

Complex Functions and Electrostatics

Kirk T. McDonald

Joseph Henry Laboratories, Princeton University, Princeton, NJ 08544

(February 28, 2000)

This note is a considerably extended version of a document in support of demonstrations given at John Witherspoon Middle School and Princeton High School in 2000 and 2001.

A complex number z can be written in terms of two real numbers x and y according to,

$$z = x + iy, \quad (1)$$

where,

$$i = \sqrt{-1}. \quad (2)$$

A function f of the complex variable z maps z into another complex number, which can be described in terms of two real numbers u and v ,

$$f(z) = u + iv. \quad (3)$$

Since u and v depend on z , and z depends on x and y , we can say that u and v are both real functions of x and y ,

$$u = u(x, y), \quad v = v(x, y). \quad (4)$$

If the function f is differentiable, some important relations of the functions u and v follow. We define $f' = df/dz$. Then we can write,

$$\frac{\partial f}{\partial x} = f' \frac{\partial z}{\partial x} = f' = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x}, \quad (5)$$

$$-i \frac{\partial f}{\partial y} = -i f' \frac{\partial z}{\partial y} = f' = -i \frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}. \quad (6)$$

Since eqs. (5) and (6) are both equal to f' , we can equate their real and imaginary parts to find the so-called Cauchy-Riemann equations,

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}. \quad (7)$$

Among other things, we learn that,

$$\frac{\partial u}{\partial x} \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \frac{\partial v}{\partial y} = 0. \quad (8)$$

If we introduce the gradient vector,

$$\nabla u = \left(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y} \right), \quad (9)$$

eq. (8) can be written in vector form as,

$$\nabla u \cdot \nabla v = 0. \quad (10)$$

If we take derivatives of eq. (7), we find something amusing,

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x \partial y}, \quad \text{and} \quad \frac{\partial^2 u}{\partial y^2} = -\frac{\partial^2 v}{\partial x \partial y}. \quad (11)$$

so that if we add the two parts of eq. (11) we find,

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \nabla^2 u = 0. \quad (12)$$

We can also take derivatives of eq. (7) in a different way,

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 v}{\partial y^2}, \quad \text{and} \quad \frac{\partial^2 u}{\partial x \partial y} = -\frac{\partial^2 v}{\partial x^2}. \quad (13)$$

which can be combined to tell us that,

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = \nabla^2 v = 0. \quad (14)$$

This mathematics relates to electrostatics where the electric field \mathbf{E} can be derived from an electric potential V according to,

$$\mathbf{E} = -\nabla V. \quad (15)$$

Furthermore, in a charge-free region Gauss law tells us that,

$$\nabla \cdot \mathbf{E} = 0. \quad (16)$$

Equations (15) and (16) combine to tell us that in a charge-free region the electric potential obeys the second-order differential equation,

$$\nabla \cdot \nabla V = \nabla^2 V = 0, \quad (17)$$

which is the same form as eqs. (12) and (14). The amazing conclusion is that **any (differentiable) function $f = u + iv$ of a complex variable gives us not one but two real functions $u(x, y)$ and $v(x, y)$ which are possible electric potential functions for some problem.** Furthermore, since the electric field lines corresponding to potential V point along the direction of ∇V , eq. (10) tells us that the field lines of potential u are orthogonal to the field lines of potential v . Since the field lines of potential u are orthogonal to the equipotentials of u , it appears that **curves of constant v map out the field lines of potential u** (and curves of constant u map out the field lines of potential v).

To compare with our demonstration of electric field lines, we wish to graph the equations,

$$u(x, y) = \text{constant}, \quad \text{and} \quad v(x, y) = \text{constant}, \quad (18)$$

for each of the four complex functions $f(z = x + iy) = u + iv$ listed below.

1. As a first, and very simple example, consider,

$$f(z) = z. \tag{19}$$

Comparing with equations (1) and (3), we see that,

$$u = x, \quad \text{and} \quad v = y. \tag{20}$$

What electrostatic problem is this the solution to? If we take the potential to be v , then equipotentials are surfaces of constant y , and the field lines have constant $u = x$. That is, we have “solved” the elementary problem of a uniform electric field in the x direction.

