

Images of Electric and Magnetic Dipole Antennas in a Conducting Plane

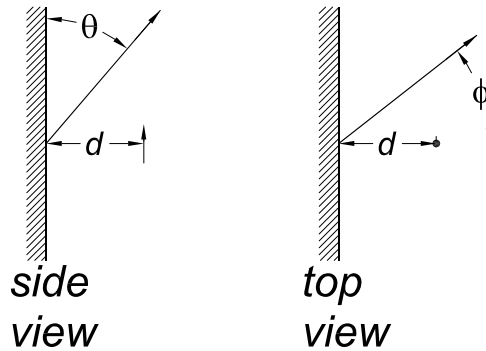
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1 Problem

A short, linear dipole antenna can be thought of as an oscillating electric dipole of angular frequency ω . Suppose such an antenna (of length much less than a wavelength $\lambda = c/\omega$) is located at distance d away from a perfectly conducting plane. In the first part of the problem, the dipole is oriented parallel to the plane, as shown below.



Show that the power radiated in the direction (θ, ϕ) is,

$$\frac{dP}{d\Omega} = 4A \sin^2 \theta \sin^2 \Delta, \quad (1)$$

where,

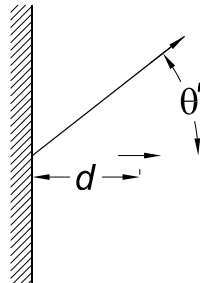
$$\Delta = \frac{2\pi\lambda}{d} \sin \theta \cos \phi, \quad (2)$$

and the power radiated by the dipole alone is,

$$\frac{dP}{d\Omega} = A \sin^2 \theta. \quad (3)$$

Sketch the shape of the radiation pattern for $d = \lambda/2$ and $d = \lambda/4$.

Suppose instead that the electric dipole was oriented perpendicular to the conducting plane.



Show that the radiated power in this case is,

$$\frac{dP}{d\Omega} = 4A \sin^2 \theta' \cos^2 \Delta, \quad (4)$$

where,

$$\Delta = \frac{2\pi\lambda}{d} \cos \theta'. \quad (5)$$

In the above, the polar angles θ and θ' are measured with respect to the axes of the dipoles.

Consider also the case of a small loop antenna, which can be thought of as a magnetic-dipole oscillator, in the two orientations illustrated above.

2 Solution

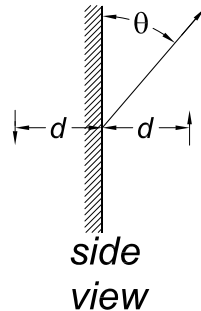
For a general discussion of electric and magnetic image methods, see

<http://kirkmcd.princeton.edu/examples/image.pdf>

2.1 Linear, Electric Dipole Antenna

2.1.1 Parallel to the Conducting Plane

Since the dipole is much less than a wavelength away from the conducting plane, the fields between the dipole and the plane are essentially the instantaneous static fields. Thus, charges arrange themselves on the plane as if there were an image dipole at distance d on the other side of the plane. The radiation from the moving charges on the plane is effectively that due to the oscillating image dipole. A distant observer sees the sum of the radiation fields from the dipole and its image.



The image dipole is inverted with respect to the original, *i.e.*, the two dipoles are 180° out of phase.

Furthermore, there is a difference s in path length between the two dipoles and the distant observer at angles (θ, ϕ) . We first calculate in a spherical coordinate system with z axis along the first dipole, and x axis pointing from the plane to that dipole. Then, the path difference is,

$$s = 2d \hat{\mathbf{x}} \cdot \hat{\mathbf{n}} = 2d \sin \theta \cos \phi. \quad (6)$$

This path difference results in an additional phase difference δ between the fields from the two dipoles at the observer, in the amount,

$$\delta = 2\pi \frac{s}{\lambda} = \frac{4\pi d}{\lambda} \sin \theta \cos \phi. \quad (7)$$

If we label the electric fields due to the original and image dipoles as \mathbf{E}_1 and \mathbf{E}_2 , respectively, then the total field is,

$$\mathbf{E} = \mathbf{E}_1 + \mathbf{E}_2 = \mathbf{E}_1(1 - e^{i\delta}), \quad (8)$$

and, recalling eq. (3), the power radiated is,

$$\begin{aligned} \frac{dP}{d\Omega} &= \frac{|\mathbf{E}|^2}{|\mathbf{E}_1|^2} \frac{dP_1}{d\Omega} = |1 - e^{i\delta}|^2 A \sin^2 \theta = 2A \sin^2 \theta (1 - \cos \delta) = 4A \sin^2 \theta \sin^2 \delta/2 \\ &= 4A \sin^2 \theta \sin^2 \Delta, \end{aligned} \quad (9)$$

where,

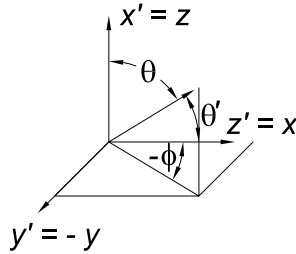
$$\Delta = \frac{\delta}{2} = \frac{2\pi d}{\lambda} \sin \theta \cos \phi. \quad (10)$$

