Image Method for Time-Dependent Charge/Current Distributions above a Perfectly Conducting Plane

Kirk T. McDonald
Joseph Henry Laboratories, Princeton University, Princeton, NJ 08544
(November 17, 2023)

1 Problem

The image method arose in electrostatics,[1] and in general cannot be applied to time-dependent examples because the boundary conditions for $\mathbf{E}$ and $\mathbf{B}$ are not satisfied at the relevant perfectly conducting surface. Show, however, that image method does apply to arbitrary time-dependent charge/current distributions above a perfectly conducting plane.

2 Solution

We recall that the boundary conditions at the surface of a perfect conductors are that the electric field $\mathbf{E}$ be perpendicular to the surface, while the magnetic field $\mathbf{B}$ must be parallel to the surface.

We consider an electric charge q in arbitrary motion above a perfectly conducting plane, say, $z = 0$. In the absence of the conducting plane, the electromagnetic fields of the charge are those given by Liénard and Wiechert [3, 4], which can be written as,

$$\mathbf{E} = q\left(\frac{\hat{\mathbf{r}} - \boldsymbol{\beta}}{\gamma^2(1 - \hat{\mathbf{r}}\cdot\boldsymbol{\beta})^3\, r^2}\right) + \frac{q}{c}\left(\frac{\hat{\mathbf{r}} \times ((\hat{\mathbf{r}} - \boldsymbol{\beta}) \times \dot{\boldsymbol{\beta}})}{(1 - \hat{\mathbf{r}}\cdot\boldsymbol{\beta})^3\, r}\right), \qquad \mathbf{B} = \hat{\mathbf{r}} \times \mathbf{E}, \tag{1}$$

in Gaussian units, where $\mathbf{r}$ is the distance vector from the charge to the observation point, $\boldsymbol{\beta} = \mathbf{v}/c$ is the velocity of the charge normalized to c, the speed of light in vacuum, $\dot{\boldsymbol{\beta}} = d\boldsymbol{\beta}/dt$ is the normalized acceleration, $\gamma = 1/\sqrt{1-\beta^2}$, and $\mathbf{r}$, $\boldsymbol{\beta}$ and $\dot{\boldsymbol{\beta}}$ are evaluated at the retarded time $t_{\rm ret} = t - r/c$.

To verify the boundary conditions at the surface, $z = 0$, of the perfectly conducting plane, we now consider an observation point on that plane.

Writing $\mathbf{r} = \mathbf{r}_\perp + \mathbf{r}_z$, $\boldsymbol{\beta} = \boldsymbol{\beta}_\perp + \boldsymbol{\beta}_z$ and $\dot{\boldsymbol{\beta}} = \dot{\boldsymbol{\beta}}_\perp + \dot{\boldsymbol{\beta}}_z$, the corresponding quantities for the image charge $q' = -q$ with respect to the plane $z = 0$ can be written as $\mathbf{r}' = \mathbf{r}_\perp - \mathbf{r}_z$, $\boldsymbol{\beta}' = \boldsymbol{\beta}_\perp - \boldsymbol{\beta}_z$ and $\dot{\boldsymbol{\beta}}' = \dot{\boldsymbol{\beta}}_\perp - \dot{\boldsymbol{\beta}}_z$. Then, $\hat{\mathbf{r}}'\cdot\boldsymbol{\beta}' = \hat{\mathbf{r}}\cdot\boldsymbol{\beta}$, $\beta' = \beta$ and $\gamma' = \gamma$, so we can write the electric fields $\mathbf{E}$ (due to q) and $\mathbf{E}'$ (due to image charge $-q$) for $z = 0$ as,

$$\begin{aligned}\mathbf{E}(z=0) &= a(\hat{\mathbf{r}} - \boldsymbol{\beta}) + b(\hat{\mathbf{r}} \times ((\hat{\mathbf{r}} - \boldsymbol{\beta}) \times \dot{\boldsymbol{\beta}}) \\ &= a(\hat{\mathbf{r}} - \boldsymbol{\beta}) + b(\hat{\mathbf{r}}\cdot\dot{\boldsymbol{\beta}})(\hat{\mathbf{r}} - \boldsymbol{\beta}) - b(1 - \hat{\mathbf{r}}\cdot\boldsymbol{\beta})\dot{\boldsymbol{\beta}} \\ &= a(\hat{\mathbf{r}}_\perp + \hat{\mathbf{r}}_z - \boldsymbol{\beta}_\perp - \boldsymbol{\beta}_z) + b(\hat{\mathbf{r}}\cdot\dot{\boldsymbol{\beta}})(\hat{\mathbf{r}}_\perp + \hat{\mathbf{r}}_z - \boldsymbol{\beta}_\perp - \boldsymbol{\beta}_z) - b(1 - \hat{\mathbf{r}}\cdot\boldsymbol{\beta})(\dot{\boldsymbol{\beta}}_\perp + \dot{\boldsymbol{\beta}}_z),\end{aligned} \tag{2}$$