That was easy! Let’s try another.

2. Now consider,

$$f(z) = z^2 = (x + iy)^2 = x^2 + 2ixy - y^2. \tag{21}$$

Comparing with equation (3), we see that,

$$u = x^2 - y^2, \quad \text{and} \quad v = 2xy. \tag{22}$$

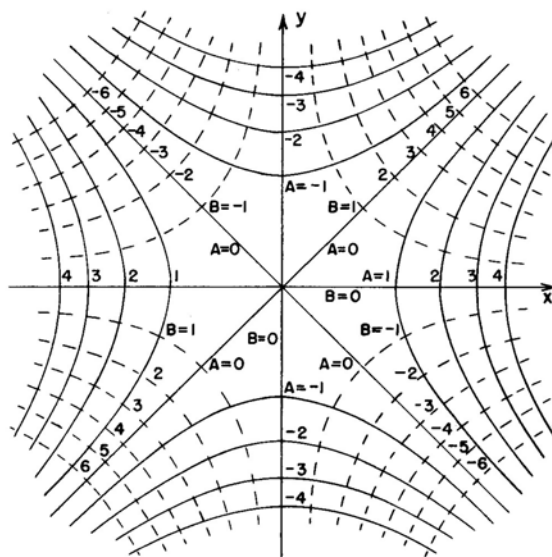
To see what problem is solved by the potentials u and v , we look at the surfaces of constant u and v .

$$u = x^2 - y^2 = A, \quad \text{and} \quad v = 2xy = B. \tag{23}$$

We rewrite these in the form $y = y(x)$,

$$u : y = \pm\sqrt{(x^2 - A)}, \quad \text{and} \quad v : y = B/(2x). \tag{24}$$

The graphs of these functions are hyperbolae. For $A > 0$, the graph exists only when $|x| \geq \sqrt{A}$.



A device based on these functions is called a quadrupole, which has conductors placed at four equipotential surfaces, such as $v = B = 1(x > 0)$, $B = 1(x < 0)$, $B = -1(y > 0)$,

and $B = -1(y < 0)$, shown as dotted lines in the figure. The field lines then run along the solid lines between pairs of adjacent “poles”. The electric field is given by,

$$\mathbf{E} = -\nabla v = (-2y, -2x), \quad (25)$$

whose magnitude, $E = 2\sqrt{x^2 + y^2} = 2r$, is proportional to the radial distance from the z axis. Devices which exert forces that are proportional to the distance from an axis can be thought of as lenses, and so an electric quadrupole is a somewhat peculiar kind of lens. A more common type of quadrupole lens is a magnetic quadrupole, which bends charged particles moving along the z axis in the x - z planes towards the z axis (focusing), but bends particles moving in the y - z plane away from the z axis (defocusing).

Now we’re ready for something a bit more difficult.

3. For a third example, consider,

$$f(z) = \sqrt{z}. \quad (26)$$

We can make progress by combining this with equations (1) and (3), which yields,

$$u + iv = \sqrt{x + iy}. \quad (27)$$

Next, we can square equation (27), which gives,

$$u^2 + 2iuv - v^2 = x + iy. \quad (28)$$

Of course, the left hand side of this has the same form as equation (21).

In equation (28), we are saying that two complex numbers are equal to one another. In detail, this means that the **real** parts of both numbers are equal, and that the **imaginary** parts of both numbers are equal. That is,

$$u^2 - v^2 = x, \quad (29)$$

$$2uv = y. \quad (30)$$

We want to solve equations (29)-(30) for u and v in terms of x and y . First, equation (30) tells us that,

$$v = \frac{y}{2u}. \quad (31)$$

Plugging this into equation (29), we find that,

$$u^2 - \frac{y^2}{4u^2} = x. \quad (32)$$

We multiply by $4u^2$, and rearrange terms to find that,

$$4u^2 - 4xu^2 - y^2 = 0. \quad (33)$$

This is a quartic equation in the variable u ! But since the only powers of u that appear are u^0 , u^2 and u^4 , we can consider this to be a quadratic equation in u^2 . So, we use the quadratic formula to learn that,

$$u^2 = \frac{4x \pm \sqrt{16x^2 + 16y^2}}{8} = \frac{x \pm \sqrt{x^2 + y^2}}{2}. \quad (34)$$

