Impedance Matching of Transmission Lines

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1 Problem

This problem considers several ways of "matching" two transmission lines such that a wave propagates from the first to the second line without reflection at the junction.

The related problem of "matching" a transmission line to a load is also considered to some extent.

In this problem, a transmission line is a device consisting of two parallel conductors such that TEM (transverse electromagnetic) waves can be propagated. Examples of transmission lines include coaxial cables, and simple 2-wire (Lecher) lines.

Take the lines to lie along the x axis. Then, the two conductors carry currents $I(x,t)$ and $-I(x,t)$, waves of angular frequency ω propagating in the $+x$ direction have the form,

$$I(x,t) = I_+ \cos(kx - \omega t) = Re(I_+ e^{i(kx-\omega t)}), \quad \text{which we abbreviate as} \quad I = I_+ e^{i(kx-\omega t)}, \tag{1}$$

where $i = \sqrt{-1}$, $k = 2\pi/\lambda$ is the wave number and ω/k is the wave velocity.[1,2] Similarly, waves propagating in the $-x$ direction have the form,

$$I(x,t) = I_- e^{i(-kx-\omega t)}. \tag{2}$$

A voltage difference $V(x,t)$ exists between the two conductors, which is related to the current in the conductors according to,

$$V = IZ, \tag{3}$$

where Z is the impedance of the transmission line. A sign-convention is required for proper use of eq. (3); we say that $V_+ = I_+ Z$ for waves propagating in the $+x$ direction, while $V_- = -I_- Z$ for waves propagating in the $-x$ direction.

If waves are propagating in both directions along a transmission line, then the voltage is related to the current by,

$$V = V_+ + V_- = (I_+ - I_-)Z. \tag{4}$$

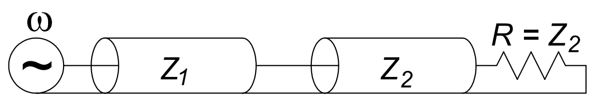
[1]If the conductors are surrounded by vacuum, the wave velocity is c, the speed of light. If the conductors are embedded in dielectric and/or permeable media with (relative) dielectric constant ϵ and (relative) permeability μ, the wave velocity is c/n where $n = \sqrt{\epsilon\mu}$ is the index of refraction. In this case, the wave number can be written $k = n\omega/c$, and the wave length is $\lambda = c/2\pi n\omega$. If several different sections of transmission line are present, they can have different indices, wave numbers and wave lengths for waves of a given angular frequency ω. While we ignore this possibility in the present note, it could be accommodated by minor changes in notation.

[2]If the currents on the two conductors are not equal and opposite, then their average $(I_1 + I_2)/2$ is called the common mode current. This case is generally undesirable as it involves transfer of energy to and from the environment of the transmission line. We do not consider common mode currents in this problem.

The transmission line may be assumed to operate in vacuum (rather than in a dielectric that can support leakage currents). The capacitance and inductance per unit length along the line are C and L, respectively. The resistance per unit length along the conductors is negligible.

(a) **Reflection Due to an Impedance Mismatch.**

Suppose that two transmission lines are directly connected to each other, as illustrated below for coaxial cables of impedances Z_1 and Z_2.[3]



Show that the power transmitted into line 2 is given by,

$$P_2 = \frac{4Z_1Z_2}{(Z_1+Z_2)^2}P_1, \tag{5}$$

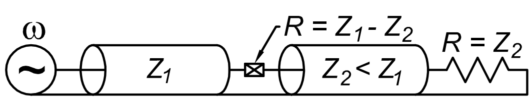
where P_1 is the power in line 1 that is incident on the junction. How much power is reflected back down line 1?

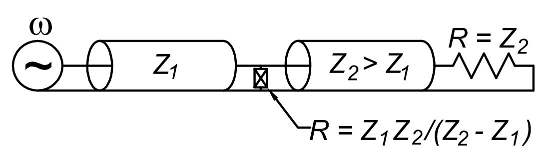
(b) **Reflection Due to a Complex Load Impedance.**

Generalize part (a) to include the case that Z_2 is a complex load impedance, rather than a transmission line with a real impedance. Show that the ratio of the total voltage to the total current in the input coaxial cable is real for a set of positions along the cable, which permits matching at these points using the techniques considered in the rest of this problem. *A graphical representation of this insight uses the so-called Smith chart [1]. For discussion of impedance matching of the voltage source to the transmission line, see [2].*

(c) **Impedance Matching via Resistors.**

Show that there will be no reflected wave from an incident in line 1 if an appropriate resistor is placed at the junction. The case that $Z_2 > Z_1$ and $Z_2 < Z_1$ must be dealt with separately if only a single resistor is to be used. Hence, this scheme for matching works only when the waves are incident from line 1.



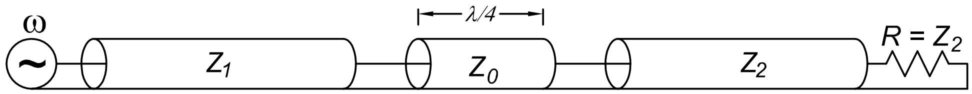


Because the resistors dissipate energy, the transmitted power is less than the incident power. Further, the resistive impedance matching scheme works only for waves transmitted in one direction.

[3]The figure includes a load resistor of value $R = Z_2$ at the end of transmission line 2. Without such a load resistor there would be reflections off the end of line 2, and the analysis would need to be modified accordingly.

(d) **A $\lambda/4$ Matching Section.**

Show that the power can be transmitted without loss at a particular frequency ω from a transmission line of impedance Z_1 into one of impedance Z_2 if the junction consists of a piece of transmission line of length $\lambda/4$ and impedance $Z_0 = \sqrt{Z_1 Z_2}$. This prescription is familiar from antireflection coatings on optical lenses.[4]

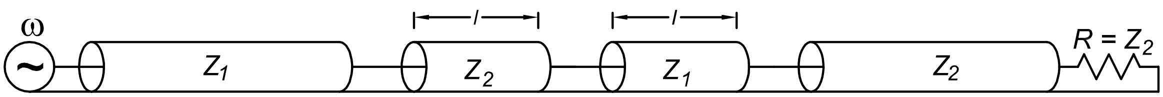


This scheme works for waves transmitted in either direction, but requires a precise impedance for the matching section, which may not be readily available in practice.

Show also that a transmission line of impedance Z_1 could be matched into a complex load $Z_2 = R + iX$ with an appropriate length $l \neq \lambda/4$ of transmission line of impedance Z_0, provided that $R \neq Z_1$.

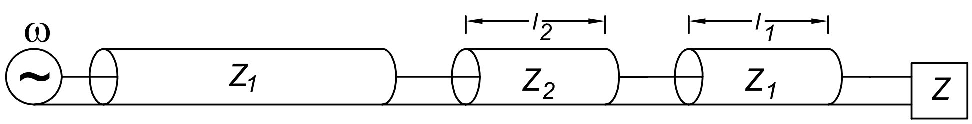
(e) **A "$\lambda/12$" Matching Section.**

Show that power can be transmitted without loss at a particular frequency ω from a transmission line of impedance Z_1 into one of impedance Z_2 if the transition consists of two pieces of transmission line of equal lengths $l \approx \lambda/12$ and impedance Z_2 and Z_1, as sketched below.



This scheme works for waves transmitted in either direction, and can be built using only pieces of the two transmission lines of interest. However, the matching is optimal only at a single frequency.

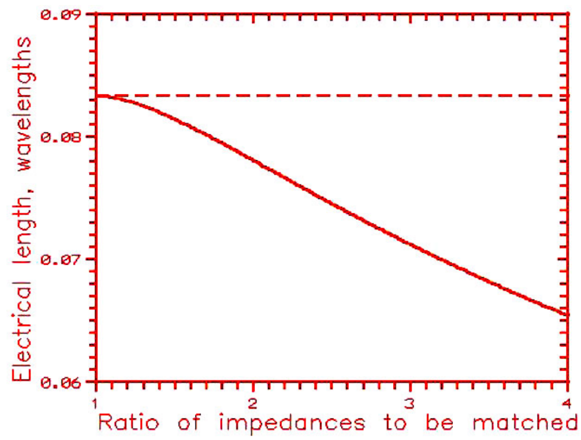
The "$\lambda/12$" matching scheme was invented by P. Bramham in 1959 [4]. In 1971, F. Regier [6] gave a generalization that permits matching a transmission line of (real) impedance Z_1 to a complex load impedance $Z = R + iX$, where R is the load resistance and X is the load reactance.



Given impedances Z, Z_1 and Z_2, deduce the lengths l_1 and l_2 of the matching sections.