$$\begin{aligned}\mathbf{E}'(z=0) &= -a(\hat{\mathbf{r}}' - \boldsymbol{\beta}') - b(\hat{\mathbf{r}}' \times ((\hat{\mathbf{r}}' - \boldsymbol{\beta}') \times \dot{\boldsymbol{\beta}}') \\ &= -a(\hat{\mathbf{r}}_\perp - \hat{\mathbf{r}}_z - \boldsymbol{\beta}_\perp + \boldsymbol{\beta}_z) - b(\hat{\mathbf{r}}\cdot\dot{\boldsymbol{\beta}})(\hat{\mathbf{r}}_\perp - \hat{\mathbf{r}}_z - \boldsymbol{\beta}_\perp + \boldsymbol{\beta}_z) - b(1 - \hat{\mathbf{r}}\cdot\boldsymbol{\beta})(\dot{\boldsymbol{\beta}}_\perp - \dot{\boldsymbol{\beta}}_z),\end{aligned} \tag{3}$$

[1] The image method was first discussed by W. Thomson (Lord Kelvin) in 1848 [1]. For a review that mentions the present problem, see [2].

where function a (and function b) has the same value at a point on the surface of the conducting plane for both charge q and image charge $-q$. Hence, the total electric field at the surface of the conducting plane is,

$$\mathbf{E}_{\rm tot}(z=0) = \mathbf{E}(z=0) + \mathbf{E}'(z=0) = 2a(\hat{\mathbf{r}}_z - \boldsymbol{\beta}_z) + 2b(\hat{\mathbf{r}}\cdot\dot{\boldsymbol{\beta}})(\hat{\mathbf{r}}_z - \boldsymbol{\beta}_z) - 2b(1-\hat{\mathbf{r}}\cdot\boldsymbol{\beta})\dot{\boldsymbol{\beta}}_z, \quad (4)$$

which has only a z-component, as required at the surface of a perfect conductor.

The magnetic fields at the surface of the perfect conductor are, recalling eqs. (2)-(3).

$$\mathbf{B}(z=0) = \hat{\mathbf{r}}\times\mathbf{E} = -a\hat{\mathbf{r}}\times\boldsymbol{\beta} - b(\hat{\mathbf{r}}\cdot\dot{\boldsymbol{\beta}})\,\hat{\mathbf{r}}\times\boldsymbol{\beta} - b(1-\hat{\mathbf{r}}\cdot\boldsymbol{\beta})\,\hat{\mathbf{r}}\times\dot{\boldsymbol{\beta}}, \quad (5)$$

$$\mathbf{B}'(z=0) = \hat{\mathbf{r}}'\times\mathbf{E}' = a\hat{\mathbf{r}}'\times\boldsymbol{\beta}' + b(\hat{\mathbf{r}}\cdot\dot{\boldsymbol{\beta}})\,\hat{\mathbf{r}}'\times\boldsymbol{\beta}' + b(1-\hat{\mathbf{r}}\cdot\boldsymbol{\beta})\,\hat{\mathbf{r}}'\times\dot{\boldsymbol{\beta}}'. \quad (6)$$

Now,

$$\hat{\mathbf{r}}\times\boldsymbol{\beta} = (\hat{r}_y\beta_z - \hat{r}_z\beta_y)\,\hat{\mathbf{x}} + (\hat{r}_z\beta_x - \hat{r}_x\beta_z)\,\hat{\mathbf{y}} + (\hat{r}_x\beta_y - \hat{r}_y\beta_x)\,\hat{\mathbf{z}}, \quad (7)$$

$$\hat{\mathbf{r}}'\times\boldsymbol{\beta}' = -(\hat{r}_y\beta_z - \hat{r}_z\beta_y)\,\hat{\mathbf{x}} - (\hat{r}_z\beta_x - \hat{r}_x\beta_z)\,\hat{\mathbf{y}} + (\hat{r}_x\beta_y - \hat{r}_y\beta_x)\,\hat{\mathbf{z}}, \quad (8)$$

since $\mathbf{r} = r_x\,\hat{\mathbf{x}} + r_y\,\hat{\mathbf{y}} + r_z\,\hat{\mathbf{z}}$ and $\boldsymbol{\beta} = \beta_x\,\hat{\mathbf{x}} + \beta_y\,\hat{\mathbf{y}} + \beta_z\,\hat{\mathbf{z}}$, while $\mathbf{r}' = r_x\,\hat{\mathbf{x}} + r_y\,\hat{\mathbf{y}} - r_z\,\hat{\mathbf{z}}$ and $\boldsymbol{\beta}' = \beta_x\,\hat{\mathbf{x}} + \beta_y\,\hat{\mathbf{y}} - \beta_z\,\hat{\mathbf{z}}$. Then,

