Charging of an Insulator in a Liquid Argon Detector

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1 Problem

Insulating material used in a gas- or liquid-filled particle detector, such as a gaseous wire chamber or a liquid-argon time-projection chamber, generally resides in a region of nonzero (nominally static) electric field. Initially, the electric field inside the insulator is that due to the "external" electric field $\mathbf{E}_0(\mathbf{x})$, taking into account the (relative) permittivities (dielectric constants) of the various materials in the device.[1] Cosmic rays, and particles from other sources such as nuclear reactors and particles accelerators, that pass through the device create electron-ion pairs whose charges may drift until intercepted by conductors or insulators. In the latter case, the insulator becomes charged, which perturbs the external field $\mathbf{E}_0$. Discuss whether such perturbations can result in electric fields much larger than E_0 inside the insulators, which might lead to their failure via electric discharges.

This problem was suggested by Bo Yu.

2 Solution

This problem is related to the larger issue of space charge in ionization detectors [1, 2], that electrons and ions created by passing charged particles move with different drift velocities leaving a net positive charge in the bulk of the ionization medium. One effect of this resulting "space charge" is that the external field $\mathbf{E}_0$ is reduced in the region close to the anode, both in the active volume of the detector, and in the surrounding volume. The present problem is concerned with the latter volume, in which insulators might be placed as part of the support structure for the active detector volume.

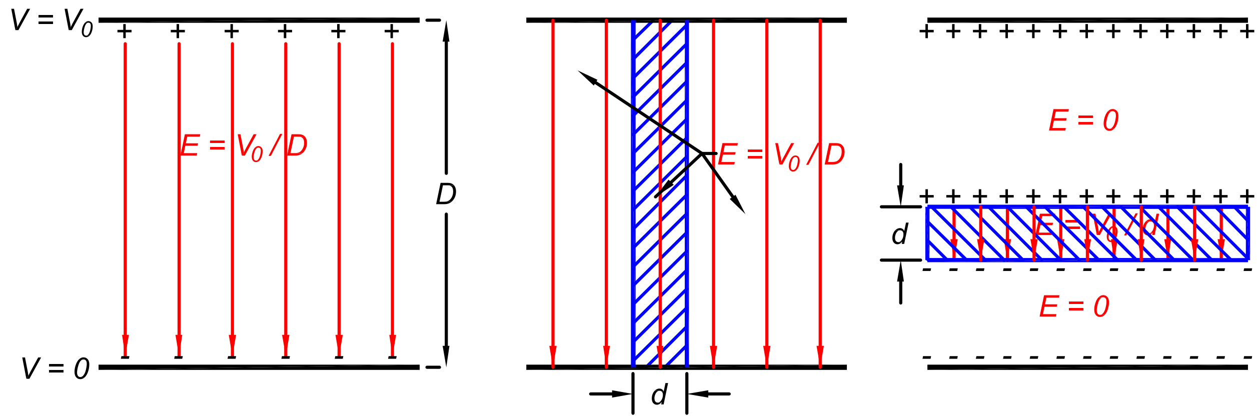
We consider examples of a planar insulator, a circular cylinder insulator, a spherical insulator, and an elliptic-cylinder insulator, and a rectangular insulator, each in a uniform, external electric field $\mathbf{E}_0$. That is, we ignore here the change in the external field due to space charge in the ionization medium (such as liquid argon), and instead investigate the effect of charge accumulation on the insulator.

A general consideration is that as long as electric field lines reach the outer surface of the insulator, its charging continues. Hence, if there exists a final, static configuration, the electric field just outside an insulator is then either zero or parallel to its surface.

2.1 Planar Insulator

The case of a planar insulator whose extent is the same as that of the field $\mathbf{E}_0$ is straightforward, as sketched below.

[1] The adjective "external" will be applied only to the field $\mathbf{E}_0$, while the total field $\mathbf{E}$ in the presence of a conducting or insulating body will be partitioned in the fields exterior and interior to that body.



If the plane of the insulator is parallel to the field $\mathbf{E}_0$, and completely fills the gap of height D between the electrodes that establish that field, then the field inside the insulator is also $\mathbf{E}_0$, independent of the permittivities of the insulator or the medium surrounding it. See the middle figure above.

On the other hand, if the plane of the insulator is perpendicular to $\mathbf{E}_0$, then positive charge will accumulate on the upper surface of the insulator, and negative charge on its lower surface, until the field outside the insulator is zero and charge ceases to flow in the medium surrounding it.[2] See the right figure above. Then, assuming that the potential difference V_0 associated with the external field $\mathbf{E}_0$ is unchanged, the steady-state field inside the insulator has magnitude $E = V_0/d = E_0 D/d$, where d is the thickness of the insulator.[3]

For a thin insulator, $d \ll D$, the resulting internal field could be very large, which might lead to failure via electric discharges.

In practice, insulators inside particle detectors are unlikely to be as large as the detector, so we next consider smaller insulators.