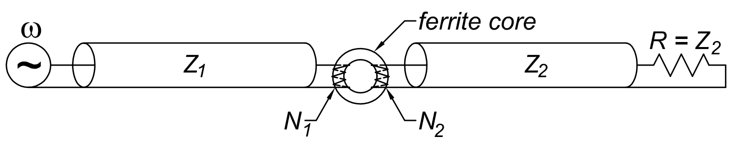
When $Z = Z_2$ is real, then the lengths of the matching sections are $l_1 = l_2 = (\lambda/4\pi)\cos^{-1}[(Z_1^2 + Z_2^2)/(Z_1 + Z_2)^2]$, which is close to $\lambda/12 = 0.08333\lambda$ as shown in the figure below (from [5]).

[4]See, for example, prob. 5 of [3].



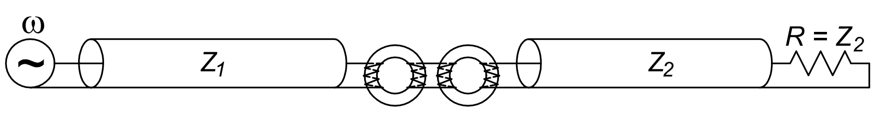
(f) **Impedance Matching via a Flux-Linked Transformer.**

Show that transmission lines 1 and 2 can be matched if each line is attached to a small coil with N_1 and N_2 turns, respectively (as shown on the next page), where $N_1/N_2 = \sqrt{Z_1/Z_2}$. All of the magnetic flux created by the current coil 1 should be linked by coil 2, and *vice versa*, for ideal transformer action. You may ignore the internal resistances and capacitances of the transformer windings. Even with these idealizations, show that a transmission-line analysis predicts a reflected power that varies as $1/\omega^2$ (at high frequency), so that a flux-linked transformer has poor performance at low frequency.



To maximize the flux linkage in practice, it is advantageous to wind the two coils around a ring or "core" of a high-permeability magnetic material, such as a ferrite. See, for example, [7]. However, the performance of magnetic materials is limited at very high frequencies, so a ferrite-based, flux-linked, impedance-matching transformer has only a finite bandwidth.

Lenz' law indicates that the induced current in line 2 will have the opposite sign to that in line 1. Hence, the voltage in lines 1 and 2 are reversed at their junction, and we speak of an inverting transformer. A noninverting transformer could be built using two ferrite cores, as shown in the figure below.

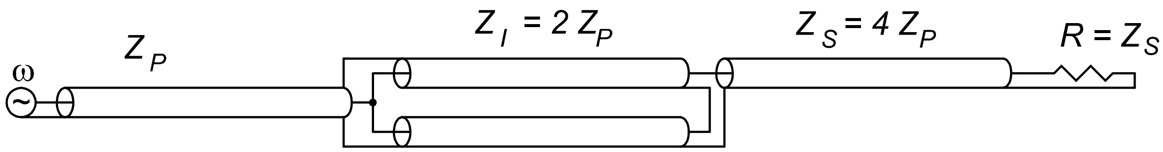


A matching transformer can be constructed with no metallic connection between the conductors of line 1 and those of line 2. In this case we speak of an isolation transformer, particularly when the impedance of lines 1 and 2 is the same.

There is some special terminology associated with isolation transformers, based on the concepts of balanced and unbalanced transmission lines. In principle, a balanced transmission line is one in which only a TEM wave propagates, and so the currents on its two conductors are equal and opposite at each point along the line. Any transmission line can become unbalanced due to coupling with electromagnetic fields in its environment, but connecting one of the conductors to an electrical "ground" also unbalances the line.[5] An isolation transformer connected to an unbalanced line 1 can result in line 2 being balanced, and therefore the name balun (balanced-unbalanced) is sometimes given to isolation transformers.

(g) **The Transmission-Line Transformer of Guanella.**

In 1944, Guanella [8] suggested a device that can match a (primary) transmission line of impedance Z_P to a (secondary) line of impedance $Z_S = 4Z_P$ using two intermediate pieces of transmission line of impedance $Z_I = 2Z_P$, as shown in the figure below.



Verify the desired functionality of this circuit, supposing that the intermediate segments of transmission lines are long enough that TEM waves propagate in them. This has the effect that the voltages at the two ends of these segments are isolated from one another, because the integral $\int \mathbf{E}\cdot d\mathbf{l}$ along the line vanishes for a TEM mode.

For the above condition to hold, the intermediate segments need to be a wavelength or more long, which may be inconvenient in practice. Guanella suggested that short intermediate segments could be used if their conductors were wound into inductive helices, rather than being simple straight wires. When his scheme was later applied to coaxial cables, the inductive isolation was provided by looping the coax cable through a ferrite core. This improves the low-frequency response of the device without changing its basic principle.

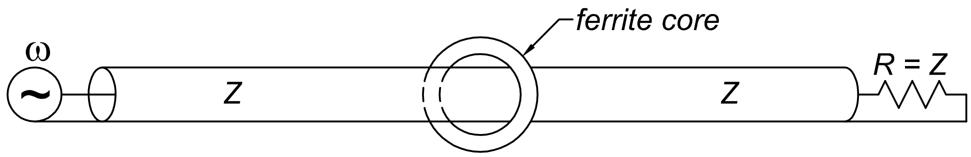
Guanella's device has come to be known as a transmission-line transformer even though it does not involve any classic transformer action as first discovered by Faraday. Because Guanella's transmission-line transformer provides a measure of isolation between the voltages at its two ends, it also serves as a balun. In some applications, the isolation is more important than the impedance match.

Many variants of the transmission-line transformer have been conceived [9, 10, 11]. For example, a piece of coax cable that passes through a ferrite core is called by some a 1:1 transmission-line transformer [12], although it might be more correct to call it a balun.

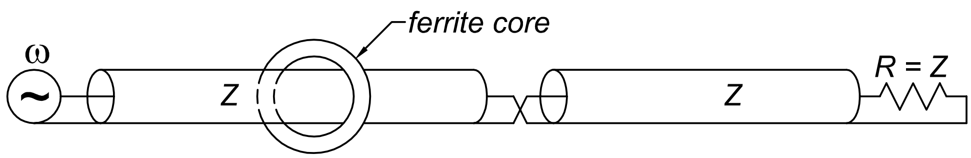
[5] If the currents on the two conductors of a transmission line are labeled I_a and I_b, then we can write,

$$I_a = \frac{I_a + I_b}{2} + \frac{I_a - I_b}{2} \equiv I_{\rm common} + I_{\rm TEM}, \qquad I_b = \frac{I_a + I_b}{2} - \frac{I_a - I_b}{2} = I_{\rm common} - I_{\rm TEM}, \tag{6}$$

where the antisymmetric currents are associated with TEM wave propagation, and the symmetric currents are often called common mode currents. Note that the symbol $\equiv$ means "is defined to be".

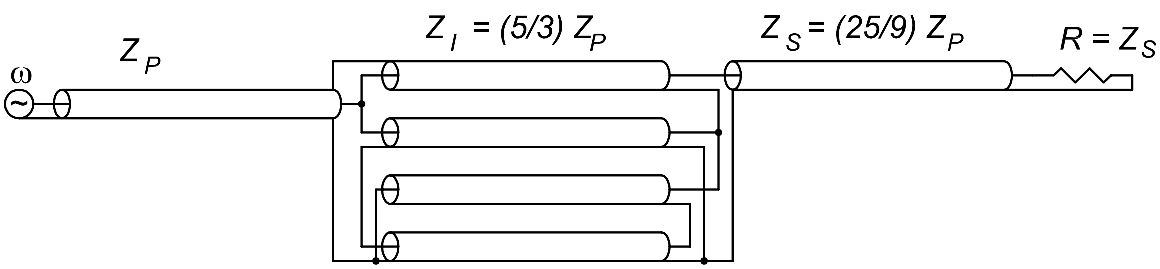


A more nontrivial variant is obtained if a second coaxial cable is attached to the first (the one that passes through the ferrite core), with the conductors of the two cables cross connected as shown below. In this case we obtain a 1:1 inverting transmission-line transformer.



Show that if m intermediate segments of impedance $Z_I = 2Z_P$ are used instead two, shorting the inner and outer conductors of adjacent cables at their output ends, the resulting transmission-line transformer makes a $1 : m^2$ impedance match ($1 : m$ voltage transformer).

If the intermediate transmission lines are shorted to one another at appropriate places at their input ends as well as their output ends, additional transformer ratios may be obtained. For example, if four intermediate sections of impedance $Z_I = (5/3)Z_P$ are connected as shown below, we can make a match to a secondary line of impedance $Z_S = (25/9)Z_P$.