Suppose we had chosen to use a spherical coordinate system (r, θ', ϕ') with the z' axis pointing from the plane to dipole 1, and the x' axis parallel to dipole 1. Then, the phase difference would have the simple form,

$$\Delta = \frac{\delta}{2} = \frac{\pi}{\lambda} 2d \hat{\mathbf{z}}' \cdot \hat{\mathbf{n}} = \frac{2\pi d}{\lambda} \cos \theta', \quad (11)$$

but the factor $\sin^2 \theta$ would now become,

$$\sin^2 \theta = n_x^2 + n_y^2 = n_{z'}^2 + n_{y'}^2 = \cos^2 \theta' + \sin^2 \theta' \cos^2 \phi' = 1 - \sin^2 \theta' \sin^2 \phi'. \quad (12)$$



If $d = \lambda/4$, then

$$\frac{dP}{d\Omega} = 4A \sin^2 \theta \sin^2 \left(\frac{\pi}{2} \sin \theta \cos \phi \right). \quad (13)$$

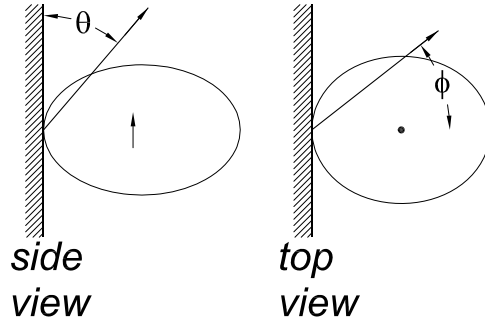
In the “side” view, $\phi = 0$, so the pattern has shape,

$$\sin^2 \theta \sin^2 \left(\frac{\pi}{2} \sin \theta \right) \quad (\text{side view}), \quad (14)$$

while in the “top” view, $\theta = \pi/2$ and the shape is,

$$\sin^2 \left(\frac{\pi}{2} \cos \phi \right) \quad (\text{top view}). \quad (15)$$

This pattern has a single lobe in the forward hemisphere, as illustrated below:



If instead, $d = \lambda/2$, then,

$$\frac{dP}{d\Omega} = 4A \sin^2 \theta \sin^2 (\pi \sin \theta \cos \phi). \quad (16)$$

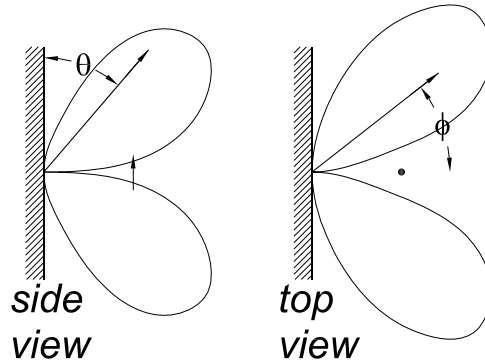
In the “side” view, $\phi = 0$, so the pattern has shape,

$$\sin^2 \theta \sin^2 (\pi \sin \theta) \quad (\text{side view}), \quad (17)$$

while in the “top” view, $\theta = \pi/2$ and the shape is,

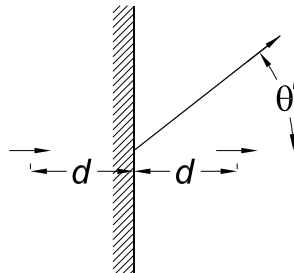
$$\sin^2 (\pi \cos \phi) \quad (\text{top view}). \quad (18)$$

This pattern does not radiate along the line from the plane to the dipole, as illustrated below:



2.1.2 Perpendicular to the Conducting Plane

If the electric dipole is aligned with the line from the plane to the dipole, its image has the same orientation.



The only phase difference between the radiation fields of the dipole and its image is that due to the path difference δ , whose value has been given in eqs. (10) and (11). It is simpler to use the angles (θ', ϕ') in this case, since the radiation pattern of a single dipole varies as $\sin^2 \theta'$. Then,

$$\mathbf{E} = \mathbf{E}_1 + \mathbf{E}_2 = \mathbf{E}_1(1 + e^{i\delta}), \quad (19)$$

and, recalling eq. (3), the power radiated is,

$$\begin{aligned} \frac{dP}{d\Omega} &= \frac{|\mathbf{E}|^2}{|\mathbf{E}_1|^2} \frac{dP_1}{d\Omega} = |1 + e^{i\delta}|^2 A \sin^2 \theta' = 2A \sin^2 \theta' (1 + \cos \delta) = 4A \sin^2 \theta' \cos^2 \delta/2 \\ &= 4A \sin^2 \theta' \cos^2 \Delta, \end{aligned} \quad (20)$$

with,

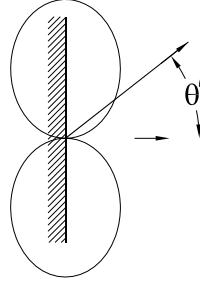
$$\Delta = \frac{\delta}{2} = \frac{2\pi d}{\lambda} \cos \theta'. \quad (21)$$

This radiation pattern is axially symmetric about the line from the plane to the dipole.