2.2 Circular-Cylinder Insulator

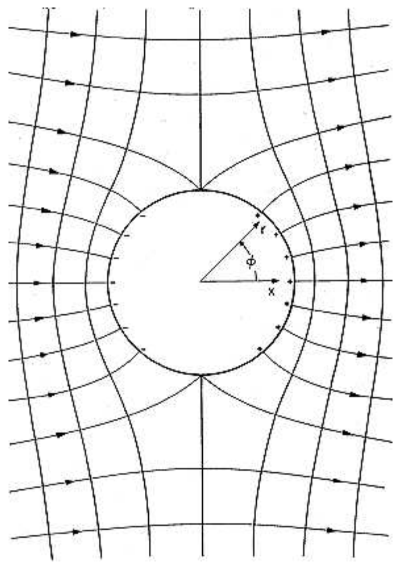
Consider a circular-cylinder insulator of radius $a \ll D$, whose axis is perpendicular to the uniform field $\mathbf{E}_0$.

If charge accumulates on this cylinder until a steady state is achieved, the electric field outside the cylinder will not be zero (as for the planar insulator in the right figure on p. 2), but rather, the field lines just outside the cylinder will be parallel to its surface, such that the electrons and ions which flow in the surrounding medium will not reach the cylinder. The challenge is to deduce the distribution of charge on the cylinder, and the perturbed, static electric field $\mathbf{E}$ that is consistent with this steady-state.

[2]While positive charge flows onto the upper surface of the insulator, negative charge flows onto the upper electrode (and similarly positive charge flows onto the lower electrode. These electrodes are assumed to be conductors held as constant potentials, so charge flows between the "battery" that maintains the potentials and the electrodes, such that the final charge distribution shown in the right figure above can be achieved.

[3]The analysis of the steady state does not depend on the relative dielectric constants ϵ_I and ϵ_A of the insulator and the liquid argon, respectively. However, the initial field $\mathbf{E}_I$ inside the insulator (before significant charge accumulation) does depend on these for the case on an "infinite" planar insulator perpendicular to $\mathbf{E}_0$. Here, $V_0 = E_I d + E_A(D - d)$, and $\epsilon_I E_I = \epsilon_A E_A$, where E_A is the initial field in the liquid argon, such that the initial field inside the insulator is $E_I = V_0/[D + (\epsilon_A/\epsilon_I - 1)d]$.

A clue as to the solution comes from consideration of a related problem, a conducting circular cylinder in an otherwise uniform electric field. The figure below[4] shows the field lines and equipotential surfaces for the case where the uniform field $\mathbf{E}_0$ points to the right.



A famous result of electrostatics in two dimensions is that for any analytic function $f(z) = u + iw$ of a complex variable $z = x + iy$, both functions $u(x, y)$ and $v((x, y)$ obey Laplace's equation, $\nabla^2 u = 0 = \nabla^2 v$.[5] Also, lines of constant u are orthogonal to lines of constant v. Hence, u is a possible potential function for some 2-d electrostatics problem, with lines of constant v corresponding to the electric field lines. And, taking v to be the potential function, we have a solution to another electrostatics problem, where now lines of constant u correspond to the field lines.

Thus, in the above figure for the case of a conducting cylinder in an otherwise uniform electric field, we can interchange the roles of the field lines and the equipotentials to obtain a solution to a different problem. In the latter problem, we see that the field lines do not reach the cylinder (except at two points), such that electrons and ions that flow along the field lines would never reach the cylinder. Hence, this second solution is the solution we desire for the steady state of a cylindrical insulator in, say, liquid argon.

It remains to deduce the electric field inside the cylinder, which does not immediately follow from the preceding argument.

For this, we note that in the case of the conducting cylinder, the induced surface-charge distribution at $r = a$ in a cylindrical coordinate system (r, ϕ, z) with origin at the center of the cylinder is, for uniform field $\mathbf{E}_0 = E_0\,\hat{\mathbf{x}} = E_0(\hat{\mathbf{r}}\cos\phi - \hat{\boldsymbol{\phi}}\sin\phi)$,

$$\sigma(\phi) = 2\epsilon_0 E_0 \cos\phi \qquad \text{(conducting cylinder)}, \tag{1}$$

and the electric field associated with this charge distribution is,

$$\mathbf{E}_\sigma = E_0 \begin{cases} \left(\frac{a}{r}\right)^2 (\hat{\mathbf{r}}\cos\phi + \hat{\boldsymbol{\phi}}\sin\phi) & (r > a), \\ -\hat{\mathbf{x}} & (r < a), \end{cases} \qquad \text{(conducting cylinder)}, \tag{2}$$

[4]From http://web.mit.edu/6.013_book/www/chapter5/5.8.html

[5]This was noted by Helmholtz in [3] in the context of 2-dimensional fluid flow. See also p. 71 of [4].

Note that the peak electric field for a conducting cylinder in an otherwise uniform external field $\mathbf{E}_0$ (perpendicular to the axis of the cylinder) is $2\mathbf{E}_0$.