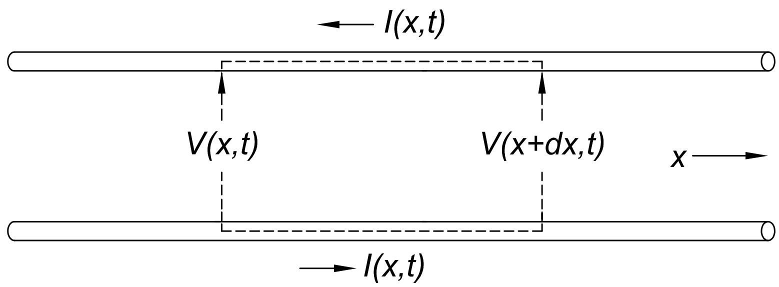
2 Solution

(a) **Reflection Due to an Impedance Mismatch.**

We first deduce the wave equations for current and voltage in a transmission line, and then relate this to the concept of impedance.

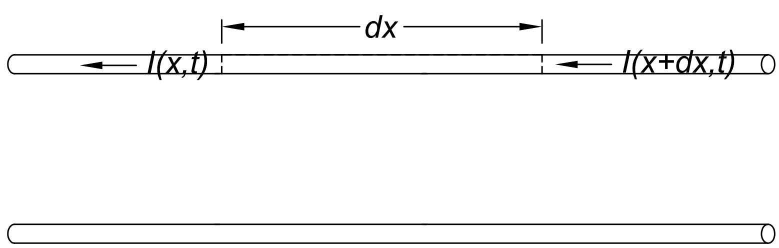
To illustrate the steps in deducing the wave equation, we use figures based on a 2-wire transmission line. Referring to the sketch below and recalling that L is the inductance per unit length of the "loop" formed by the two lines, Kirchhoff's rule for the circuit of length dx shown by dashed lines tells us,[6]

$$V(x) - V(x + dx) - (L dx)\frac{\partial I}{\partial t} = 0, \qquad \text{or} \qquad -\frac{\partial V}{\partial x} = L\frac{\partial I}{\partial t}\,. \tag{7}$$



Next, the charge dQ that accumulates on length dx of the upper wire during time dt is $(C dx)dV$ in terms of the change of voltage dV between the wires and the capacitance C per unit length between the two wires. The charge can also can be written in terms of currents at the two ends of segment dx, so that,

$$Q = (C dx)dV = (I(x) - I(x + dx))dt, \qquad \text{so} \qquad -\frac{\partial I}{\partial x} = C\frac{\partial V}{\partial t}\,. \tag{8}$$



Together, eqs. (7) and (8) imply the desired wave equations,

$$\frac{\partial^2 I}{\partial x^2} = LC\frac{\partial^2 I}{\partial t^2}\,, \qquad \frac{\partial^2 V}{\partial x^2} = LC\frac{\partial^2 V}{\partial t^2}\,. \tag{9}$$

The well-known solutions for waves of angular frequency ω are that,

$$I = I_\pm e^{i(\pm kx - \omega t)}, \qquad V = V_\pm e^{i(\pm kx - \omega t)}, \tag{10}$$

where the wave velocity v is,

$$v = \frac{\omega}{k} = \frac{1}{\sqrt{LC}}\,. \tag{11}$$

It can be shown that the capacitance C and inductance L for an arbitrary 2-conductor line in vacuum obey the relation $1/\sqrt{LC} = c$, where c is the speed of light.[7]

[6]Here, we apply Kirchhoff's circuit law to a short segment of a transmission line. However, Kirchhoff's analysis does not well apply to transmission-line circuits that are large compared to a wavelength, as discussed in [13].

[7]See, for example, pp. 16-17 of http://kirkmcd.princeton.edu/examples/ph501lecture13.pdf

If we insert the solutions (10) in either of eqs. (8) or (9), we find that,

$$V_\pm = \pm\sqrt{\frac{L}{C}} I_\pm \equiv \pm Z I_\pm . \tag{12}$$

Thus, the impedance of an ideal 2-conductor transmission line is the positive real number given by,

$$Z = \sqrt{\frac{L}{C}} . \tag{13}$$

The time-average power carried by either the + or the − wave is given by,

$$P_\pm = \frac{Re(V_\pm I_\pm^\star)}{2} = \frac{V_\pm I_\pm}{2} = \pm\frac{Z I_\pm^2}{2} = \pm\frac{V_\pm^2}{2Z}, \tag{14}$$

where the sign of the power indicates the direction of propagation of the wave along the x axis.

After these lengthy preliminaries, we return to the question of the power transmitted across the junction between transmission lines of impedances Z_1 and Z_2.

Assuming that the incident wave arrives from $x = -\infty$ on line 1, this wave is described by current and voltage I_{1+} and $V_{1+} = I_{1+} Z_1$. At the junction with line 2, the incident wave splits into a transmitted wave described by current and voltage I_{2+} and $V_{2+} = I_{2+} Z_2$, and a reflected wave described by current and voltage I_{1-} and $V_{1-} = -I_{1-} Z_1$.

Both current and voltage are continuous at the junction, so we have,

$$\begin{aligned} V_{2+} &= V_{1+} + V_{1-}, & (15) \\ I_{2+} &= I_{1+} + I_{1-}. & (16) \end{aligned}$$

Using eq. (12) to eliminate the voltages in favor of the currents, eq. (15) becomes,

$$I_{2+}\frac{Z_2}{Z_1} = I_{1+} - I_{1-}. \tag{17}$$

Solving eqs. (16) and (17) we find,

$$I_{1-} = \frac{Z_1 - Z_2}{Z_1 + Z_2} I_{1+}, \qquad I_{2+} = \frac{2Z_1}{Z_1 + Z_2} I_{1+}. \tag{18}$$

So, whenever there is an impedance mismatch, *i.e.*, whenever $Z_1 \neq Z_2$, there is a reflected wave created by the junction.

Using eq. (14) we calculate the reflected and tranmitted power to be,

$$P_{1-} = \left(\frac{Z_1 - Z_2}{Z_1 + Z_2}\right)^2 P_{1+}, \qquad P_{2+} = \frac{4Z_1 Z_2}{(Z_1 + Z_2)^2} P_{1+}. \tag{19}$$

These powers obey $P_{1-} + P_{2+} = P_{1+}$, which also follows from conservation of energy at the junction, which has been assumed to be lossless.

(b) **Reflection Due to a Complex Load Impedance.**

The analysis of eqs. (15)-(18) does not actually depend on the assumption that the impedances Z_1 and Z_2 are real numbers. For the case of complex impedances, eq. (14) becomes $P_\pm = \pm Re(Z)\left|I_\pm\right|^2/2$, so that the generalization of eq. (19) for Z_1 real but Z_2 complex is,

$$P_{1-} = \left|\frac{Z_2 - Z_1}{Z_2 + Z_1}\right|^2 P_{1+}, \qquad P_{2+} = \frac{4Z_1\, Re(Z_2)}{\left|Z_2 + Z_1\right|^2} P_{1+}. \tag{20}$$

The first of eq. (18) is often rewritten as,

$$I_r = \frac{1 - Z_2/Z_1}{1 + Z_2/Z_2} I_i = \frac{1-z}{1+z} I_i \equiv -\gamma I_i, \tag{21}$$

where subscript i means incident, subscript r means reflected, $z = Z_2/Z_1$ is the normalized load impedance, and,

$$\gamma = \frac{z-1}{z+1} \tag{22}$$

is the complex voltage reflection coefficient, since $V_r = \gamma V_i$ according to the sign convention (12). We note that,

$$z = \frac{1+\gamma}{1-\gamma}\,. \tag{23}$$

At a point $x < 0$ on line 1, the total voltage is,[8]

$$V_1(x,t) = V_{1+}\, e^{i\,(kx-\omega t)} + V_{1-} e^{i(-kx-\omega t)} = Z_1 I_{1+}(e^{ikx} + \gamma e^{-i\,kx})e^{-i\,\omega t}, \tag{26}$$

and the total current is,

$$I_1(x,t) = I_{1+} e^{i(kx-\omega t)} + I_{1-} e^{i(-kx-\omega t)} = I_{1+}(e^{i\,kx} - \gamma e^{-i\,kx})e^{-i\,\omega t}, \tag{27}$$

[8]A practical diagnostic of the modulated traveling wave (26) can be obtained by placing a "square-law" detector across the transmission line at position x, which measures the DC root-mean-square (rms) voltage,

$$V_{\rm rms}(x) = \sqrt{\frac{Re(V_1 V_1^\star)}{2}} = V_{1+}\sqrt{\frac{1 + \left|\gamma\right|^2 + 2\left|\gamma\right|\cos(\phi - 2kx)}{2}}, \tag{24}$$

where we write the complex reflection coefficient (22) as $\gamma = \left|\gamma\right| e^{i\phi}$. The maxima and minima of $V_{\rm rms}(x)$ occur at values of x separated by $\lambda/4$ and have magnitudes $V_{1+}(1 \pm \left|\gamma\right|)/\sqrt{2}$. The ratio of the maximum rms voltage to the minimum is called the VSWR,

$$\text{VSWR} = \frac{1 + \left|\gamma\right|}{1 - \left|\gamma\right|}\,. \tag{25}$$

VSWR is an acronym for voltage standing wave ratio, although strictly speaking the waveform (26) is not a pure standing wave unless $\gamma = 1$. An applet that illustrates the waveform (26) for real values of γ is available at http://www.bessernet.com/Ereflecto/tutorialFrameset.htm.

