Total and Frustrated Reflection of a Gaussian Optical Beam

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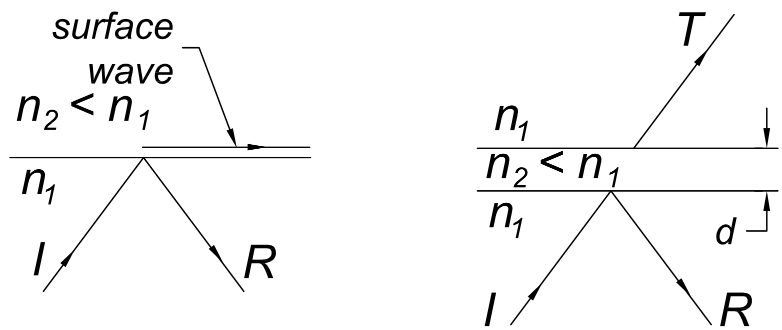
1 Problem

When an electromagnetic wave in a medium with index of refraction n_1 encounters an interface with a region of index $n_2 < n_1$ the wave can be totally reflected, with only an evanescent (surface) wave excited in the region of lower index. In this case, energy is (largely) transported parallel to the interface in the region of lower index.[1]

Discuss the flow of energy when the incident wave has limited transverse extent. In particular, consider a weakly focused, linearly polarized Gaussian beam that is incident from the medium with lower index.



Also discuss the flow of energy when a Gaussian beam is incident from a medium of index n_1 onto a region of index $n_2 < n_1$ of thickness d beyond which the medium has index n_1.

2 Solution

We will use the time-average Poynting vector, $\langle \mathbf{S} \rangle = Re(\mathbf{E} \times \mathbf{B}^\star)/2\mu$ (in SI units), to discuss the flow of energy in waves with electric field $\mathbf{E}$ and magnetic field $\mathbf{B}$.

2.1 Weakly Focused, Linearly Polarized Gaussian Optical Beams

We use so-called Gaussian beams to describe approximate wave solutions to Maxwell's equations that have limited transverse extent. However, even if the incident beam is cylindrically symmetric about its axis of propagation, the refracted beam will in general have an elliptical cross section. Hence, in Appendix A we make a small generalization of typical presentations of Gaussian beams (see, for example, sec. 2.4 of [1]) to consider elliptical beams. The first-order Gaussian beam of angular frequency ω that propagates along the z-axis with

[1]See, for example, prob. 11 of http://kirkmcd.princeton.edu/examples/ph501set6.pdf

y-polarization in a medium with permittivity ϵ and permeability μ has fields,

$$\begin{aligned} E_x &\approx 0, \\ E_y &\approx \frac{E_0\, e^{-\rho^2/(1+z^2/z_0^2)}}{\sqrt{1+z^2/z_0^2}} e^{i\{kz[1+z_0\rho^2/k(z^2+z_0^2)]-\omega t-\tan^{-1}(z/z_0)\}}, \\ E_z &\approx -\frac{2iy}{kw_{0y}^2} E_y\, \frac{e^{-i\tan^{-1} z/z_0}}{\sqrt{1+z^2/z_0^2}}, \end{aligned} \tag{1}$$

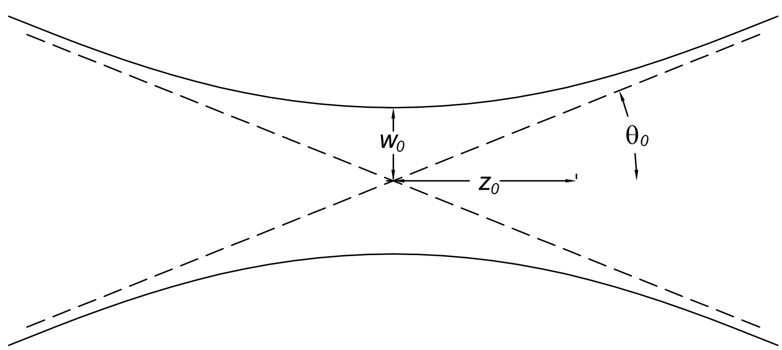
$$B_x = -\frac{n}{c} E_y, \qquad B_y = 0, \qquad B_z = \frac{2ix}{kw_{0x}^2} \frac{n}{c} E_y\, \frac{e^{-i\tan^{-1} z/z_0}}{\sqrt{1+z^2/z_0^2}}, \tag{2}$$