In the problem of the cylindrical insulator we desire that the radial electric field be zero just outside the cylinder. This is achieved by reversing the sign of the charge distribution (1),[6] which reverses the sign of the electric field (2). Hence, the desired solution is,[7]

$$\sigma(\phi) = -2\epsilon_0 E_0 \cos\phi \qquad \text{(insulating cylinder)}, \tag{3}$$

and the electric field associated with this charge distribution is,[8]

$$\mathbf{E}_\sigma = E_0 \begin{cases} -\left(\frac{a}{r}\right)^2 (\hat{\mathbf{r}}\cos\phi + \hat{\boldsymbol{\phi}}\sin\phi) & (r > a), \\ \hat{\mathbf{x}} & (r < a), \end{cases} \qquad \text{(insulating cylinder)}. \tag{4}$$

The total electric field for the steady-state solution for the insulating cylinder is,

$$\mathbf{E} = \mathbf{E}_0 + \mathbf{E}_\sigma = E_0 \begin{cases} \left[1 - \left(\frac{a}{r}\right)^2\right] \hat{\mathbf{r}}\cos\phi - \left[1 + \left(\frac{a}{r}\right)^2\right] \hat{\boldsymbol{\phi}}\sin\phi & (r > a), \\ 2\,\hat{\mathbf{x}} & (r < a), \end{cases} \qquad \text{(insulating cylinder)}. \tag{5}$$

Hence, the steady-state field inside an insulating cylinder is only double the external field $\mathbf{E}_0$ (independent of its radius a so long as this is small compared to D), and electrical discharges are unlikely in this case. Note that this result is also independent of the dielectric constants of the insulator and the medium surrounding it.[9]

2.3 Spherical Insulator

We can build on the solution for a circular-cylinder insulator to discuss the case of a spherical insulator of radius $a \ll D$.

Again we first consider the case of a conducting sphere in the otherwise uniform electric field $\mathbf{E}_0 = E_0\,\hat{\mathbf{z}} = E_0(\hat{\mathbf{r}}\cos\theta - \hat{\boldsymbol{\theta}}\sin\theta)$ in a spherical coordinate system (r, θ, ϕ), for which the induced surface-charge distribution is,

$$\sigma(\phi) = 3\epsilon_0 E \cos\theta \qquad \text{(conducting sphere)}, \tag{6}$$

[6]Positive charge accumulates on the side of the cylinder that intercepts lines of $\mathbf{E}_0$, *i.e.*, the side with $x < 0$. The charge distribution (3) has positive charge on this side (and negative charge on the side with $x > 0$) as expected.

[7]The charge densities in eqs. (1) and (3) are those of the total charge at the interface $r = a$, which consist of both "free" and "bound" charge, where the latter is associated with the electric dipoles in the dielectric media, such as liquid argon and the insulator. That is, $\sigma = \sigma_{\rm free} + \sigma_A + \sigma_I$ for the case of an insulator in liquid argon. In the steady state, the radial field is zero in the liquid argon just outside the insulator, so $\sigma_A = 0$. The bound charge density on the surface of the insulator, with relative dielectric constant ϵ_I, is given by $\sigma_I = \mathbf{P}_I \cdot \hat{\mathbf{r}} = (\epsilon_I - 1)\epsilon_0 \mathbf{E}_I \cdot \hat{\mathbf{r}} = 2(\epsilon_I - 1)\epsilon_0 E_0 \cos\phi$, using eq. (5), such that $\sigma_{\rm free} = \sigma - \sigma_I = 2\epsilon_I\epsilon_0 E_0 \cos\phi$. This is the charge density that must accumulate on the insulator to achieve the steady-state.

[8]Since the total static field inside a conductor is zero, the interior field due to charge density σ is just $-\mathbf{E}_0$. The exterior field is given, for example, in the link of footnote 2.

[9]We have not needed the initial field strength E_I inside the circular-cylinder insulator of relative dielectric constant ϵ_I, when surrounded by a medium of relative dielectric constant ϵ_A with initially uniform electric field $\mathbf{E}_0$ in the absence of the insulator, but its value is $E_I = E_0/(\epsilon_I/\epsilon_A + 1)$. See, for example, p. 49 of [5].

and the electric field associated with this charge distribution is,[10]

$$\mathbf{E}_\sigma = E_0 \begin{cases} \left(\frac{a}{r}\right)^3 (2\hat{\mathbf{r}}\cos\theta + \hat{\boldsymbol{\theta}}\sin\theta) & (r > a), \\ -\hat{\mathbf{z}} & (r < a), \end{cases} \qquad \text{(conducting sphere)}, \tag{7}$$