Note that $\sqrt{2}V_{\rm rms}(x)$ is the envelope of the waveform $V_1(x,t)$.

We introduce the complex, position-dependent impedance $Z(x)$ defined so that $V_1(x,t) = I_1(x,t)Z(x)$,

$$Z(x) = Z_1 \frac{1 + \gamma e^{-2i\,kx}}{1 - \gamma e^{-2i\,kx}}\,. \tag{28}$$

If line 1 were terminated at position x by a load of impedance $Z(x)$, the reflected wave would be just the same as when the line is terminated at $x = 0$ by impedance Z_2.

As a check, note that $Z(0) = Z_1(1+\gamma)/(1-\gamma) = Z_1 z = Z_2$ using eq. (23).

If $Z(x)$ is real, then a "match" could be made at position x using techniques appropriate for real, *i.e.*, purely resistive, loads as considered in parts (c) and (d). From eq. (28) we see that $Z(x)$ will be real if $\gamma e^{-2i\,kx}$ is real also. After some algebra, we find the desired value(s) of x to be,

$$x = \frac{\lambda}{4\pi}\tan^{-1}\left(\frac{2\,Im(z)}{1-|z|^2}\right) = \frac{\lambda}{4\pi}\tan^{-1}\left(\frac{2Z_1\,Im(Z_2)}{Z_1^2-|Z_2|^2}\right) = \frac{\lambda}{4\pi}\tan^{-1}\left(\frac{2Z_1X_2}{Z_1^2-R_2^2-X_2^2}\right), \tag{29}$$

where we write $Z_2 = R_2 + i\,X_2$. Recall that we desire x to be negative, so we rewrite eq. (29) as,

$$-x = \frac{\lambda}{4\pi}\tan^{-1}\left(\frac{2Z_1X_2}{R_2^2+X_2^2-Z_1^2}\right), \tag{30}$$

so that we can use the smallest positive value of the arctangent as our solution.

As expected, if the load reactance X_2 is zero, we obtain $x = 0$ as the position closest to the load at which the impedance $Z(x)$ is real.

The real value of the impedance $Z(x)$ for x given by eq. (30) is,

$$Z(x) = Z_1 \frac{\sqrt{Z_1^2+R_2^2+X_2^2+2Z_1R_2} - \sqrt{Z_1^2+R_2^2+X_2^2-2Z_1R_2}}{\sqrt{Z_1^2+R_2^2+X_2^2+2Z_1R_2} + \sqrt{Z_1^2+R_2^2+X_2^2-2Z_1R_2}}\,. \tag{31}$$

If $X_2 = 0$ then according to eq. (31), $Z(x) = R_2$ as expected.

The Smith Chart.

Before computers and pocket calculators were common, P.H. Smith [1] gave a graphical method for finding the values of x for which $Z(x)$ is real, as well as the real values for $Z(x)$.

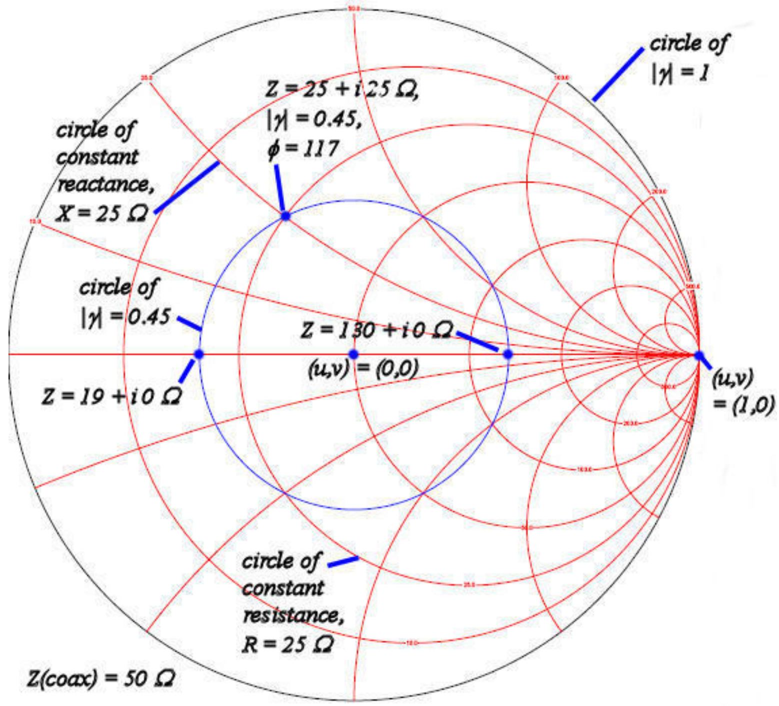
Converting eq. (28) to a normalized impedance, we write,

$$z(x) = \frac{1+\gamma e^{-2i\,kx}}{1-\gamma e^{-2ikx}} = \frac{z + i\,\tan(kx)}{1 + i\,z\tan(kx)}\,, \tag{32}$$

using eq. (22). We can also define a complex, position-dependent reflection coefficient according to,

$$\gamma(x) = \frac{z(x)-1}{z(x)+1} = \gamma e^{-2i\,kx} = |\gamma|\,e^{i(\phi_\gamma - 2kx)}. \tag{33}$$

The complex, position-dependent reflection coefficient $\gamma(x)$ moves in a circle of radius $|\gamma| \leq 1$ on the complex plane, and completes one clockwise rotation around this circle when x increases by $\lambda/2$. This insight is represented graphically on a so-called Smith chart, with is a plot of the complex reflection coefficient $\gamma = u + i\,v$ on the complex u-v plane.



For ease of use, curve of constant resistance R_2 and constant reactance X_2 are also shown on the Smith chart, for a specified value of the (real) impedance Z_1. For this, we write,

$$z = \frac{R_2 + i\,X_2}{Z_1} = \frac{1+\gamma}{1-\gamma} = \frac{1+u+i\,v}{1-u-i\,v} = \frac{1-u^2-v^2+2i\,v}{(u-1)^2+v^2}\,. \tag{34}$$

Thus,

$$\frac{R_2}{Z_1} = \frac{1-u^2-v^2}{(1-u)^2+v^2} \qquad \text{and} \qquad \frac{X_2}{Z_1} = \frac{2v}{(1-u)^2+v^2}\,. \tag{35}$$

Bringing all u's and v's into the numerator and rearranging terms a bit, we find,

$$\left(u - \frac{R_2}{Z_1+R_2}\right)^2 + v^2 = \left(\frac{Z_1}{Z_1+R_2}\right)^2 \qquad \text{and} \qquad (u-1)^2 + \left(v - \frac{Z_1}{X_2}\right)^2 = \left(\frac{Z_1}{X_2}\right)^2. \tag{36}$$

These are both equations of circles. In particular, circles of constant R_2 have their center on the u (horizontal) axis, and these circles all pass through the point $(u,v) = (1,0)$. Circles of constant X_2 all have their centers on the vertical line $u = 1$, and these circles also all pass through the point $(u,v) = (1,0)$.

In the Smith chart above,[9] the transmission line impedance is $Z_1 = 50\,\Omega$ and the load impedance is $Z_2 = 25 + i\,25\,\Omega$. We plot Z_2 using the two sets of circles of constant resistance and constant reactance to locate this point, which happens to correspond to the complex reflection coefficient $\gamma = 0.45e^{i\,117^\circ}$. The circle of $|\gamma| = 0.45$ crosses the real axis at points corresponding to $Z(x) = 19\ \Omega$ if we move $180 - 117 = 63^\circ$ counterclockwise from Z_2, and to $Z(x) = 130\ \Omega$ if we move counterclockwise by 243°. The corresponding distances along the coaxial cable are $x = (\lambda/2)(63/360) = 0.05\lambda$, and $x = (\lambda/2)(243/360) = 0.34\lambda$.

We leave it to the reader to judge whether the graphic construction is more or less convenient than using the algebraic relations (30)-(31). Either way, one could "match" the complex load impedance Z_2 to the coaxial cable of impedance Z_1 by cutting into the cable at position x and adding an appropriate matching resistor either in series or parallel, depending on whether $Z(x)$ is greater or less than Z_1, as discussed in the following section.

(c) **Impedance Matching via Resistors.**

According to eq. (18) or (19), there will be no reflected wave at a junction if the total impedance beyond the junction is the same as that before the junction.