$$\hat{\mathbf{r}}\times\boldsymbol{\beta} - \hat{\mathbf{r}}'\times\boldsymbol{\beta}' = 2(\hat{r}_y\beta_z - \hat{r}_z\beta_y)\,\hat{\mathbf{x}} + 2(\hat{r}_z\beta_x - \hat{r}_x\beta_z)\,\hat{\mathbf{y}}, \quad (9)$$

and similarly for $\hat{\mathbf{r}}\times\dot{\boldsymbol{\beta}} - \hat{\mathbf{r}}'\times\dot{\boldsymbol{\beta}}'$, both of which have no z-component. Hence, the total magnetic field $\mathbf{B}_{\rm tot}(z=0) = \mathbf{B}(z=0) + \mathbf{B}'(z=0)$ at the surface of the conducting plane has no z-component, as required at the surface of a perfect conductor.

Thus, the image method is valid for a single electric charge with arbitrary motion above a perfectly conducting plane. And, via the superposition principle, this also holds for any charge/current distribution above an infinite, perfectly conducting plane.

However, for arbitrary, time-dependent motion of the charge, the electromagnetic-field boundary conditions cannot be satisfied for any other type of surface of a perfect conductor, and the time-dependent image method does not in general.

3 Comments

The time-dependent image method has its main application to antennas over a perfectly conducting "ground" plane, to which the surface of the Earth is a good-enough approximation in many places.[2] This was discussed in [7, 8] for "electric" dipole antennas parallel and perpendicular to the "ground" surface. These examples were reviewed by the author in [9] for both "electric" and "magnetic" antennas. The more practical case of antennas above an imperfectly conducting plane has been discussed extensively, with approximate image methods reviewed, for example, in [10].

Another application of the time-dependent image method concerns current-carrying wires and metallic strips parallel to a perfectly conducting plane. These systems can be considered as 2-conductor transmission lines, with one of the conductors being the image (with respect to the perfectly conducting plane) of the physical wire/strip.

This note was inspired by e-discussions with Li Pan.

[2] Maxwell discussed a time-dependent image method for electric currents above an infinite conducting plane in 1872 [5], and in Arts. 660-667 of his *Treatise* [6].

References

[1] W. Thomson, *Geometrical Investigations with Reference to the Distribution of Electricity on Spherical Conductors*, Camb. Dublin Math. J. **3**, 131 (1848), http://kirkmcd.princeton.edu/examples/EM/thomson_cdmj_3_131_48.pdf

[2] P. Hammond, *Electric and Magnetic Images*, Proc. IEE **107**, 306 (1960), http://kirkmcd.princeton.edu/examples/EM/hammond_piee_107_306_60.pdf

[3] A. Liénard, *Champ électrique et magnétique produit par une charge électrique contentreé en un point et animée d'un mouvement quelconque*, L'Éclairage Élect. **16**, 5, 53, 106 (1898), http://kirkmcd.princeton.edu/examples/EM/lienard_ee_16_5_98.pdf

[4] E. Wiechert, *Elektrodynamishe Elementargesetze*, Arch. Néerl. **5**, 549 (1900); Ann. d. Phys. **73**, 667 (1901), http://kirkmcd.princeton.edu/examples/EM/wiechert_ap_73_667_01.pdf

[5] J.C. Maxwell, *On the Induction of Electric Currents in an Infinite Plane Sheet of uniform conductivity*, Proc. Roy. Soc. London **20**, 160 (1872)m http://kirkmcd.princeton.edu/examples/EM/maxwell_prsl_20_160_72.pdf

[6] J.C. Maxwell, *A Treatise on Electricity and Magnetism*, Vol. 2 (Clarendon Press, 1873), http://kirkmcd.princeton.edu/examples/EM/maxwell_treatise_v2_73.pdf

[7] J. Aharoni, *Antennae: An Introduction to Their Theory* (Clarendon Press, 1946), http://kirkmcd.princeton.edu/examples/EM/aharoni_46_sec21.pdf

[8] E.R. Moullin, *Radio Aerials* (Clarendon Press, 1949), http://kirkmcd.princeton.edu/examples/EM/moullin_49_sec3.2.pdf

[9] K.T. McDonald, *Images of Electric and Magnetic Dipole Antennas in a Conducting Plane* (Dec. 5, 1998), http://kirkmcd.princeton.edu/examples/image_antenna.pdf

[10] I.V. Lindell, E. Alanen and K. Mannersalo, *Exact Image Method for Impedance Computation of Antennas Above the Ground*, IEEE Trans. Ant. Prop. **33**, 937 (1985), http://kirkmcd.princeton.edu/examples/EM/lindell_ieeetap33_937_85.pdf