To find u , we take the square root of equation (34). This will give us two solutions! Here, mathematics is wonderful: the $+$ solution is for u and the $-$ solution is for iv :

$$u = \sqrt{\frac{x + \sqrt{x^2 + y^2}}{2}} = \sqrt{\frac{\sqrt{x^2 + y^2} + x}{2}}, \quad (35)$$

and,

$$iv = \sqrt{\frac{x - \sqrt{x^2 + y^2}}{2}}, \quad \text{so} \quad v = \sqrt{\frac{\sqrt{x^2 + y^2} - x}{2}}. \quad (36)$$

The trick we used to get equation (36) is not so obvious, but you could always plug equation (35) into equation (31) to find v .

Altogether,

$$\sqrt{z} = \sqrt{x + iy} = \sqrt{\frac{\sqrt{x^2 + y^2} + x}{2}} + i\sqrt{\frac{\sqrt{x^2 + y^2} - x}{2}}. \quad (37)$$

You can square equation (37) to verify that the solution is correct.

As an example, we can calculate the \sqrt{i} . Writing $i = z = x + iy$, we see that in this case, $x = 0$, and $y = 1$. Plugging these values into equation (37), we find that,

$$\sqrt{i} = \sqrt{\frac{1}{2}} + i\sqrt{\frac{1}{2}} = \frac{1+i}{\sqrt{2}}. \quad (38)$$

We check this,

$$\left(\frac{1+i}{\sqrt{2}}\right)^2 = \frac{1+2i-1}{2} = i. \quad (39)$$

To relate the function $f = u + iv$ to electric potentials, we again consider surfaces of constant u and v .

$$f(z) = \sqrt{z}, \quad \Rightarrow \quad u = \sqrt{\frac{\sqrt{x^2 + y^2} + x}{2}} = A, \quad \text{and} \quad v = \sqrt{\frac{\sqrt{x^2 + y^2} - x}{2}} = B. \quad (40)$$

These are readily rearranged into the forms

$$u : y = 2\sqrt{(\sqrt{A}(\sqrt{A} - x))}, \quad \text{and} \quad v : y = 2\sqrt{(\sqrt{B}(\sqrt{B} + x))}, \quad (41)$$

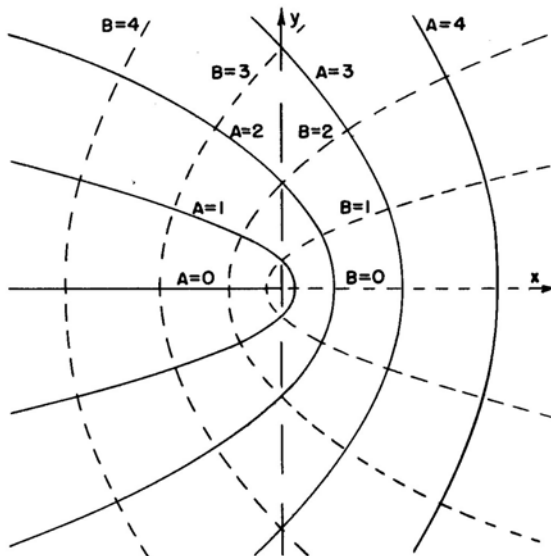
where A and B are positive. [The notation follows that needed to plot these curves on a graphing calculator.]

The graphs of these are curves (shown in the figure on the next page) that loop around the origin; the u curves look like U's on their sides pointing to the left and crossing the x axis at $x = \sqrt{A}$; the v curves are mirror images of the u curves, reflected about the y axis. The u graph exists only for $x \leq \sqrt{A}$, and the v graph exists only for $x \geq -\sqrt{B}$.

