Fig, 3. These wires will carry charge of opposite sign to that on the anode wire, say $-\epsilon q$. Since only half the charge on these cathode wire contributes to the field inside a particular cell, we can write the matching condition at the plane $x = 0$ as,

$$
\frac{\partial \phi(0^+, y)}{\partial x} = -2\pi q \delta(y) + \pi\epsilon q \delta(y - b/2) + \pi\epsilon q \delta(y + b/2). \quad (37)
$$

The potential is again symmetric in both x and y, so we can again use the series expansion (24). The boundary condition (37) implies that the Fourier coefficients are,

$$A_0 = \frac{2\pi q(1-\epsilon)}{b}, \qquad A_n = \frac{2q[1-(-1)^n\epsilon]}{n\cosh\frac{n\pi a}{b}}\,, \tag{38}$$

which again vanish for odd n. Hence, we replace n by $2n$ in eqs. (24) and (38) and sum over all n to write the potential as,

$$\phi(x,y) = \frac{2\pi q(1-\epsilon)(a/2-|x|)}{b} + 2q\sum_{n=1}^{\infty}\frac{[1-(-1)^n\epsilon]}{n}\frac{\sinh\frac{2n\pi(a/2-|x|)}{b}}{\cosh\frac{n\pi a}{b}}\cos\frac{2n\pi y}{b}\,. \tag{39}$$

The coefficient ϵ is determined by the requirement that the potential on the cathode wires be zero,

$$\phi(0,\pm b/2) = 0 = \frac{\pi q(1-\epsilon)a}{b} + 2q\sum_{n=1}^{\infty}\frac{[(-1)^n-\epsilon]}{n}\tanh\frac{n\pi a}{b}\,. \tag{40}$$

2.5 Array of Cells with One Anode and One Cathode Wire

The charge distribution in this problem is specified, so we can use the formal expression

$$\phi(x,y) = \int\frac{\rho}{r}\,d\mathrm{Vol} \tag{41}$$

to evaluate the potential. The anode wires are located at $(x,y) = ((2m+1)a,(2n+1)b)$, where integers m and n take on both positive and negative values. The cathode wires are at $(x,y) = (2ma,2nb)$. The potential at a point that is at distance r from a wire that carries charge q per unit length is $-2q\ln r$, so the potential of the wire array of Fig. 4 is,

$$\begin{aligned}\phi(x,y) &= 2q\sum_{m,n}\left(\ln\sqrt{(x-2ma)^2+(y-2nb)^2} - \ln\sqrt{(x-(2m+1)a)^2+(y-(2n+1)b)^2}\right)\\ &= q\sum_{m,n}\ln\frac{(x-2ma)^2+(y-2nb)^2}{(x-(2m+1)a)^2+(y-(2n+1)b)^2}\,.\end{aligned} \tag{42}$$

The potential (42) is zero at points $(x,y) = (ma+a/2, nb+b/2)$ and diverges at the wires.

2.6 Array of Cells with 1 Anode and 3 Cathode Wires

The array of cells shown in Fig. 5 differs from that in Fig. 4 by the addition of cathode wires at $(x,y) = (2ma,(2n+1)b)$ and at $((2m+1)a,2nb)$. Again, we sum the logarithmic potentials of the wires to find,

$$\begin{aligned}\phi(x,y) = q\sum_{m,n}\big(&\epsilon_1\ln[(x-2ma)^2+(y-2nb)^2] + \epsilon_2\ln[(x-2ma)^2+(y-(2n+1)b)^2]\\ &+\epsilon_3\ln[(x-(2m+1)a)^2+(y-2nb)^2] - \ln[(x-(2m+1)a)^2+(y-(2n+1)b)^2]\big)\,,\end{aligned} \tag{43}$$

where the sum of the charges per unit length, $-\epsilon_1 q$, $-\epsilon_2 q$ and $-\epsilon_3 q$, on the cathode wires is the negative of the charge per unit length q on the anode wire.

References

[1] E. Iarocci, *Plastic Streamer Tubes and Their Applications in High Energy Physics*, Nucl. Instr. and Meth. **217**, 30 (1983),
http://kirkmcd.princeton.edu/examples/detectors/iarocci_nim_217_30_83.pdf

[2] K.T. McDonald, *Notes on Electrostatic Wire Grids* (Mar. 5, 2003)
http://kirkmcd.princeton.edu/examples/grids.pdf

[3] GARFIELD, a computer program for simulation of gaseous detectors,
http://consult.cern.ch/writeup/garfield/

[4] See, for example, problem 1 of Princeton Ph501 Set 3,
http://kirkmcd.princeton.edu/examples/ph501set3.pdf

[5] See, for example, J.C. Maxwell, *A Treatise on Electricity and Magnetism*, 3rd ed. (Macmillan, 1904), Vol. 1, p. 310,
http://kirkmcd.princeton.edu/examples/EM/maxwell_treatise_v1_04.pdf

[6] T. Tomitani, *Analysis of Potential Distribution in a Gaseous Counter of Rectangular Cross-Section*, Nucl. Instr. and Meth. **100**, 179 (1972),
http://kirkmcd.princeton.edu/examples/EM/tomitani_nim_100_179_72.pdf

[7] G.A. Erskine, *Electrostatic Problems in Multiwire Proportional Chambers*, Nucl. Instr. and Meth. **105**, 565 (1972), http://kirkmcd.princeton.edu/examples/EM/erskine_nim_105_565_72.pdf