where c is the speed of light in vacuum,

$$k_0 = \frac{\omega}{c}, \qquad k = nk_0, \qquad n = c\sqrt{\epsilon\mu}, \qquad \rho = \sqrt{\frac{x^2}{w_{0x}^2} + \frac{y^2}{w_{0y}^2}}, \tag{3}$$

w_{0j} for $j = x$ or y is the characteristic radius of the beam at its waist (focus), θ_{0j} is the diffraction angle and z_0 is the Rayleigh range, as shown in the figure below, which are related by,

$$\theta_{0j} = \frac{w_{0j}}{z_0}, \qquad \frac{1}{z_0} = \frac{1}{kw_{0x}^2} + \frac{1}{kw_{0y}^2} \qquad \left(z_0 = \frac{kw_0^2}{2} = \frac{2}{k\theta_0^2} \quad \text{if} \quad w_{0x} = w_{0y} = w_0 \right). \tag{4}$$



Near the focus ($\rho \lesssim 1, |z| < z_0$), the beam (1)-(2) can be approximated as the plane wave,[2]

$$E_x = 0, \qquad E_y = E_0\, e^{-\rho^2}\, e^{i(kz-\omega t)}, \qquad E_z = -\frac{2iy}{kw_{0y}^2} E_y, \tag{5}$$

$$B_x = -\frac{n}{c} E_y, \qquad B_y = 0 \qquad B_z = \frac{2ix}{kw_{0x}^2} \frac{n}{c} E_y, \tag{6}$$

which obeys $\boldsymbol{\nabla} \cdot \mathbf{E} = 0 = \boldsymbol{\nabla} \cdot \mathbf{B}$ recalling eq. (4). The equations $\boldsymbol{\nabla} \times \mathbf{E} = -\partial \mathbf{B}/\partial(ct)$ and $\boldsymbol{\nabla} \times \mathbf{H} = \partial \mathbf{D}/\partial(ct)$ are satisfied up to terms of order $\rho^2 w_{0j}/z_0$. We are interested in transverse distances $\rho \approx 1$, so the approximation (5)-(6) is a good solution to Maxwell's

[2]The forms (5)-(6) could also be deduced quickly by first assuming E_y and B_x to be a plane wave with a Gaussian transverse modulation, and then enforcing conditions $\boldsymbol{\nabla} \cdot \mathbf{E} = 0 = \boldsymbol{\nabla} \cdot \mathbf{B}$ to determine E_z and B_z.

equations provided $w_{0j} \ll z_0$, *i.e.*, $\theta_{0j} \ll 1$. This is the case in the present problem, where we wish to explore the behavior of very weakly focused optical beams.

The flow of energy in this elliptical beam is described by the (real) Poynting vector,

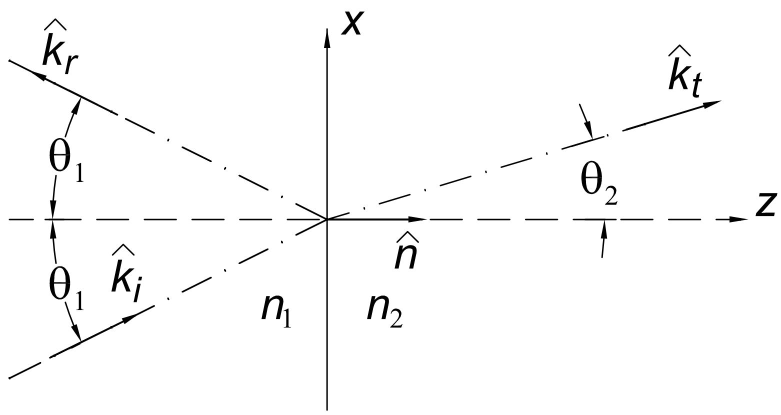
$$\mathbf{S} = \frac{Re\mathbf{E} \times Re\mathbf{B}}{\mu} = \sqrt{\frac{\epsilon}{\mu}} E_0^2 \, e^{-2\rho^2} \left(-\frac{2x}{kw_{0x}^2} \sin[2(kz - \omega t)], -\frac{2y}{kw_{0y}^2} \sin[2(kz - \omega t)], \cos^2(kz - \omega t) \right). \tag{7}$$