So, if $Z_2 < Z_1$, placing a resistor of value $R = Z_1 - Z_2$ in series with line 2 brings the impedance beyond the junction to Z_1, and no reflection will occur.

The currents in both lines are equal: $I_{1+} = I_{2+}$. However, the series resistance reduces the voltage in line 2 to,

$$V_{2+} = I_{2+} Z_2 = I_{1+} Z_2 = \frac{Z_2}{Z_1} V_{1+}. \tag{37}$$

Hence, the transmitted power is,

$$P_{2+} = \frac{V_{2+} I_{2+}}{2} = \frac{Z_2}{Z_1} \frac{V_{1+} I_{1+}}{2} = \frac{Z_2}{Z_1} P_{1+} \qquad (Z_2 < Z_1). \tag{38}$$

Similarly, if $Z_2 > Z_1$, then the impedance beyond the junction can be reduced to Z_1 by adding a resistor in parallel (*i.e.*, between the two conductors at the junction) of value R given by,

$$\frac{1}{R} = \frac{1}{Z_1} - \frac{1}{Z_2}, \qquad \text{so that} \qquad R = \frac{Z_1 Z_2}{Z_2 - Z_1}. \tag{39}$$

In this case the voltages on the two lines are equal, $V_{2+} = V_{1+}$, while the currents are related by,

$$I_{2+} = \frac{V_{2+}}{Z_2} = \frac{V_{1+}}{Z_2} = \frac{Z_1}{Z_2} I_{1+}, \tag{40}$$

and the transmitted power is only,

$$P_{2+} = \frac{V_{2+} I_{2+}}{2} = \frac{Z_1}{Z_2} \frac{V_{1+} I_{1+}}{2} = \frac{Z_1}{Z_2} P_{1+} \qquad (Z_2 > Z_1). \tag{41}$$

[9]The Smith chart was generated using the Smith V2.03 software package available at http://www.fritz.dellsperger.net/

(d) **A $\lambda/4$ Matching Section.**

We suppose that the transmission line of impedance Z_1 occupies the region $x < 0$, the transition section of impedance Z_0 runs from $x = 0$ to $x = l$, and the line of impedance Z_2 extends over the region $x > l$.

The incident wave in line 1 moves in the $+x$ direction, and we desire no reflected wave this line. The matching section can support waves in both direction, while the wave in line 2 should move only in the $+x$ direction.

We can write the various waves as,[10]

$$
\begin{aligned}
V_{1+} &= V_{1+}e^{i(kx-\omega t)} = I_{1+}Z_1, &(42)\\
V_{0+} &= V_{0+}e^{i(kx-\omega t)} = I_{0+}Z_0, &(43)\\
V_{0-} &= V_{0-}e^{i(-kx-\omega t)} = -I_{0-}Z_0, &(44)\\
V_{2+} &= V_{2+}e^{i(kx-\omega t)} = I_{2+}Z_2. &(45)
\end{aligned}
$$

Continuity of the current and voltage at the junction $x = 0$ tells us that,

$$
\begin{aligned}
I_{1+} &= I_{0+} + I_{0-}, &(46)\\
V_{1+} &= V_{0+} + V_{0-}. &(47)
\end{aligned}
$$

Eliminating voltage in favor of current in eq. (47), we have,

$$
I_{1+}\frac{Z_1}{Z_0} = I_{0+} - I_{0-}. \tag{48}
$$

Solving eqs. (46) and (48), we find the currents in the matching section to be,

$$
I_{0\pm} = \frac{Z_0 \pm Z_1}{2Z_0} I_{1+}. \tag{49}
$$

Similarly, continuity of the current and voltage at the junction $x = l$ tells us that,

$$
\begin{aligned}
I_{0+}e^{ikl} + I_{0-}e^{-ikl} &= I_{2+}e^{ikl}, &(50)\\
V_{0+}e^{ikl} + V_{0-}e^{-ikl} &= V_{2+}e^{ikl}. &(51)
\end{aligned}
$$

Eliminating voltage in favor of current in eq. (51), we have,

$$
\frac{Z_0}{Z_2}(I_{0+}e^{ikl} - I_{0-}e^{-ikl}) = I_{2+}e^{ikl}. \tag{52}
$$

Using eq. (49) in eq. (50), we find current I_{2+} to be,

$$
I_{2+}e^{ikl} = \frac{(Z_0 + Z_1)e^{ikl} + (Z_0 - Z_1)e^{-ikl}}{2Z_0} I_{1+}, \tag{53}
$$

[10] It is not necessary to represent the behavior in line 2 as a traveling wave; all that is needed for the analysis is the relation $V_{2+} = I_{2+}Z_2$. Line 2 could be replaced by a resistor of value $R = Z_2$.

while using eq. (49) in eq. (52), we find,

$$I_{2+}e^{ikl} = \frac{(Z_0 + Z_1)e^{ikl} - (Z_0 - Z_1)e^{-ikl}}{2Z_2} I_{1+}. \tag{54}$$

Combining eqs. (53) and eq. (54), we must have,

$$Z_0 Z_2 + Z_1 Z_2 - Z_0^2 - Z_0 Z_1 = -(Z_0^2 - Z_0 Z_1 + Z_0 Z_2 - Z_1 Z_2)e^{-2i\,kl}. \tag{55}$$

Since the impedances Z_i are real, eq. (55) can be consistent only if e^{-2ikl} is real also. The options are $e^{-2i\,kl} = \pm 1$, corresponding to length l of the matching section being even or odd integer multiples of $\lambda/4$, recalling that $k = 2\pi/\lambda$. When $l = m\lambda/2$ for integer m, we can have only the trivial case that $Z_1 = Z_2$ (for any value of Z_0).

The nontrivial solution is that $l = m\lambda/4$ for odd integer m, in which case eq. (55) reduces to the condition that,

$$Z_0^2 = Z_1 Z_2, \qquad Z_0 = \sqrt{Z_1 Z_2}. \tag{56}$$

We now explore the option that $Z_2 = R + iX$ is a load with complex impedance rather than a transmission line with real impedance. The analysis up to and including eq. (55) holds whether Z_2 is real or complex. So, in the latter case we multiply eq. (55) by e^{ikl} and rearrange it to find,

$$Z_0(Z_1 - Z_2) = i\tan(kl)(Z_1 Z_2 - Z_0^2), \tag{57}$$

and substituting $Z_2 = R + iX$, we have,

$$Z_0(Z_1 - R) + Z_1 X \tan(kl) = i[Z_1 X + \tan(kl)(Z_1 R - Z_0^2)]. \tag{58}$$

The lefthand side of eq. (58) is real while the righthand side is imaginary. Therefore, both sides must vanish, which requires that,

$$\tan(kl) = \frac{Z_0(R - Z_1)}{Z_1 X} = \frac{Z_1 X}{Z_0^2 - Z_1 R}. \tag{59}$$

Thus, the impedance Z_0 obeys the cubic equation,

$$Z_0^3 - Z_1 R Z_0 - \frac{Z_1^2 X^2}{R - Z_1} = 0. \tag{60}$$

So long as R does not equal Z_1 there is at least one real solution for Z_0. The three roots of the cubic equation (60) are,

$$Z_0 = A_+ + A_-, \qquad -\frac{A_+ + A_-}{2} + \pm i\sqrt{3}\frac{A_+ - A_-}{2} \tag{61}$$

where,

$$A_\pm = \sqrt[3]{\frac{Z_1^2 X^2}{2(R - Z_1)}\left(1 \pm \sqrt{1 - \frac{4}{27}\frac{R^3(R - Z_1)^2}{Z_1 X^4}}\right)}. \tag{62}$$

Of course, the solution could be implemented with a physical transmission line only if Z_0 is both real and positive.

(e) **A "$\lambda/12$" Matching Section.**

We analyze the more general case of a load impedance $Z = R + iX$.

The input line of impedance Z_1 extends over the region $x < 0$. This connects to the first matching section, of impedance Z_2 and length l_2, followed by the second matching section, of impedance Z_1 and length l_1. The load impedance is located at $x = l_1 + l_2$.

We desire that the wave in the input line 1 be only in the $+x$ direction. In the matching sections there are waves moving in both directions. These waves can be written as,

$$\begin{aligned} V_{1+} &= V_{1+}e^{i(kx-\omega t)} = I_{1+}Z_1, & (63) \\ V_{2+} &= V_{2+}e^{i(kx-\omega t)} = I_{2+}Z_2, & (64) \\ V_{2-} &= V_{2-}e^{i(-kx-\omega t)} = -2_{0-}Z_2, & (65) \\ V'_{1+} &= V'_{1+}e^{i(kx-\omega t)} = I'_{1+}Z_1, & (66) \\ V'_{1-} &= V'_{1-}e^{i(kx-\omega t)} = I'_{1-}Z_1, & (67) \end{aligned}$$

where the currents and voltages $I'_{1\pm}$ and $V'_{1\pm}$ refer to the matching section of impedance Z_1. The current and voltage in the load are related by $V = IZ$.