In the problem of the spherical insulator we desire that the radial electric field be zero just outside the sphere. This is achieved by reversing the sign of the charge distribution (6), and dividing by 2. Hence, the desired solution is,

$$\sigma(\theta) = -\frac{3}{2}\epsilon_0 E\cos\theta \qquad \text{(insulating sphere)}, \tag{8}$$

and the electric field associated with this charge distribution is,

$$\mathbf{E}_\sigma = E_0 \begin{cases} -\left(\frac{a}{r}\right)^3 \left(\hat{\mathbf{r}}\cos\theta + \frac{\hat{\boldsymbol{\theta}}}{2}\sin\theta\right) & (r > a), \\ \frac{\hat{\mathbf{z}}}{2} & (r < a), \end{cases} \qquad \text{(insulating sphere)}. \tag{9}$$

The total electric field for the steady-state solution for the insulating cylinder is,

$$\mathbf{E} = \mathbf{E}_0 + \mathbf{E}_\sigma = E_0 \begin{cases} \left[1 - \left(\frac{a}{r}\right)^3\right]\hat{\mathbf{r}}\cos\theta - \left[1 + \frac{1}{2}\left(\frac{a}{r}\right)^3\right]\hat{\boldsymbol{\theta}}\sin\theta & (r > a), \\ \frac{3}{2}\,\hat{\mathbf{z}} & (r < a), \end{cases} \qquad \text{(insulating sphere)}. \tag{10}$$

Hence, the steady-state field inside an insulating sphere is only 1.5 times the external field $\mathbf{E}_0$ (independent of radius a so long as this is small compared to D), and electrical discharges are unlikely in this case. Qualitatively, the steady-state field inside an insulating sphere is smaller than inside an insulating cylinder because there are more ways for the perturbed field lines to pass around a sphere than a cylinder, so the desired field perturbation can be generated by a smaller accumulated charge, which also leads to a smaller field inside the insulator.

2.4 Elliptic-Cylinder Insulator

The method used to deal with circular cylinders and spheres can be extended to the cases of elliptic cylinders and ellipsoids, but for this we need to work in appropriate elliptical coordinate systems. Here, we only consider elliptic cylinders.

One of the first efforts involving elliptic cylinders in electromagnetism was [7],[11] although the axes of the elliptic cylinders were only parallel or perpendicular to the external (magnetic) field. The case of the axes of a conducting elliptical cylinder at a general angle to the external field $\mathbf{E}_0$ has been treated briefly on p. 1199 of [10], and in problem 433, p. 204 of [11]. A different approach is illustrated in sec. 4.261 of [12]. We adopt the notation of [11], and transcribe eq. (10.1.27) of [10] into this.

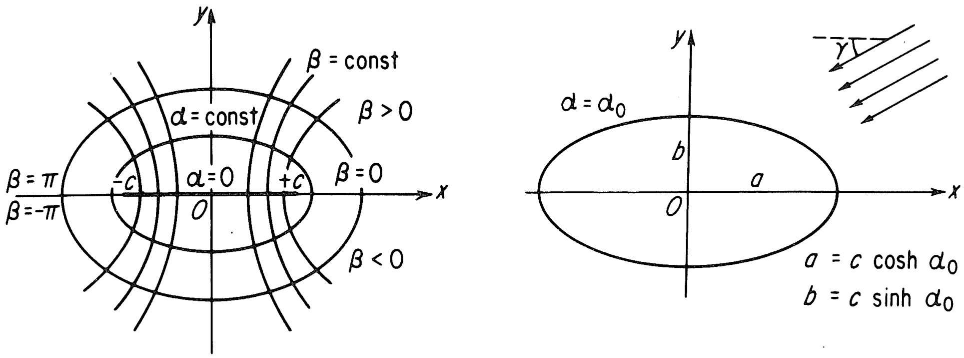
[10]The exterior field $\mathbf{E}_\sigma(r > a)$ is the same as that of a point electric dipole, as can be deduced from the potential, given, for example, in sec. 2.5 of [6]. Note that the peak electric field for a conducting sphere in an otherwise uniform external field $\mathbf{E}_0$ is $3\mathbf{E}_0$.

[11]Consideration of elliptic cylinders in fluid dynamics may have begun with [8].

The elliptic cylinder is centered on the origin, with semimajor axis a along the x-axis, and semiminor axis b along the y-axis. The focal distance is c,

$$c = \sqrt{a^2 - b^2}, \qquad a = c\cosh\alpha_0, \qquad b = c\sinh\alpha_0. \tag{11}$$

The external electric field $\mathbf{E}_0$ makes angle γ to the x-axis as shown in the figure below [11].