If we consider the equipotentials to be surfaces of constant v , then we see the half plane ($x > 0, y = 0$) has potential $v = B = 0$, and corresponds to a semi-infinite grounded,

conducting plane. The electric fields lines are along the solid curves in the figure, and the field is given by,

$$\mathbf{E} = -\nabla v = \frac{\sqrt{2}}{4\sqrt{x^2 + y^2}} \left(\sqrt{\sqrt{x^2 + y^2} - x}, \frac{y}{\sqrt{\sqrt{x^2 + y^2} - x}} \right). \quad (42)$$



The electric field is very strong near the origin, *i.e.*, near the sharp edge of the conducting sheet. For example, along the x axis to the left of the sheet,

$$\mathbf{E}(-\epsilon, 0) \approx \left(\frac{1}{2\sqrt{\epsilon}}, 0 \right). \quad (43)$$

Also, just above the sheet and just to the right of its edge, we find,

$$\mathbf{E}(\epsilon, \delta) \approx \left(0, \frac{1}{2\sqrt{\epsilon}} \right). \quad (44)$$

According to Gauss' law, the surface charge density on the conducting sheet is proportional to the normal component of the electric field, so at small distance ϵ to the right of the edge of the sheet, we have (in Gaussian units),

$$\sigma = \frac{E_y}{4\pi} = \frac{1}{8\pi\sqrt{\epsilon}}. \quad (45)$$

That is, the charge density near a sharp edge of a conductor varies as the inverse of the square root of the distance in from that edge.

A comment on multiple square roots.

We know that a positive real number has two real square roots: $\sqrt{4} = 2$ and -2 .

Likewise, a negative real number has two imaginary square roots: $\sqrt{-4} = 2i$ and $-2i$.

Hence, we expect that the complex number $z = x + iy$ has two complex square roots. But so far we have displayed only one of these in equation (37). Looking back to equation (34), we see that this is also satisfied by $-u$ of (35). Then, equation (31) is solved by $-v$ of (36) when we use $-u$. We conclude that,

$$\sqrt{z} = u + iv, \quad \text{and} \quad -u - iv. \quad (46)$$

That is,

$$\begin{aligned} \sqrt{z} &= \sqrt{x + iy} = \sqrt{\frac{\sqrt{x^2 + y^2} + x}{2}} + i\sqrt{\frac{\sqrt{x^2 + y^2} - x}{2}} \\ \text{and} \quad &-\sqrt{\frac{\sqrt{x^2 + y^2} + x}{2}} - i\sqrt{\frac{\sqrt{x^2 + y^2} - x}{2}}. \end{aligned} \quad (47)$$

Amazingly, we can do more!. Since all four of u , $-u$, v and $-v$ have been useful, we might also consider the combinations,

$$u - iv, \quad \text{and} \quad -u + iv = -(u - iv). \quad (48)$$

Are these the square roots of something? es, by squaring these two quantities we find that they obey,

$$\begin{aligned} \sqrt{x - iy} &= u - iv = \sqrt{\frac{\sqrt{x^2 + y^2} + x}{2}} - i\sqrt{\frac{\sqrt{x^2 + y^2} - x}{2}} \\ \text{and} \quad &-u + iv = -\sqrt{\frac{\sqrt{x^2 + y^2} + x}{2}} + i\sqrt{\frac{\sqrt{x^2 + y^2} - x}{2}}. \end{aligned} \quad (49)$$

The quantity $x - iy$ is called the **complex conjugate** of $x + iy$. Similarly, $u - iv$ is the complex conjugate of $u + iv$. People sometimes use the symbol \star to indicate the complex conjugate,

$$z^\star = x - iv \quad \text{when} \quad z = x + iy. \quad (50)$$

Comparing equations (47) and (49), we see that,

$$\sqrt{z^\star} = (\sqrt{z})^\star. \quad (51)$$

Can you show that $(z^\star)^2 = (z^2)^\star$?