The time-average flow of energy is, of course, only in the direction of propagation of the wave. In addition, there is an oscillatory transverse flow of energy.[3]

2.2 Reflection and Refraction at a Single Interface

The classic formalism for plane waves is reviewed in Appendix B.

We first consider the refraction of a weakly focused Gaussian beam at a single interface between linear, isotropic media of indices $n_1 = c\sqrt{\epsilon_1 \mu_1}$ and $n_2 = c\sqrt{\epsilon_2 \mu_2}$, using notation as shown in the figure below. For simplicity, we restrict our attention to the case that the electric field is perpendicular to the plane of incidence (*i.e.*, to the x-z plane).



The incident, reflected and transmitted beams each have the Gaussian form (5)-(6), with respect to axes (x_i, y_i, z_i), (x_r, y_r, z_r) and (x_t, y_t, z_t), where the z-axes are in the directions of propagation of the various beams.

The transformation between the axes (x_i, y_i, z_i) of the incident beam and the laboratory axes (x, y, z) is,

$$x_i = \cos\theta_1 \, x - \sin\theta_1 \, z, \qquad y_i = y, \qquad z_i = \sin\theta_1 \, x + \cos\theta_1 \, z, \tag{8}$$

and,

$$\rho_i^2 = \frac{x_i^2}{w_{ix}^2} + \frac{y_i^2}{w_{iy}^2} = \frac{\cos^2\theta_1 \, x^2 - \sin 2\theta_1 \, xz + \sin^2\theta_1 \, z^2}{w_{ix}^2} + \frac{y^2}{w_{iy}^2}, \tag{9}$$

where θ_1 is the angle between the axis of the incident beam and the z-axis, and we assume that axis of the incident beam passes through the origin. The components of a vector $\mathbf{A}$ with respect to the laboratory frame are related to those with respect to axes (x_i, y_i, z_i) by,

$$A_x = \cos\theta_1 \, A_{x_i} + \sin\theta_1 \, A_{z_i}, \qquad A_y = A_{y_i}, \qquad A_z = -\sin\theta_1 \, A_{x_i} + \cos\theta_1 \, A_{z_i}. \tag{10}$$

[3]Near its waist, a Gaussian beam is similar to a wave inside a conducting wave guide, which latter case also exhibits steady longitudinal, and oscillatory transverse, flow of energy [2].

Combining eqs. (5)-(10), the components of the incident beam in the laboratory frame are,

$$E_{ix} = -i\frac{\sin\theta_1\, y}{z_0} E_{iy}, \tag{11}$$

$$E_{iy} = E_{0i}\, e^{-\rho_i^2} e^{i(n_1 \sin\theta_1\, k_0 x + n_1 \cos\theta_1\, k_0 z - \omega t)}, \tag{12}$$

$$E_{iz} = -i\frac{\cos\theta_1\, y}{z_0} E_{iy}, \tag{13}$$

$$B_{ix} = \frac{n_1}{c}\left[\cos\theta_1 - i\sin\theta_1 \frac{\cos\theta_1\, x - \sin\theta_1\, z}{z_0}\right] E_{iy}, \tag{14}$$

$$B_{iy} = 0 \tag{15}$$

$$B_{iz} = \frac{n_1}{c}\left[\sin\theta_1 + i\cos\theta_1 \frac{\cos\theta_1\, x - \sin\theta_1\, z}{z_0}\right] E_{iy}. \tag{16}$$