The elliptic-cylinder coordinates (α, β, z) are defined by,[12]

$$x + iy = c\ \cosh(\alpha + i\beta), \qquad x = c\ \cosh\alpha\cos\beta, \qquad y = c\ \sinh\alpha\sin\beta. \tag{12}$$

The hyperbolae of constant β are orthogonal to the ellipses of constant α; the surface of the physical insulator is at $\alpha = \alpha_0$. Unit vectors $\boldsymbol{\alpha}$ lie along lines of constant β, and *vice versa.*

The 2-d line element is,

$$ds^2 = c^2(\cosh^2\alpha - \cos^2\beta)(d\alpha^2 + d\beta^2), \tag{13}$$

from which one infers that the gradient of the electric scalar potential $V(\alpha, \beta)$ is,

$$\boldsymbol{\nabla} V = \frac{1}{c\sqrt{\cosh^2\alpha - \cos^2\beta}} \left(\hat{\boldsymbol{\alpha}} \frac{\partial V}{\partial\alpha} + \hat{\boldsymbol{\beta}} \frac{\partial V}{\partial\beta} \right), \tag{14}$$

and that Laplace's equation for the potential is,[13]

$$\nabla^2 V = \frac{1}{c^2\left(\cosh^2\alpha - \cos^2\beta\right)} \left(\frac{\partial V}{\partial\alpha^2} + \frac{\partial V}{\partial\beta^2} \right) = 0. \tag{15}$$

The external electric field $\mathbf{E}_0 = -E_0\,\hat{\mathbf{x}}\cos\gamma - E_0\,\hat{\mathbf{y}}\sin\gamma$ has potential,

$$V_0 = E_0(x\ \cos\gamma + y\sin\gamma) = c\,E_0(\cosh\alpha\cos\beta\cos\gamma + \sinh\alpha\sin\beta\sin\gamma), \tag{16}$$

and hence the external electric field can be written in elliptic coordinates as,

$$\begin{aligned} \mathbf{E}_0 = -\boldsymbol{\nabla} V_0 \quad = \quad & -\frac{E_0}{\sqrt{\cosh^2\alpha - \cos^2\beta}} [\hat{\boldsymbol{\alpha}}(\sinh\alpha\cos\beta\cos\gamma + \cosh\alpha\sin\beta\sin\gamma) \\ & + \hat{\boldsymbol{\beta}}(-\cosh\alpha\sin\beta\cos\gamma + \sinh\alpha\cos\beta\sin\gamma)]. \end{aligned} \tag{17}$$

[12]Note that the origin is at $(\alpha, \beta) = (0, \pi/2)$, where $\cosh^2\alpha - \cos^2\beta = 1$.

[13]General expressions for Laplace's equation in curvilinear coordinate systems were first given in English in [9].

The total potential $V = V_0 + V_\sigma$ for the conducting elliptic cylinder in a uniform field, where V_σ is the potential associated with the charges on the elliptic cylinder, can be taken as zero inside the cylinder ($\alpha < \alpha_0$), so,

$$V_\sigma(\alpha < \alpha_0) = -V_0(\alpha < \alpha_0) = -c\, E_0(\cosh\alpha\cos\beta\cos\gamma + \sinh\alpha\sin\beta\sin\gamma) \quad \text{(conductor)}. \tag{18}$$

The potential outside the cylinder ($\alpha > \alpha_0$) is given by eq. (10.1.27) of [10] for $\epsilon \to \infty$,

$$V_\sigma(\alpha > \alpha_0) = -c\, E_0\, e^{\alpha_0 - \alpha}(\cosh\alpha_0\cos\beta\cos\gamma + \sinh\alpha_0\sin\beta\sin\gamma) \qquad \text{(conductor)}, \tag{19}$$

such that the on the surface of the conducting elliptic cylinder, $V(\alpha_0) = V_0(\alpha_0) + V_\sigma(\alpha_0) = 0$. The exterior electric field components are,

$$\begin{aligned}
E_{\sigma,\alpha}(\alpha > \alpha_0) &= -\frac{1}{c\sqrt{\cosh^2\alpha - \cos^2\beta}}\frac{\partial V_\sigma}{\partial\alpha} \qquad \text{(conductor)} \\
&= -\frac{E_0\, e^{\alpha_0-\alpha}}{\sqrt{\cosh^2\alpha - \cos^2\beta}}(\cosh\alpha_0\cos\beta\cos\gamma + \sinh\alpha_0\sin\beta\sin\gamma), \qquad (20) \\
E_{\sigma,\beta}(\alpha > \alpha_0) &= -\frac{1}{c\sqrt{\cosh^2\alpha - \cos^2\beta}}\frac{\partial V_\sigma}{\partial\beta} \qquad \text{(conductor)} \\
&= \frac{E_0\, e^{\alpha_0-\alpha}}{\sqrt{\cosh^2\alpha - \cos^2\beta}}(-\cosh\alpha_0\sin\beta\cos\gamma + \sinh\alpha_0\cos\beta\sin\gamma). \qquad (21)
\end{aligned}$$

At the surface of the conducting elliptic cylinder, $\alpha = \alpha_0$, the β-components of $\mathbf{E}_0$ and $\mathbf{E}_\sigma$ cancel, so the total field has only an α-component (which is perpendicular to the surface of the conductor), as expected.[14]