The analysis proceeds as in part (c). Continuity of the current and voltage at the junction $x = 0$ tells us that,

$$\begin{aligned} I_{1+} &= I_{2+} + I_{2-}, & (68) \\ V_{1+} &= V_{2+} + V_{2-}. & (69) \end{aligned}$$

Eliminating voltage in favor of current in eq. (69), we have,

$$I_{1+}\frac{Z_1}{Z_2} = I_{2+} - I_{2-}. \qquad (70)$$

Solving eqs. (68) and (70), we find the currents in the matching section to be,

$$I_{2\pm} = \frac{Z_2 \pm Z_1}{2Z_2} I_{1+}. \qquad (71)$$

Similarly, continuity of the current and voltage at the junction $x = l_2$ tells us that,

$$\begin{aligned} I_{2+}e^{ikl_2} + I_{2-}e^{-ikl_2} &= I'_{1+}e^{ikl_2} + I'_{1-}e^{-ikl_2}, & (72) \\ V_{2+}e^{ikl_2} + V_{2-}e^{-ikl_2} &= V'_{1+}e^{ikl_2} + V'_{1-}e^{-ikl_2}. & (73) \end{aligned}$$

Eliminating voltage in favor of current in eq. (73), we have,

$$\frac{Z_2}{Z_1}(I_{2+}e^{i\,kl_2} - I_{2-}e^{-i\,kl_2}) = I'_{1+}e^{i\,kl_2} - I'_{1-}e^{-i\,kl_2}. \qquad (74)$$

Solving eqs. (72) and eq. (74), and using eq. (71), we find,

$$I'_{1+} = \frac{(Z_1 + Z_2)I_{2+} + (Z_1 - Z_2)I_{2-}e^{-2ikl_2}}{2Z_1} = \frac{(Z_1 + Z_2)^2 - (Z_1 - Z_2)^2 e^{-2ikl_2}}{4Z_1Z_2} I_{1+}, \qquad (75)$$

$$I'_{1-} = \frac{(Z_1 - Z_2)I_{2+}e^{2ikl_2} + (Z_1 + Z_2)I_{2-}}{2Z_1} = \frac{Z_1^2 - Z_2^2}{4Z_1Z_2}(e^{2ikl_2} - 1)I_{1+}. \tag{76}$$

Finally, continuity of the current and voltage at the junction $x = l_1 + l_2$ tells us that,

$$I'_{1+}e^{ik(l_1+l_2)} + I'_{1-}e^{-ik(l_1+l_2)} = I, \tag{77}$$

$$V'_{1+}e^{ik(l_1+l_2)} + V'_{1-}e^{-ik(l_1+l_2)} = V. \tag{78}$$

Eliminating voltage in favor of current in eq. (78), we have,

$$\frac{Z_1}{Z}(I'_{1+}e^{ik(l_1+l_2)} - I'_{1-}e^{-ik(l_1+l_2)}) = I. \tag{79}$$

Substituting eqs. (75) and (76) in eq. (77), we find,

$$I = \frac{(Z_1 + Z_2)^2 e^{ik(l_1+l_2)} - (Z_1 - Z_2)^2 e^{ik(l_1-l_2)} + (Z_1^2 - Z_2^2)(e^{-ik(l_1-l_2)} - e^{-ik(l_1+l_2)})}{4Z_1Z_2} I_{1+}, \tag{80}$$

while from eq. (79) we find,

$$I = \frac{(Z_1 + Z_2)^2 e^{ik(l_1+l_2)} - (Z_1 - Z_2)^2 e^{ik(l_1-l_2)} - (Z_1^2 - Z_2^2)(e^{-ik(l_1-l_2)} - e^{-ik(l_1+l_2)})}{4ZZ_2} I_{1+}, \tag{81}$$

Consistency of eqs. (80) and (81) requires that,

$$\begin{aligned} &(Z - Z_1)\left[(Z_1 + Z_2)^2 e^{ik(l_1+l_2)} - (Z_1 - Z_2)^2 e^{ik(l_1-l_2)}\right] \\ &\quad = -(Z + Z_1)(Z_1^2 - Z_2^2)(e^{-ik(l_1-l_2)} - e^{-ik(l_1+l_2)}). \end{aligned} \tag{82}$$

We first consider the case that $Z = Z_2$ and is therefore real, since Z_2 is the (real) impedance of a section of transmission line. Then, we can divide eq. (82) by $Z_2 - Z_1$ to obtain,

$$(Z_1 + Z_2)^2 e^{ik(l_1+l_2)} - (Z_1 - Z_2)^2 e^{ik(l_1-l_2)} = (Z_1^2 + Z_2^2)(e^{-ik(l_1-l_2)} - e^{-ik(l_1+l_2)}), \tag{83}$$

which we rearrange as,

$$(Z_1 + Z_2)^2(e^{ik(l_1+l_2)} + e^{-ik(l_1+l_2)}) = (Z_1 - Z_2)^2 e^{ik(l_1-l_2)} + (Z_1 + Z_2)^2 e^{-ik(l_1-l_2)}, \tag{84}$$

and so,

$$(Z_1 + Z_2)^2 \cos[k(l_1 + l_2)] = (Z_1^2 + Z_2^2)\cos[k(l_1 - l_2)] + iZ_1Z_2 \sin[k(l_1 - l_2)]. \tag{85}$$

Since the lefthand side of eq. (85) is real, we must have that $\sin[k(l_1 - l_2)] = 0$, which is most simply satisfied by taking,

$$l_1 = l_2 \equiv l \qquad (Z = Z_2). \tag{86}$$

Then, the real part of eq. (85) become,

$$\cos(2kl) = \cos\frac{4\pi l}{\lambda} = \frac{Z_1^2 + Z_2^2}{(Z_1 + Z_2)^2} \qquad (Z = Z_2). \tag{87}$$

For $Z_1 \approx Z_2$, the righthand side of eq. (87) is close to 1/2, so $4\pi l/\lambda \approx \pi/3$, and,

$$l \approx \frac{\lambda}{12} \qquad (Z = Z_2). \tag{88}$$

Returning to the case of matching into a load impedance of $Z = R + i\,X$, we rewrite eq. (82) as,

$$\begin{aligned}(R - Z_1 + i\,X)e^{i\,kl_1}\left[(Z_1 + Z_2)^2 e^{i\,kl_2} - (Z_1 - Z_2)^2 e^{-i\,kl_2}\right] \\ = -(R + Z_1 + i\,X)e^{-ikl_1}(Z_1^2 - Z_2^2)(e^{ikl_2} - e^{-ikl_2}),\end{aligned} \tag{89}$$

and further as,

$$\begin{aligned}(R - Z_1 + iX)e^{i\,kl_1}\left[2Z_1 Z_2 \cos(kl_2) + i(Z_1^2 + Z_2^2)\sin(kl_2)\right] \\ = -i(R + Z_1 + i\,X)e^{-i\,kl_1}(Z_1^2 - Z_2^2)\sin(kl_2).\end{aligned} \tag{90}$$

Since the length l_1 appears only as a phase factor, we can eliminate it by taking the absolute square of eq. (90). Thus,

$$\begin{aligned}[(R - Z_1)^2 + X^2]\left[4Z_1^2 Z_2^2 \cos^2(kl_2) + (Z_1^2 + Z_2^2)^2 \sin^2(kl_2)\right] \\ = [(R + Z_1)^2 + X^2](Z_1^2 - Z_2^2)^2 \sin^2(kl_2),\end{aligned} \tag{91}$$

and hence,

$$\begin{aligned}\tan^2(kl_2) &= \frac{4Z_1^2 Z_2^2[(R - Z_1)^2 + X^2]}{[(R + Z_1)^2 + X^2](Z_1^2 - Z_2^2)^2 - [(R - Z_1)^2 + X^2](Z_1^2 + Z_2^2)^2} \\ &= \frac{(R - Z_1)^2 + X^2}{RZ_1\left(\frac{Z_2}{Z_1} - \frac{Z_1}{Z_2}\right)^2 - (R - Z_1)^2 - X^2}.\end{aligned} \tag{92}$$

The righthand side of eq. (92) must be positive, so if $(R - Z_1)^2 + X^2$ is large, the ratio Z_2/Z_1 must be large also.

To determine the length l_1 we return to eq. (90) and equate the phases of the lefthand and righthand sides. Thus,

$$\tan^{-1}\frac{X}{R - Z_1} + kl_1 + \tan^{-1}\left(\frac{Z_1^2 + Z_2^2}{2Z_1 Z_2}\tan(kl_2)\right) = -\frac{\pi}{2} + \tan^{-1}\frac{X}{R + Z_1} - kl_1, \tag{93}$$

so that,

$$2kl_1 = -\frac{\pi}{2} + \tan^{-1}\frac{X}{R + Z_1} - \tan^{-1}\frac{X}{R - Z_1} - \tan^{-1}\left(\frac{Z_1^2 + Z_2^2}{2Z_1 Z_2}\tan(kl_2)\right). \tag{94}$$

If the righthand side of eq. (94) is negative, add as many 2π's as are needed to make it positive.