Similarly, the reflected beam is related by,

$$x_r = -\cos\theta_1\, x - \sin\theta_1\, z, \qquad y_r = y, \qquad z_r = -\sin\theta_1\, x - \cos\theta_1\, z, \tag{17}$$

$$\rho_r^2 = \frac{x_r^2}{w_{rx}^2} + \frac{y_r^2}{w_{rx}^2} = \frac{\cos^2\theta_1\, x^2 + \sin 2\theta_1\, xz + \sin^2\theta_1\, z^2}{w_{rx}^2} + \frac{y^2}{w_{rx}^2}\,,\,, \tag{18}$$

$$A_x = -\cos\theta_1\, A_{x_i} + \sin\theta_1\, A_{z_i}, \qquad A_y = A_{y_i}, \qquad A_z = -\sin\theta_1\, A_{x_i} - \cos\theta_1\, A_{z_i}, \tag{19}$$

$$E_{rx} = -i\frac{\sin\theta_1\, y}{z_0} E_{ry}, \tag{20}$$

$$E_{ry} = E_{0r}\, e^{-\rho_r^2} e^{i(n_1 \sin\theta_1\, k_0 x - n_1 \cos\theta_1\, k_0 z - \omega t)}, \tag{21}$$

$$E_{rz} = i\frac{\cos\theta_1\, y}{z_0} E_{iy}, \tag{22}$$

$$B_{rx} = \frac{n_1}{c}\left[-\cos\theta_1 + i\sin\theta_1 \frac{\cos\theta_1\, x + \sin\theta_1\, z}{z_0}\right] E_{ry}, \tag{23}$$

$$B_{ry} = 0 \tag{24}$$

$$B_{rz} = \frac{n_1}{c}\left[\sin\theta_1 + i\cos\theta_1 \frac{\cos\theta_1\, x + \sin\theta_1\, z}{z_0}\right] E_{ry}, \tag{25}$$

where we have assumed that the reflected beam also makes angle θ_1 with respect to the z-axis, and that the axis of the reflected beam passes through the origin.

Likewise, the transmitted beam is related by,

$$x_t = \cos\theta_2\, x - \sin\theta_2\, z, \qquad y_t = y, \qquad z_t = \sin\theta_2\, x + \cos\theta_2\, z, \tag{26}$$

$$\rho_t^2 = \frac{x_t^2}{w_{tx}^2} + \frac{y_t^2}{w_{ty}^2} = \frac{\cos^2\theta_2\, x^2 - \sin 2\theta_2\, xz + \sin^2\theta_2\, z^2}{w_{tx}^2} + \frac{y^2}{w_{ty}^2}\,, \tag{27}$$

$$A_x = \cos\theta_2\, A_{x_t} + \sin\theta_2\, A_{z_t}, \qquad A_y = A_{y_t}, \qquad A_z = -\sin\theta_2\, A_{x_t} + \cos\theta_2\, A_{z_t}, \tag{28}$$

$$E_{tx} = -i\frac{\sin\theta_2\, y}{z_{0t}} E_{ty}, \tag{29}$$

$$E_{ty} = E_{0t}\, e^{-\rho_t^2} e^{i(n_2 \sin\theta_2\, k_0 x + n_2 \cos\theta_2\, k_0 z - \omega t)}, \tag{30}$$

$$E_{tz} = -i \frac{\cos\theta_2\, y}{z_{0t}} E_{ty}, \tag{31}$$

$$B_{tx} = \frac{n_2}{c} \left[\cos\theta_2 - i \sin\theta_2 \frac{\cos\theta_2\, x - \sin\theta_2\, z}{z_{0t}} \right] E_{ty}, \tag{32}$$

$$B_{ty} = 0 \tag{33}$$

$$B_{tz} = \frac{n_2}{c} \left[\sin\theta_2 + i \cos\theta_2 \frac{\cos\theta_2\, x - \sin\theta_2\, z}{z_{0t}} \right] E_{ty}. \tag{34}$$