For the steady-state with an insulating elliptic cylinder, we desire that the α-component of the total field be zero at its surface, such that the field is parallel to the surface (*i.e.*, has only a β-component there). For this, a suitable exterior potential is,[15]

$$V_\sigma(\alpha > \alpha_0) = c\, E_0\, e^{\alpha_0-\alpha}(\sinh\alpha_0\cos\beta\cos\gamma + \cosh\alpha_0\sin\beta\sin\gamma) \qquad \text{(insulator)}, \tag{23}$$

which obeys $\nabla^2 V_\sigma = 0$. The exterior electric field components are,

$$\begin{aligned}
E_{\sigma,\alpha}(\alpha > \alpha_0) &= \frac{E_0\, e^{\alpha_0-\alpha}}{\sqrt{\cosh^2\alpha - \cos^2\beta}}(\sinh\alpha_0\cos\beta\cos\gamma + \cosh\alpha_0\sin\beta\sin\gamma), \qquad (24) \\
E_{\sigma,\beta}(\alpha > \alpha_0) &= -\frac{E_0\, e^{\alpha_0-\alpha}}{\sqrt{\cosh^2\alpha - \cos^2\beta}}(-\sinh\alpha_0\sin\beta\cos\gamma + \cosh\alpha_0\cos\beta\sin\gamma), \qquad (25)
\end{aligned}$$

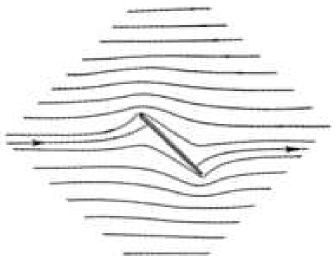
for which $E_{0,\alpha}(\alpha_0) + E_{\sigma,\alpha}(\alpha_0) = 0$ as desired.

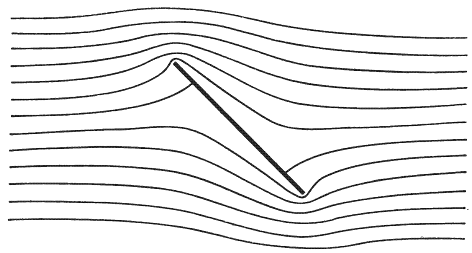
[14] The total electric field $\mathbf{E}(\alpha = \alpha_0^+) = \mathbf{E}_0(\alpha_0^+) + \mathbf{E}_\sigma(\alpha_0^+)$ just outside the surface of the conductor, and the surface-charge density σ, are, from eqs. (17) and (20)-(21), and recalling eq. (11),

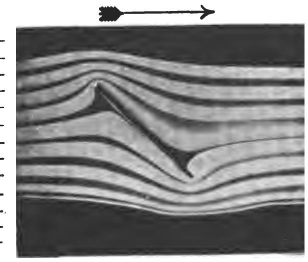
$$E_\alpha(\alpha_0^+) = \frac{E_0\, e^{\alpha_0}\cos(\beta-\gamma)}{\sqrt{\cosh^2\alpha_0 - \cos^2\beta}} = \frac{(a+b)E_0\cos(\beta-\gamma)}{\sqrt{a^2\sin^2\beta + b^2\cos^2\beta}}, \qquad E_\beta(\alpha_0^+) = 0, \qquad \sigma = \epsilon_0 E_\alpha(\alpha_0^+). \tag{22}$$

[15] The total exterior potential $V = V_0 + V_\sigma$ is given as the hydrodynamic potential ψ in eq. (10.1.28) of [10], which is consistent with eq. (23).

The total steady-state exterior field $\mathbf{E} = \mathbf{E}_0 + \mathbf{E}_\sigma$ is illustrated in the figures below (fig. 10.7 of [10], p. 94 of [13] and fig. 27 of [14]) for a nearly flat elliptic cylinder at 45° to the external field $\mathbf{E}_0$.







We also need the potential and the fields inside the insulating cylinder. Recalling the relation between the potentials (18) and (19), we infer that the extrapolation of the potential (23) inside the insulator is,

$$V_\sigma(\alpha < \alpha_0) = c\, E_0(\sinh\alpha\cos\beta\cos\gamma + \cosh\alpha\sin\beta\sin\gamma) \qquad \text{(insulator)}, \tag{26}$$

which satisfies Laplace's equation, and matches potential (23) at $\alpha = \alpha_0$. The interior electric field components associated with this potential are,

$$E_{\sigma,\alpha}(\alpha < \alpha_0) = -\frac{E_0}{\sqrt{\cosh^2\alpha - \cos^2\beta}}(\cosh\alpha\cos\beta\cos\gamma + \sinh\alpha\sin\beta\sin\gamma), \tag{27}$$

$$E_{\sigma,\beta}(\alpha < \alpha_0) = -\frac{E_0}{\sqrt{\cosh^2\alpha - \cos^2\beta}}(-\sinh\alpha\sin\beta\cos\gamma + \cosh\alpha\cos\beta\sin\gamma), \tag{28}$$

