Image Method for an Electric Charge above a Topological Insulator with a Quantum Hall Surface

Kirk T. McDonald
Joseph Henry Laboratories, Princeton University, Princeton, NJ 08544
(January 14, 2010)

1 Problem

The method of images provides a solution to the problem of an electric charge q in vacuum above a dielectric half space. See, for example, secs. 404-406 of [1], sec. 4.4 of [2], or [3]. Consider the case that the dielectric is a so-called topological insulator[1] with relative permittivity ϵ and relative permeability μ, and that its planar interface with vacuum is coated with a thin layer of magnetic material [7]. This surface exhibits quantum Hall effects, some features of which can be modeled classically. In particular, although this surface is a nonconductor in the usual Ohmic sense, in the presence of an electric field $\mathbf{E}$ it supports a surface current $\mathbf{K}$ related by,

$$\mathbf{K} = P\,\hat{\mathbf{n}} \times \mathbf{E}, \tag{1}$$

where $P = \pm\alpha c/4\pi$ $(= \pm e^2/4\pi\hbar$ in Gaussian units), and the sign of P depends on the direction of the magnetization of the coating material relative to $\hat{\mathbf{n}}$.

Show that the images "charges" induced by an electric charge q in vacuum above the topological insulator are dyons, *i.e.*, particles with both electric and magnetic charge.[2]

2 Solution

2.1 General Considerations

Electric charge is conserved on the surface layer, so in the presence of an electric field it also supports a surface electric charge density σ related by,

$$\frac{\partial \sigma}{\partial t} = -\boldsymbol{\nabla}\cdot\mathbf{K} = P\,\hat{\mathbf{n}}\cdot\boldsymbol{\nabla}\times\mathbf{E} = -\frac{P\,\hat{\mathbf{n}}}{c}\cdot\frac{\partial \mathbf{B}}{\partial t}\,, \tag{2}$$

and hence,

$$\sigma = -\frac{P}{c}\hat{\mathbf{n}}\cdot\mathbf{B}, \tag{3}$$

where $\mathbf{B}$ is the magnetic field associated with the surface current $\mathbf{K}$.

In this static example, $\boldsymbol{\nabla}\times\mathbf{E} = 0$, so the electric field can be related to a scalar potential V according to $\mathbf{E} = -\boldsymbol{\nabla}V$. The tangential component of the electric field is continuous across the interface between the topological insulator and vacuum, which implies that the potential V is continuous across the interface, which we take to be the surface $z = 0$. The insulator occupies the space $z < 0$.

[1]For qualitative introductions by Kane and Moore, see [4, 5]. For a longer review, see, for example, [6].
[2]Feb. 8, 2022. Similar behavior is possible in magnetoelectric materials such as Cr_2O_3 [8].

Inside the topological insulator with relative permittivity ϵ, the electric field $\mathbf{E}$ and the displacement field $\mathbf{D}$ are related by $\mathbf{D} = \epsilon\mathbf{E}$ (in Gaussian units). We assume that there are no free charges inside the insulator, such that within it $\boldsymbol{\nabla} \cdot \mathbf{D} = 0$, and hence $\boldsymbol{\nabla} \cdot \mathbf{E} = 0$ there also. That is, there are no bound/polarization charges inside the insulator, although a bound/polarization charge density may also reside on the surface, in addition to the "free" surface charge density (3).

Because of the "free" surface charge density σ, the normal components of $\mathbf{D}$ and $\mathbf{E}$ at the interface obey,

$$\begin{aligned} E_z(x, y, 0^+) &= D_z(x, y, 0^+) = D_z(x, y, 0^-) + 4\pi\sigma(x, y) = D_z(x, y, 0^-) - \frac{4\pi P}{c} B_z(x, y, 0) \\ &= \epsilon E_z(x, y, 0^-) \mp \alpha B_z(x, y, 0). \end{aligned} \tag{4}$$

Away from the interface, $\boldsymbol{\nabla} \times \mathbf{B} = 0$, so the magnetic field can be related to a scalar potential U according to $\mathbf{B}(z \neq 0) = -\boldsymbol{\nabla} U$. However, the magnetic scalar potential U is not continuous across the interface.

Of course, B_z is continuous across the interface since there are no physical magnetic charges.