If $d = \lambda/4$, then,

$$\frac{dP}{d\Omega} = 4A \sin^2 \theta' \cos^2 \left(\frac{\pi}{2} \cos \theta' \right). \quad (22)$$

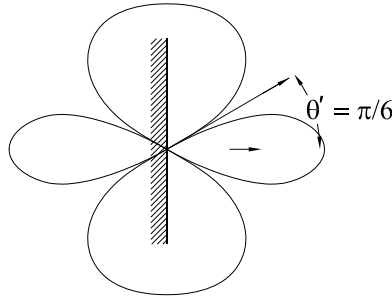
This pattern is a flattened version of the “donut” pattern $\sin^2 \theta'$, as illustrated below:



If instead, $d = \lambda/2$, then,

$$\frac{dP}{d\Omega} = 4A \sin^2 \theta' \cos^2 (\pi \cos \theta'). \quad (23)$$

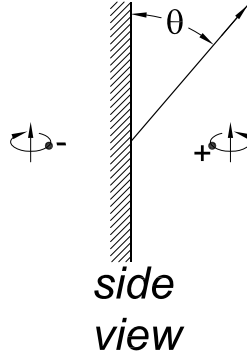
This pattern has a forward lobe for $\theta' < \pi/6$ and a “donut” for $\pi/6 < \theta' < \pi/2$, as illustrated below:



2.2 Small, Magnetic Loop Antenna

2.2.1 Axis Parallel to the Conducting Plane

For a magnetic dipole with axis parallel to the conducting plane, the image dipole has the same orientation, the image consists of the opposite charge rotating in the opposite direction, as shown below:



We use angles (θ, ϕ) and modify the argument of part 2.1.1 to find,

$$\mathbf{E} = \mathbf{E}_1 + \mathbf{E}_2 = \mathbf{E}_1(1 + e^{i\delta}), \quad (24)$$

and, recalling eq. (3), the power radiated is,

$$\begin{aligned} \frac{dP}{d\Omega} &= \frac{|\mathbf{E}|^2}{|\mathbf{E}_1|^2} \frac{dP_1}{d\Omega} = |1 + e^{i\delta}|^2 A \sin^2 \theta = 2A \sin^2 \theta (1 + \cos \delta) = 4A \cos^2 \theta \sin^2 \delta / 2 \\ &= 4A \sin^2 \theta \cos^2 \Delta, \end{aligned} \quad (25)$$

where,

$$\Delta = \frac{\delta}{2} = \frac{2\pi d}{\lambda} \sin \theta \cos \phi. \quad (26)$$

If $d = \lambda/4$, then,

$$\frac{dP}{d\Omega} = 4A \sin^2 \theta \cos^2 \left(\frac{\pi}{2} \sin \theta \cos \phi \right). \quad (27)$$

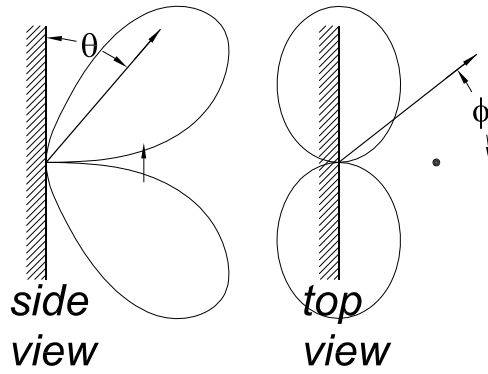
In the “side” view, $\phi = 0$, so the pattern has shape,

$$\sin^2 \theta \cos^2 \left(\frac{\pi}{2} \sin \theta \right) \quad (\text{side view}), \quad (28)$$

while in the “top” view, $\theta = \pi/2$ and the shape is,

$$\cos^2 \left(\frac{\pi}{2} \cos \phi \right) \quad (\text{top view}). \quad (29)$$

This pattern, shown below, is somewhat similar to that of part a) for, $d = \lambda/2$.