The interior field (27)-(28) is independent of parameters α_0 and c, *i.e.*, independent of the semimajor axis a and the semiminor axis b of the elliptic cylinder. Further, the total interior field follows from eqs. (17) and (27)-(28) as,

$$\mathbf{E}(\alpha < \alpha_0) = \mathbf{E}_0 + \mathbf{E}_\sigma = -\frac{E_0\, e^\alpha}{\sqrt{\cosh^2\alpha - \cos^2\beta}}[\hat{\boldsymbol{\alpha}}\cos(\beta - \gamma) - \hat{\boldsymbol{\beta}}\sin(\beta - \gamma)], \tag{29}$$

$$E(\alpha < \alpha_0) = \frac{E_0\, e^\alpha}{\sqrt{\cosh^2\alpha - \cos^2\beta}} = \frac{2E_0}{\sqrt{1 - 2\,e^{-2\alpha}\cos 2\beta + e^{-4\alpha}}} \approx 2E_0. \tag{30}$$

The field strength at the origin, $(\alpha, \beta) = (0, \pi/2)$, is just E_0,[16] but away from the origin, where $e^{-2\alpha} \ll 1$, the field strength is approximately $2E_0$.[17] Thus, the total, steady-state interior field in the case of an insulating elliptical cylinder has magnitude $\approx 2E_0$ (as previously found in sec. 2.2 for a circular cylinder), largely independent of the shape of the ellipse.

[16]While it is often said that a circle is an ellipse with zero focal distance, a circle has only one focus/center, while an ellipse has 2 foci. So, no matter how close these two foci are to one another, there remains a distinction between a nearly circular ellipse and a circle. Apparently, a consequence of this distinction is that the mathematically exact, steady-state electric field at the center of a nearly circular insulating elliptic cylinder (between its foci) is not the same as that at the center of an insulating circular cylinder. However, at distances r from its center, where $2c \lesssim r < a \approx b$ in a nearly circular elliptic-cylinder insulator, the steady-state field strength is $\approx 2E_0$, as for the steady state of circular-cylinder insulator.

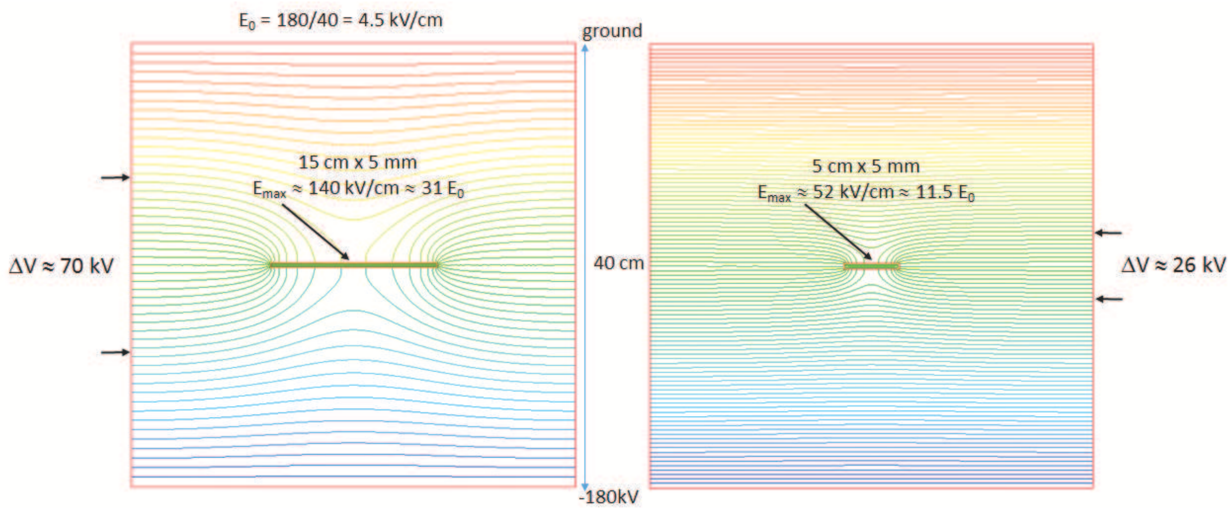
[17]The field strength at the ends of the major axis, $(\alpha, \beta) = (\alpha_0, 0 \text{ or } \pi)$, is $2E_0/(1 - e^{-2\alpha_0}) \approx 2E_0$, and the field strength at the ends of the minor axis, $(\alpha, \beta) = (\alpha_0, 0 \text{ or } 3\pi/2)$, is $2E_0/(1 + e^{-2\alpha_0}) \approx 2E_0$.

That is, if the width of the insulator is small compared to the width (transverse extent) of the external field $\mathbf{E}_0$, such that field lines can pass around the insulator as it charges up, the field inside the insulator rises to only double the value of the external field.[18] This is in significant contrast to the case of the right figure of sec. 2.1, where the insulator has the same (infinite) width as the external field, and so field lines could not "go around" the insulator and the internal field of a thin insulator would become very large compared to E_0.