Other expressions for l_1 can be given. Regier [6] prefers the form,

$$\tan(kl_1) = \frac{\left(Z_2 - R\frac{Z_1}{Z_2}\right)\tan(kl_2) + X}{R - Z_1 + X\frac{Z_2}{Z_1}\tan(kl_2)}\,. \tag{95}$$

We can verify that Bramham's impedance matching scheme is a special case of eqs. (92) and (95) when $Z = Z_2$, *i.e.*, when $R = Z_2$ and $X = 0$. Then, eq. (95) reduces to $\tan(kl_1) = \tan(kl_2)$, so that $l_1 = l_2 \equiv l$. Further, eq. 92) reduces to,

$$\tan^2(kl) = \frac{Z_1Z_2}{Z_1^2 + Z_1Z_2 + Z_2^2}, \tag{96}$$

so that,

$$\sin^2(kl) = \frac{Z_1Z_2}{(Z_1+Z_2)^2} \qquad \text{and} \qquad \cos^2(kl) = \frac{Z_1^2 + Z_1Z_2 + Z_2^2}{(Z_1+Z_2)^2}\,, \tag{97}$$

and finally,

$$\cos(2kl) = \cos^2(kl) - \sin^2(kl) = \frac{Z_1^2 + Z_2^2}{(Z_1+Z_2)^2} \qquad (Z = Z_2), \tag{98}$$

as found in eq. (87).

The quarter-wave matching scheme of part (c) is also a special case of eqs. (92) and (95) where $Z = R$ is real and $Z_2 = \sqrt{ZZ_1}$. Then, eq. (95) tells us that $\tan(kl_1) = 0$, so that the second matching section is not needed. And, eq. (92) tells us that $\tan(kl_2) = \infty$, so $kl_2 = \pi/2$ and $l_2 = \lambda/4$.

(f) **Impedance Matching via a Flux-Linked Transformer.**

The coils 1 and 2 of the flux-linked transformer have self inductances L_1 and L_2 and mutual inductance M.

We first relate these inductances to the numbers of turns in the coils. Suppose that current I_1 creates magnetic flux Φ_0 in a single turn of coil 1,

$$\Phi_0 = \int B_0 d\text{Area} = L_0 I_1, \tag{99}$$

where B_0 is the magnetic field due to a single turn, and L_0 is the self inductance of a single turn. If coil 1 contains N_1 turns, then the total magnetic field is $N_1 B_0$, and the total flux due to current I_1 that is linked by coil 1 is,

$$\Phi_1 = N_1^2 L_0 I_1 \equiv L_1 I_1, \tag{100}$$

assuming that all the flux from each turn of the coil passes through all of the other turns as well. Winding the turns around a ferrite core helps to make this assumption valid.

Similarly for coil 2, the flux Φ_2 that it links due to its own current I_2 is related by,

$$\Phi_1 = N_1^2 L_0 I_2 \equiv L_2 I_2. \tag{101}$$

We also suppose that all the flux created by coil 1 is linked by coil 2, and *vice versa*,

$$\Phi_{12} = N_1 N_2 L_0 I_1 \equiv M I_1, \qquad \text{and likewise} \qquad \Phi_{21} = N_2 N_1 L_0 I_2 \equiv M I_2. \tag{102}$$

In sum, the inductances are related by,

$$L_1 = N_1^2 L_0, \qquad L_2 = N_2^2 L_0, \qquad M = N_1 N_2 L_0. \tag{103}$$

In general, the self and mutual inductances of two coils obey the inequality $L_1 L_2 \geq M^2$, but in the ideal case (103) we have,

$$L_1 L_2 = M^2. \tag{104}$$

We now perform a "circuit analysis" of the transformer, being careful to note that line 1 can contain waves that move in either direction, corresponding to currents $I_{1\pm}$ and voltages $V_{1\pm} = \pm I_{1\pm} Z_1$, while line 2 carries only a wave moving away from the transformer (assuming that line 2 is properly terminated elsewhere). Then, the voltage across coil 1 obeys,

$$V_1 = (I_{1+} - I_{1-}) Z_1 = L_1((\dot{I}_{1+} - \dot{I}_{1-}) + M \dot{I}_2 = -i\omega L_1 (I_{1+} - I_{1-}) - i\omega M I_2. \tag{105}$$

For coil 2 we note that our convention for directions of currents implies that the current I_2 flows through coil 2 in the opposite sense to the flow of current I_1 through coil 1. Hence, the voltage across coil 2 is related by,

$$V_2 = I_2 Z_2 = -M((\dot{I}_{1+} - \dot{I}_{1-}) - L_2 \dot{I}_2 = i\omega M (I_{1+} - I_{1-}) + i\omega L_2 I_2. \tag{106}$$

We rearrange eqs. (105) and (106) to emphasize that we wish to solve for currents I_{1-} and I_2 in terms of I_{1+},

$$\begin{aligned} (Z_1 - i\omega L_1) I_{1-} \qquad\quad - i\omega M I_2 &= (Z_1 + i\omega L_1) I_{1+}, \qquad (107) \\ -i\omega M I_{1-} + (Z_2 - i\omega L_2) I_2 &= \qquad\quad i\omega M I_{1+}. \qquad (108) \end{aligned}$$

The determinant of the coefficients of the lefthand side is,

$$\Delta = Z_1 Z_2 + \omega^2 (M^2 - L_1 L_2) - i\omega (L_1 Z_2 + L_2 Z_1) = Z_1 Z_2 - i\omega (L_1 Z_2 + L_2 Z_1), \tag{109}$$

recalling eq. (104). Solving for I_{1-}, we find,

$$I_{1-} = \frac{Z_1 Z_2 + i\omega (L_1 Z_2 - L_2 Z_1)}{\Delta} I_{1+}. \tag{110}$$

To minimize the reflected current, we must have,

$$L_1 Z_2 = L_2 Z_1, \tag{111}$$

and hence eq. (103) tells us that the ratio of the number of turns in the transformer coils must be,

$$\frac{N_1}{N_2} = \sqrt{\frac{Z_1}{Z_2}}. \tag{112}$$

The reflected current is then,

$$I_{1-} = \frac{Z_1 Z_2}{Z_1 Z_2 - 2i\omega L_1 Z_2} I_{1+} \approx i \frac{Z_1}{\omega L_1} I_{1+}, \tag{113}$$

where the approximation holds for high frequencies. The reflected power, $P_{1-} = I_{1-}^2 Z_1/2$ varies as $1/\omega^2$ at high frequencies, but for low frequencies $I_{1-} \approx I_{1+}$ and all the power is reflected rather than transmitted by the transformer.

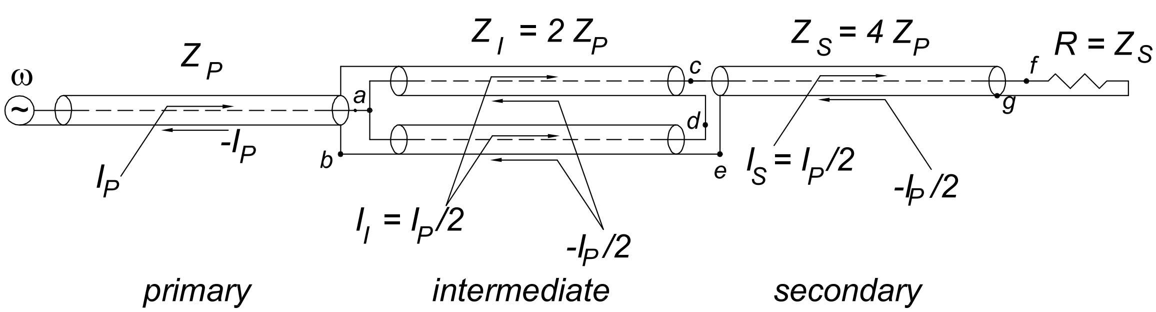
The current I_2 is given by,

$$I_2 = \frac{2i\omega M Z_1}{Z_1 Z_2 - 2i\omega L_1 Z_2} I_{1+} \approx -\frac{M Z_1}{L_1 Z_2} I_{1+} = -\frac{N_2 Z_1}{N_1 Z_2} I_{1+} = -\sqrt{\frac{Z_1}{Z_2}} I_{1+}, \tag{114}$$

where the approximation holds at high frequencies, for which the transmitted power $I_2^2 Z_2/2$ equals the incident power $I_{1+}^2 Z_1/2$ as desired. The minus sign in eq. (114) reminds us that the flux-linked transformer inverts the currents and voltages.