The boundary conditions at the interface $z = 0$ are that $E_\perp$, $D_z = \epsilon E_z$, B_z and $H_\perp = B_\perp/\mu$ are continuous. Thus, continuity of E_y,

$$E_{0i}\, e^{-\rho_i^2} e^{i(n_1 \sin\theta_1\, k_0 x - \omega t)} + E_{0r}\, e^{-\rho_r^2} e^{i(n_1 \sin\theta_1\, k_0 x - \omega t)} = E_{0t}\, e^{-\rho_t^2} e^{i(n_2 \sin\theta_2\, k_0 x - \omega t)}, \tag{35}$$

for all x and y, which confirms the assumption that $\theta_r = \theta_i = \theta_i$, verifies Snell's law,

$$n_1 \sin\theta_1 = n_2 \sin\theta_2, \tag{36}$$

tells us that beam waists are related by,

$$w_{ix} = w_{rx} = \frac{\cos\theta_1}{\cos\theta_2} w_{tx}, \qquad w_{iy} = w_{ry} = w_{ty}, \tag{37}$$

and that the wave amplitudes are related by,

$$E_{0i} + E_{0r} = E_{0t}. \tag{38}$$

Similarly, the continuity of $H_x = B_x/\mu$ tells us that,

$$\frac{n_1 \cos\theta_1}{\mu_1} (E_{0i} - E_{0r}) = \frac{n_2 \cos\theta_2}{\mu_2} E_{0t}. \tag{39}$$

Combining eqs. (38) and (39) we obtain the usual Fresnel relations,[4] for polarization perpendicular to the plane of incidence,

$$\frac{E_{0r}}{E_{0i}} = \frac{n_1 \cos\theta_1 - \frac{\mu_1}{\mu_2} n_2 \cos\theta_2}{n_1 \cos\theta_1 + \frac{\mu_1}{\mu_2} n_2 \cos\theta_2}, \qquad \frac{E_{0t}}{E_{0i}} = \frac{2 n_1 \cos\theta_1}{n_1 \cos\theta_1 + \frac{\mu_1}{\mu_2} n_2 \cos\theta_2}. \tag{40}$$

Furthermore, the continuity of E_x (or $D_z = \epsilon E_z$) tells us that the Rayleigh ranges are related by,

$$z_{0i} = z_{0r} = \frac{\sin\theta_1}{\sin\theta_2} z_{0t} = \frac{n_2}{n_1} z_{0t}. \tag{41}$$

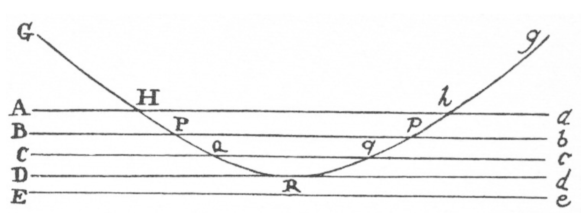
Equation (41) is not quite consistent with eqs. (4) and (37).

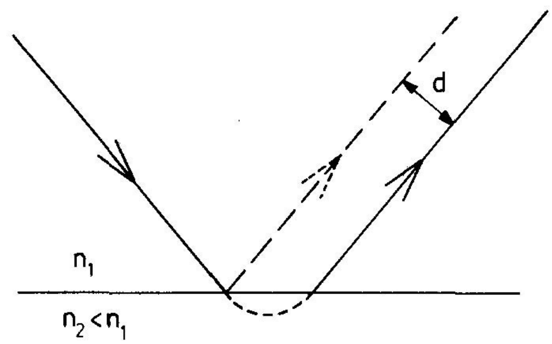
In the remainder of the body of this note we suppose that $\mu_1 = \mu_2 = \mu_0$, where the latter is the permeability of the vacuum. We also suppose that the incident Gaussian beam is circularly symmetric, and write the incident waist as $w_{ix} = w_{iy} = w_1$. It follows that the reflected Gaussian beam is also circularly symmetric with the same waist w_1. The incident and reflected beams have Rayleigh range $z_{0i} = k_1 w_1^2/2 = n_1 k_0 w_1^2/2$, while that of the transmitted beam is $z_{0t} = n_1 z_{0i}/n_2$.

[4]See Appendix B

2.3 Total Reflection at a Single Interface

The phenomenon of total reflection at an interface between optically denser and lighter media was discussed by Newton in Proposition 96 of his *Principia* [3], and in further detail in his *Opticks* [4]. Newton's model was that particles of light are attracted by the denser medium, and so follow curved rays that result in an offset between the incident and reflected path, as shown in the figure on the left below (from the *Principia*).