Note that $z + z^\star = 2x$ is a real number. Hence $\sqrt{z} + \sqrt{z^\star} = 2u$ is also a real number. But now we have a “complex” way of writing a real number! A famous example is

$$\sqrt{3 + 4i} + \sqrt{3 - 4i} = 4. \quad (52)$$

4. The fourth example is defined in a backwards way. We consider the relation,

$$z = f + e^f, \quad (53)$$

which cannot be simply inverted to the form $f(z)$. However, we can learn a bit more about this function by using $z = x + iy$ and $f = u + iv$,

$$z = x + iy = u + iv + e^{u+iv} = u + iv + e^u e^{iv} = u + iv + e^u (\cos v + i \sin v). \quad (54)$$

The real and imaginary parts of eq. (54) are separately equal, so we have,

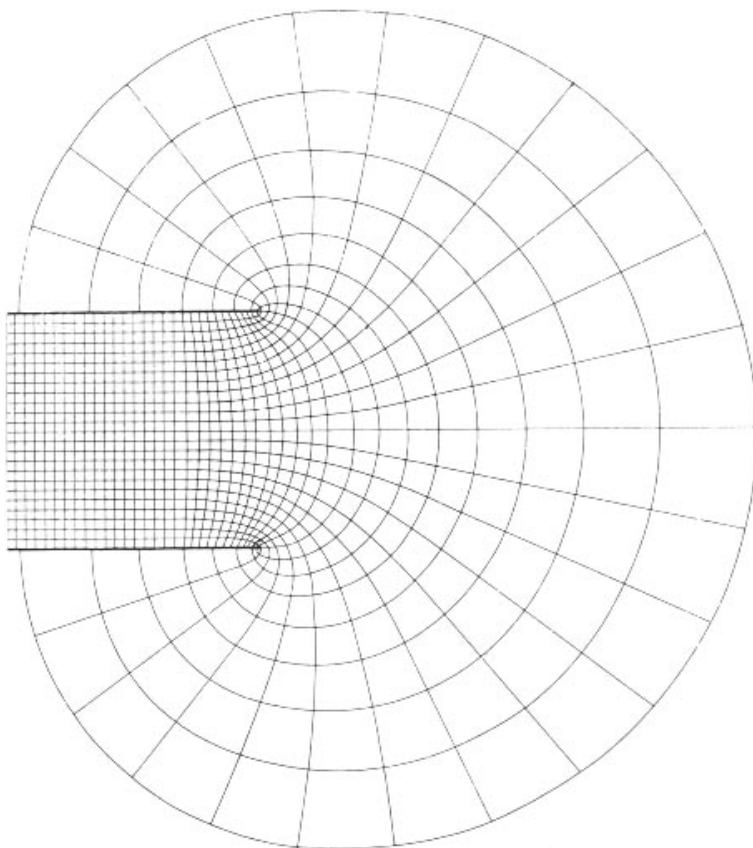
$$x = u + e^u \cos v, \quad (55)$$

$$y = v + e^u \sin v. \quad (56)$$

This form is handy for graphing. For example, if we take the equipotentials to be surfaces of constant $v = A$, these can be written in terms of the continuous parameter u as,

$$x = u + e^u \cos A, \quad y = A + e^u \sin A \quad (\text{equipotentials}). \quad (57)$$

Note that the interval of values $-\pi \leq A \leq \pi$ suffices to fill the x - y plane with curves, so we restrict A to this interval. The pattern of equipotentials is symmetric about the plane $y = 0$, corresponding to $A = 0$. The only other planar equipotential surfaces occur for $A = \pm\pi$, for which $-\infty < x \leq -1$ and $y = \pm\pi$. Thus we see that our functional relation (53) corresponds to two semi-infinite planes that form a capacitor.



From Art. 202 of *A Treatise on Electricity and Magnetism* by J.C. Maxwell.

The electric field lines correspond to surfaces of constant $u = B$, so these can be written in terms of the continuous parameter v ($-\pi \leq v \leq \pi$) as,

$$x = B + e^B \cos v, \quad y = v + e^B \sin v \quad (\text{field lines}). \quad (58)$$

For $B \ll 0$, $x \approx B$, and the lines are vertical with $-\pi \leq y \leq \pi$. This corresponds to the uniform field region in the interior of the capacitor. For $B \gtrsim 0$ we can map out the fringe field.