If instead, $d = \lambda/2$, then,

$$\frac{dP}{d\Omega} = 4A \sin^2 \theta \cos^2 (\pi \sin \theta \cos \phi). \quad (30)$$

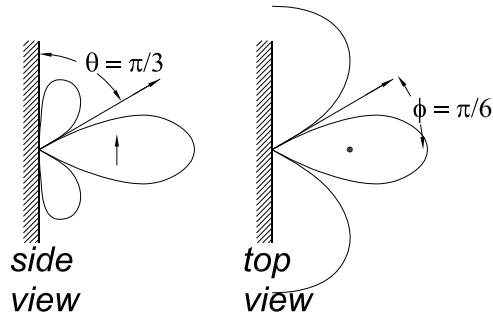
In the “side” view, $\phi = 0$, so the pattern has shape,

$$\sin^2 \theta \cos^2 (\pi \sin \theta) \quad (\text{side view}), \quad (31)$$

while in the “top” view, $\theta = \pi/2$ and the shape is,

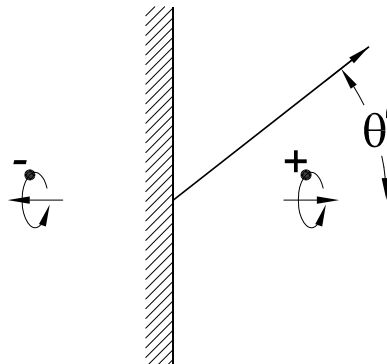
$$\cos^2 (\pi \cos \phi) \quad (\text{top view}). \quad (32)$$

This pattern, shown below, is somewhat similar to that of part b) for $d = \lambda/2$.



2.2.2 Axis Perpendicular to the Conducting Plane

Finally, we consider the case of a magnetic dipole aligned with the line from the plane to the dipole, in which case its image has the opposite orientation.



As in part 2.1.2, the only phase difference between the radiation fields of the dipole and its image is that due to the path difference δ , whose value has been given in eqs. (10) and (11). We use the angles (θ', ϕ') in this case, since the radiation pattern of a single dipole varies as $\sin^2 \theta'$. Then,

$$\mathbf{E} = \mathbf{E}_1 + \mathbf{E}_2 = \mathbf{E}_1(1 - e^{i\delta}), \quad (33)$$

and, recalling eq. (3), the power radiated is,

$$\begin{aligned} \frac{dP}{d\Omega} &= \frac{|\mathbf{E}|^2}{|\mathbf{E}_1|^2} \frac{dP_1}{d\Omega} = |1 - e^{i\delta}|^2 A \sin^2 \theta' = 2A \sin^2 \theta' (1 - \cos \delta) = 4A \sin^2 \theta' \sin^2 \delta/2 \\ &= 4A \sin^2 \theta' \sin^2 \Delta, \end{aligned} \quad (34)$$

with,

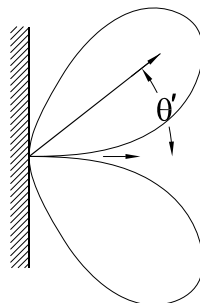
$$\Delta = \frac{\delta}{2} = \frac{2\pi d}{\lambda} \cos \theta'. \quad (35)$$

This radiation pattern is axially symmetric about the line from the plane to the dipole.

If $d = \lambda/4$, then,

$$\frac{dP}{d\Omega} = 4A \sin^2 \theta' \sin^2 \left(\frac{\pi}{2} \cos \theta' \right). \quad (36)$$

This pattern, shown below, is somewhat similar to that of part 2.1.1 for $d = \lambda/2$.



If instead, $d = \lambda/2$, then,

$$\frac{dP}{d\Omega} = 4A \sin^2 \theta' \sin^2 (\pi \cos \theta'). \quad (37)$$

This pattern is qualitatively similar to that for $d = \lambda/4$, shown just above, but the maximum occurs at a larger value of θ' .

3 Comments *(Oct. 24, 2023)*

The image method (based on statics) enforces the boundary condition that the electric field just outside a perfect conductor is perpendicular to the surface of that conductor. This happens to hold even in the case of time dependence for an infinite plane surface of the conductor, but not for other geometries.¹

¹K.T. McDonald, *Image Method for Time-Dependent Charge/Current Distributions above a Perfectly Conducting Plane* (Nov. 17, 2023), http://kirkmcd.princeton.edu/examples/image_dipole.pdf

For time-dependent dipoles, a magnetic field is also present, and the preceding discussion neglected to consider that a magnetic field must be tangential just outside the surface of a perfect conductor. However, this condition is satisfied by the image method for time-dependent, short dipoles, both “electric” and “magnetic”, in the case of an infinite plane surface (but again, not for other geometries).

Another issue not explicitly discussed above is that the image of a moving charge is at the instantaneous image location even though the speed of light is finite. Then, the fields at the surface of the perfect conductor at some time t are those due to the source and image charges at the same retarded time, and the boundary conditions at the surface are thereby satisfied.