(g) **The Transmission-Line Transformer of Guanella.**

To analyze the 1:2 voltage (1:4 impedance) transmission-line transformer of Guanella, it is helpful to identify seven points, a, b, c, d, e, f and g, in the circuit as shown in the figure below.



Then, the voltage V_{ab} between points a and b is,

$$V_{ab} = I_P Z_P. \tag{115}$$

The primary transmission line is connected in parallel to two intermediate transmission lines of equal impedance, so the current in each of the intermediate lines, ac and ad, is,

$$I_I = \frac{I_P}{2}. \tag{116}$$

The center conductor ac of the upper intermediate line connects directly to the center conductor cf of the secondary line, so we have,

$$I_S = I_I = \frac{I_P}{2}. \tag{117}$$

The voltage difference V_{cd} between the center conductor and the outer conductor of the upper intermediate line is equal to the voltage difference V_{ab} across the primary line. Likewise, the voltage difference V_{de} across the lower intermediate line is equal to V_{ab},

$$V_{cd} = V_{de} = V_{ab}. \tag{118}$$

Because the outer conductor of the upper intermediate line is shorted to the inner conductor of the lower intermediate line, we have,

$$V_{ce} = V_{cd} + V_{de} = 2V_{ab}. \tag{119}$$

And, the voltage difference $V_S = V_{fg}$ across the secondary line is equal to V_{ce}. Hence, the impedance Z_S of the secondary line should be,

$$Z_S = \frac{V_S}{I_S} = \frac{V_{ce}}{I_P/2} = \frac{2V_{ab}}{I_P/2} = \frac{2I_P Z_P}{I_P/2} = 4Z_P. \tag{120}$$

Similarly, the impedance Z_I of the intermediate lines is given by,

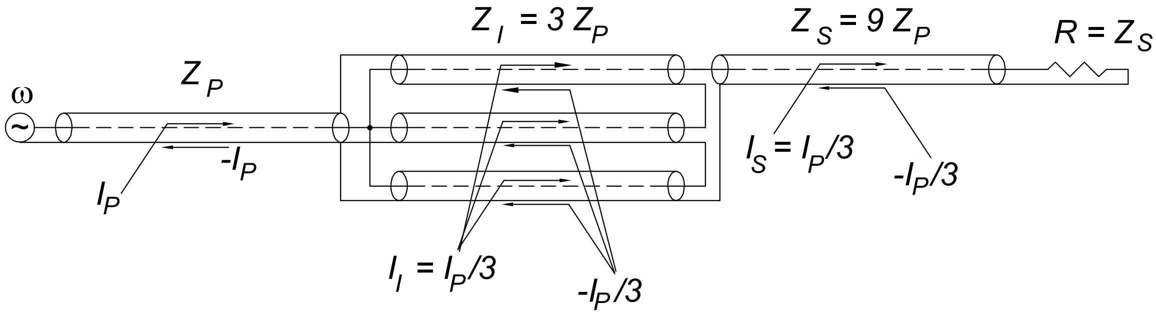
$$Z_I = \frac{V_{cd}}{I_I} = \frac{V_{ab}}{I_P/2} = \frac{I_P Z_P}{I_P/2} = 2Z_P. \tag{121}$$

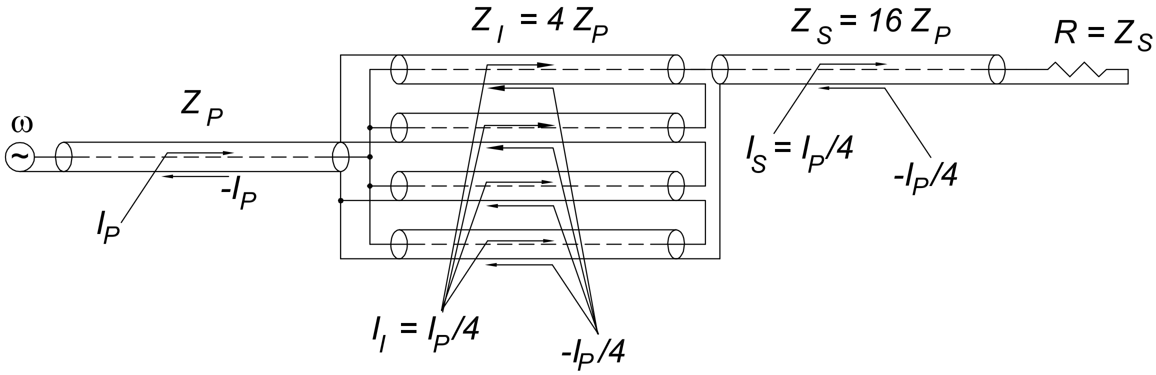
The above analysis tacitly assume that the voltages at points a and d are not equal, even though these points are connected by a conductor. While these voltages would be equal in a DC circuit, they need not be equal at points a and d which are separated by transmission lines that carry TEM waves. However, if the intermediate lines are too short, the fields in these lines will not be purely TEM, and the voltages at the two ends will not be independent of one another. This is the case when the length L of the intermediate lines is less than a wavelength, so we expect that Guanella's scheme fails at low frequencies, $\omega \lesssim c/L$. The addition of inductive isolation, either via external ferrite cores or by winding the intermediate conductors into choke coils, will improve the low-frequency performance of a transmission-line transformer.

If there had been m intermediate lines of equal impedance $Z_I = mZ_P$, then the intermediate currents would all be $I_I = I_P/m$. The voltage across each of the intermediate lines would still be $V_I = V_P = I_P Z_P$, since they are connected in parallel to the primary line. If the center and outer conductors of adjacent intermediate lines were shorted together, then the total maximum voltage between intermediate conductors is $mV_I = mV_P = mI_P Z_P$. Since this is also the voltage delivered to the secondary line, we have $V_S = mV_P$, and we have a 1:m voltage transformer. The (matched) impedance of the secondary line is therefore,

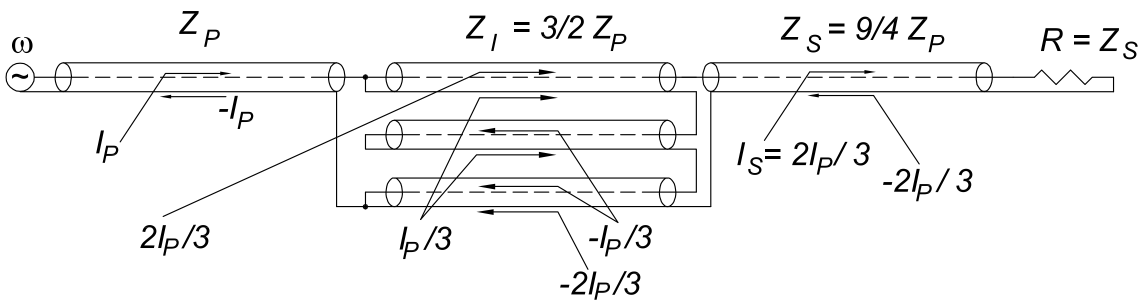
$$Z_S = \frac{V_S}{I_S} = \frac{mI_P Z_P}{I_P/m} = m^2 Z_P. \tag{122}$$

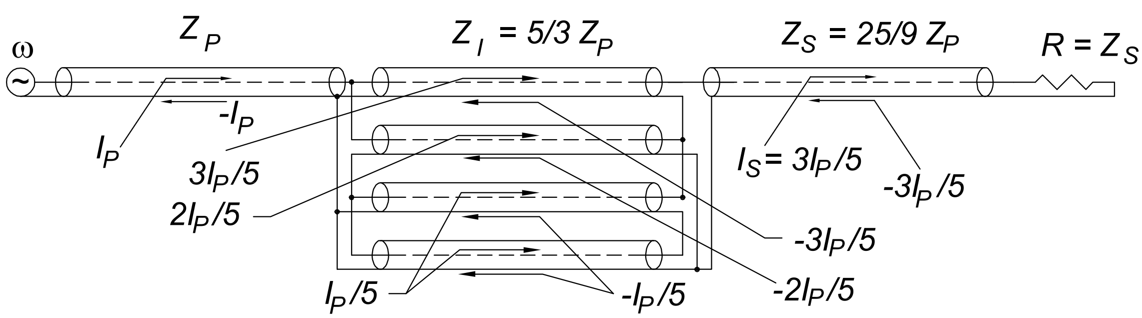
The figures below illustrate 1:3 and 1:4 voltage transmission-line transformers.





When three intermediate lines are used there is another arrangement possible, illustrated below, which provides a 2:3 voltage transformer. When four intermediate lines are user there are two additional arrangements, one of which provides a 3:5 voltage transformer, as illustrated below. See also [9, 10, 11].





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