To obtain a condition on the tangential component of the magnetic field at the interface we consider the magnetic field $\mathbf{H} = \mathbf{B}/\mu$ which obeys, $\boldsymbol{\nabla} \times \mathbf{H} = 4\pi\mathbf{K}/c = \pm\alpha\, \hat{\mathbf{n}} \times \mathbf{E}/c$, since the only "free" currents in this problem are the surface currents (1). As a consequence,

$$\hat{\mathbf{n}} \times \mathbf{H}(x, y, 0^+) = \hat{\mathbf{n}} \times \mathbf{H}(x, y, 0^-) + \frac{4\pi}{c}\mathbf{K} = \hat{\mathbf{n}} \times \mathbf{H}(x, y, 0^-) \pm \alpha\, \hat{\mathbf{n}} \times \mathbf{E}(x, y, 0), \tag{5}$$

and hence,

$$\begin{aligned} \mathbf{B}_\parallel(x, y, 0^+) &= \mathbf{H}_\parallel(x, y, 0^+) = \mathbf{H}_\parallel(x, y, 0^-) \pm \alpha\, \mathbf{E}_\parallel(x, y, 0) \\ &= \frac{\mathbf{B}_\parallel(x, y, 0^-)}{\mu} \pm \alpha\, \mathbf{E}_\parallel(x, y, 0). \end{aligned} \tag{6}$$

2.2 Point Charge Above a Half Space of Topological Insulator

We now consider the case of a point electric charge q at $(x, y, z) = (0, 0, a)$ in vacuum, above the topological insulator at $z < 0$. The image method is to suppose that the electric scalar potential V in the region $z > 0$ is that due to the original point charge q at $(0, 0, a)$ plus an image charge q' at $(0, 0, -b)$, and that the potential V in the region $z < 0$ is that due to the original point charge plus a point charge q'' at $(0, 0, c)$. Likewise, we suppose that the magnetic scalar potential U in the region $z > 0$ is that due to an image magnetic charge p' at $(0, 0, -d)$, and that the potential U in the region $z < 0$ is that due to an image magnetic charge p'' at $(0, 0, e)$.

According to the suggested image method, the electric scalar potential at $(x, 0, z > 0)$ is,

$$V(x, 0, z > 0) = \frac{q}{[x^2 + (z - a)^2]^{1/2}} + \frac{q'}{[x^2 + (z + b)^2]^{1/2}}\,, \tag{7}$$

and that at $(x, 0, z < 0)$ is,

$$V(x, 0, z < 0) = \frac{q}{[x^2 + (z - a)^2]^{1/2}} + \frac{q''}{[x^2 + (z - c)^2]^{1/2}}\,, \tag{8}$$

Then, continuity of the potential V across the plane $z = 0$ requires that,

$$b = c, \qquad \text{and} \qquad q'' = q'. \tag{9}$$

Similarly, according to the suggested image method, the magnetic scalar potential at $(x, 0, z > 0)$ is,

$$U(x, 0, z > 0) = \frac{p'}{[x^2 + (z + d)^2]^{1/2}}\,, \tag{10}$$

and that at $(x, 0, z < 0)$ is,

$$U(x, 0, z < 0) = \frac{p''}{[x^2 + (z - e)^2]^{1/2}}\,, \tag{11}$$

Continuity of B_z across the plane $z = 0$ implies that,

$$\frac{\partial U(x, 0, 0^+)}{\partial z} = -\frac{p'd}{[x^2 + d^2]^{3/2}} = \frac{\partial U(x, 0, 0^-)}{\partial z} = \frac{p''e}{[x^2 + e^2]^{3/2}}\,, \tag{12}$$

and hence,

$$d = e, \qquad \text{and} \qquad p'' = -p'. \tag{13}$$

The condition (4) for E_z at the interface implies that,

$$\frac{\partial V(x, 0, 0^+)}{\partial z} = \epsilon \frac{\partial V(x, 0, 0^-)}{\partial z} \mp \alpha\, \frac{\partial U(x, 0, 0)}{\partial z}\,, \tag{14}$$

$$\frac{qa}{[x^2 + a^2]^{3/2}} - \frac{q'b}{[x^2 + b^2]^{3/2}} = \epsilon \left(\frac{qa}{[x^2 + a^2]^{3/2}} + \frac{q'b}{[x^2 + b^2]^{3/2}} \right) \mp \alpha\, \frac{p'd}{[x^2 + d^2]^{3/2}}\,, \tag{15}$$

which requires that,

$$a = b = c = d = e, \qquad \text{and} \qquad q - q' = \epsilon(q + q') \mp \alpha\, p'. \tag{16}$$