2.5 Rectangular-Prism Insulator

We have seen that for elliptic-cylinder insulators, which have no sharp corners, the maximum field inside the insulator in the steady-state after charge accumulation is only about twice the external field, independent of the shape of the elliptic cylinder.

In some situations, prisms with rectangular cross section, say $2a \times 2b$ with $a > b$ are reasonably well approximated by elliptic cylinders of semimajor axis a and semiminor axis b. However, this seems not to be the case in the present problem. As shown in the figures below from a finite-element analysis by Bo Yu [15], the peak, steady-state field inside a charged-up, rectangular-prism insulator is $\approx E_0 a/b \gg 2E_0$.



The figures indicate that the charging of the sharp corners of the insulator leads to closely spaced equipotentials there, such that the total potential difference between the "top" and "bottom" sides of the insulator is large ($\Delta V \approx 70$ kV in the example on the left above, and ≈ 26 kV on the right).

The lesson is that insulators, like conductors, in strong fields should not have sharp corners. As seen in secs. 2.2-4, insulators with rounded cross sections have steady-state internal fields of only $\approx 2E_0$.

Insulators in high-voltage applications are sometimes protected from effects of charge

[18] Even if the elliptic cylinder were aligned with its major axis parallel to the external field $\mathbf{E}_0$, the effect of the charge accumulation, and of the field lines going around the cylinder, would be to raise the steady state internal field to $\approx 2\mathbf{E}_0$. This contrasts with the case shown in the middle figure on p. 1, where a planar insulator fills the entire gap between the electrodes that generate the field $\mathbf{E}_0$, such that no field lines are deformed, and the interior field remains $\mathbf{E}_0$ at all times.

accumulation due to corona discharge by so-called grading rings.[19] *These are conductors that surround a region of the insulator, creating an equipotential region in the plane of the grading ring, which reduces the electric field near this plane almost to zero.*

2.5.1 Conducting and Insulating Corners

An analytic treatment of the exterior solution for a rectangular conductor in an otherwise uniform electric field seems not to be available. However, some insight can be obtained by consideration of case of two intersecting, conducting planes, as in sec. 2.11 of [6], or p. 254 of [16].

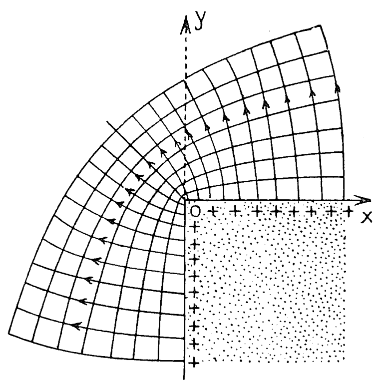
We are interested in the case of planes that intersect at 90°. Taking the line of intersection to be the z-axis, and the conducting planes to line along the positive x-axis and the negative y-axis, the opening angle for the exterior field is $\beta = 3\pi/2$ in the notation of [6]. The potential near the corner is approximately,[20]

$$V(r, \phi, z) \approx A + Br^{2/3} \sin 2\phi/3 = Re(A - iBr^{2/3}\, e^{2i\phi/3}) = Re(A - iBz^{2/3}), \tag{31}$$

where A and B are constants, and $z = x + iy = r\, e^{i\phi}$. Hence, the conjugate potential,

$$V(r, \phi, z) = Im(A - iBr^{2/3}\, e^{2i\phi/3}) = A - Br^{2/3} \cos 2\phi/3, \tag{32}$$

is the (approximate) potential for another electrostatics problem.



The electric field lines for this new problem are identical with the equipotentials (and *vice versa*) for the original problem of a conducting corner. From the figure above (p. 254 of [16]), we see that the potential (32) corresponds to an electric field that is parallel to the surface of the corner, as desired for the steady-state of a charged, rectangular insulator.

For a rectangular insulator of dimensions $2a \times 2b$, the greatest electric field will be at its center $(x, y) = (a, -b)$ in the coordinate system used above. If we define the potential to be zero on the midplane $y = -b$ of the insulator, the potential will be symmetric in y about this plane, and the peak field will be approximately $E_{\rm max} \approx V(a, 0)/b$.

For the potential to be zero at $(x, y) = (0, -b)$; $(r, \phi) = (b, 3\pi/2)$ requires that $A = -B\, b^{2/3}$, such that $E_{\rm max} = -B(a^{2/3} + b^{2/3})/b$. By dimensional analysis, $B = -E_0 d^{1/3}$ for

[19] See, for example, https://en.wikipedia.org/wiki/Corona_ring#Grading_rings

[20] See eq. (2.73) of [6].

some relevant distance $d(a, b)$, where E_0 is the field in the absence of the insulator. If we take $d = a$ (which is only an "educated guess"), we obtain,

$$E_{\rm max} \approx E_0 \left[\frac{a}{b} + \left(\frac{a}{b} \right)^{1/3} \right], \tag{33}$$

in reasonable agreement with the numerical analysis [15] reported in the figure on p. 9 above.

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