The condition (6) for $\mathbf{B}_\parallel(x, 0, 0) = B_x(x, 0, 0)$ at the interface implies that,

$$\frac{\partial U(x, 0, 0^+)}{\partial x} = \frac{1}{\mu} \frac{\partial U(x, 0, 0^-)}{\partial x} \pm \alpha\, \frac{\partial V(x, 0, 0)}{\partial x}\,, \tag{17}$$

$$\frac{p'x}{[x^2 + a^2]^{3/2}} = \frac{1}{\mu} \frac{-p'x}{[x^2 + a^2]^{3/2}} \pm \alpha\, \frac{(q + q')x}{[x^2 + a^2]^{3/2}}\,, \tag{18}$$

which requires that,

$$p' = \pm\alpha\, \frac{\mu}{\mu + 1}(q + q'). \tag{19}$$

Combining eqs. (16) and (19), we find that,

$$q' = q'' = -q\frac{A - 1}{A + 1}\,, \quad p' = -p'' = \pm\alpha\, \frac{\mu}{\mu + 1}\frac{2q}{A + 1}\,, \quad \text{where} \quad A = \epsilon - \alpha^2 \frac{\mu}{\mu + 1} \approx \epsilon. \tag{20}$$

To a good approximation the electric fields are the same as for a charge in vacuum above an ordinary dielectric medium of permittivity ϵ.

The electric field in the region $z < 0$ is as if that region were vacuum and the original electric charge q were replaced by an effective electric charge $q_{\rm eff} = q+q'' = q+q' = 2q/(A+1)$. In addition, there is a magnetic field for $z < 0$ as if the original electric charge also had a magnetic charge $p'' = \mp\alpha\, q_{\rm eff}\mu/(\mu+1)$. That is, the fields for $z < 0$ are as if the original electric charge q were replaced by the dyon $(q_{\rm eff}, p'')$ and the insulator replaced by vacuum.

The fields in the region $z > 0$ are the sum of the electric field of the original charge, plus the electric and magnetic fields of the image dyon (q', p').

Of course, the are no physical magnetic charges p' and p'', and the magnetic fields of these apparent poles are due to the surface currents (1). Likewise, the actual electric charge distribution consists only of the original charge q, a polarization charge density on the interface whose total charge is $q' = -q(A-1)/(A+1)$, plus the surface charge density (3) associated with the quantum Hall effect whose total charge is $-(\alpha/4\pi)p' = \mp\alpha^2 q\mu/[2\pi(\mu+1)(A+1)]$.

In the limit that $a = 0$ (such that charge q lies on the interface between the dielectric and vacuum) the electric field in vacuum is also as if the region $z < 0$ were vacuum but the electric charge were $2q/(A+1)$. The magnetic field in this limit is that due to a point magnetic charge (monopole) on the surface, but with the peculiar property that the sign of this charge appears to be opposite as viewed from the two sides of the surface.

References

[1] W.R. Smythe, *Static and Dynamic Electricity*, 3rd ed. (McGraw-Hill, 1968),
http://kirkmcd.princeton.edu/examples/EM/smythe_50.pdf

[2] J.D. Jackson, *Classical Electrodynamics*, 3rd ed. (Wiley, 1999),
http://kirkmcd.princeton.edu/examples/EM/jackson_ce3_99.pdf

[3] K.T. McDonald, *Dielectric (and Magnetic) Image Methods* (Nov. 21, 2009),
http://kirkmcd.princeton.edu/examples/image.pdf

[4] C. Kane and J. Moore, *Topological Insulators*, Phys. World **24**(2), 32 (2011),
http://kirkmcd.princeton.edu/examples/CM/kane_pw_24-2_32_11.pdf

[5] J.E. Moore, *The Birth of Topological Insulators*, Nature **464**, 194 (2010),
http://kirkmcd.princeton.edu/examples/CM/moore_nature_464_194_10.pdf

[6] S. Rachel, *Interacting topological insulators: a review*, Rep. Prog. Phys. **81**, 116501 (2018), http://kirkmcd.princeton.edu/examples/CM/rachel_rpp_81_116501_18.pdf

[7] X.-L. Qi, R. Li, J. Zang and S.-C. Zhang, *Inducing a Magnetic Monopole with Topological Surface States*, Science **323**, 1184 (2009),
http://kirkmcd.princeton.edu/examples/CM/qi_science_323_1184_09.pdf

[8] Q. N. Meier *et al.*, *Search for the Magnetic Monopole at a Magnetoelectric Surface*, Phys. Rev. X **9**, 011011 (2019), http://kirkmcd.princeton.edu/examples/EM/meir_prx_9_011011_19.pdf