Newton's prediction was largely forgotten after the success of the wave theory of light in the early 1800's, but was occasionally discussed in the first half of the 20$^{\text{th}}$ century [5, 6, 7, 8]. Then, in 1947 Goos and Hänchen [9] provided experimental evidence for what is now called the Goos-Hänchen shift, d, illustrated in the figure on the above right.

Total reflection occurs when the indices of the two media obey $n_1 > n_2$ and the angle of incidence, θ_1, is large enough that $\sin\theta_2 = (n_1/n_2)\sin\theta_1 > 1$. In this case,

$$\cos\theta_2 = \sqrt{1 - \sin^2\theta_2} = i\frac{\sqrt{n_1^2\sin^2\theta_1 - n_2^2}}{n_2}, \tag{42}$$

is purely imaginary. As a consequence, the amplitudes (40) of the reflected and transmitted waves include a phase shift relative to that of the incident wave, while,

$$|E_{0r}| = |E_{0i}|, \qquad \text{and} \qquad \frac{|E_{0t}|^2}{|E_{0i}|^2} = \frac{4n_1^2\cos^2\theta_1}{n_1^2 - n_2^2}. \tag{43}$$

The quantity ρ_t^2 of eq. (27) can now be written as,

$$\rho_t^2 = \frac{(n_2^2 - n_1^2\sin^2\theta_1)\,x^2 - 2in_1\sin\theta_1\sqrt{n_1^2\sin^2\theta_1 - n_2^2}\,xz + n_1^2\sin^2\theta_1\,z^2}{n_2^2 w_{tx}^2} + \frac{y^2}{w_{ty}^2}. \tag{44}$$

Hence, matching of this form to that of ρ_i^2 at $z = 0$ requires that,

$$\rho_t^2 = \frac{\cos^2\theta_1 x^2}{w_1^2} + \frac{y^2}{w_1^2} - \frac{n_1^2\sin^2\theta_1\cos^2\theta_1\,z^2}{(n_1^2\sin^2\theta_1 - n_2^2)w_1^2} + 2i\frac{n_1\sin\theta_1\cos^2\theta_1\,xz}{\sqrt{n_1^2\sin^2\theta_1 - n_2^2}\,w_1^2}, \tag{45}$$

recalling eq. (9). The factor $e^{-Re(\rho_t^2)}$ is constant on surfaces that are elliptical hyperboloids of 1 sheet about the z-axis, with asymptotic opening angle $\theta_{t_{xz}}$ in the x-z plane given by,

$$\tan\theta_{t_{xz}} = \frac{n_1\sin\theta_1}{\sqrt{n_1^2\sin^2\theta_1 - n_2^2}} > 1. \tag{46}$$

This potentially dramatic behavior is, however, masked by the damping of the transmitted wave in z, as follows from the factor,

$$e^{i(n_2 \sin\theta_2\, k_0 x + n_2 \cos\theta_2\, k_0 z - \omega t)} = e^{-\sqrt{n_1^2 \sin^2\theta_1 - n_2^2} k_0 z} e^{i(n_1 \sin\theta_1\, k_0 x - \omega t)}. \qquad (47)$$

The time-average flow of energy in medium 2 is given by,

$$\begin{aligned}
\langle \mathbf{S} \rangle &= \frac{Re(\mathbf{E}_t \times \mathbf{B}_t^\star)}{2\mu_0} = \frac{1}{2\mu_0} Re\left[E_{ty} B_{tz}^\star\, \hat{\mathbf{x}} + (E_{tz} B_{tx}^\star - E_{tx} B_{tz}^\star)\, \hat{\mathbf{y}} - E_{ty} B_{tx}^\star\, \hat{\mathbf{z}}\right] \\
&= \frac{n_2 \left|E_{0t}\right|^2}{2\mu_0 c} e^{-2Re(\rho_t^2)}\, e^{-2\sqrt{n_1^2 \sin^2\theta_1 - n_2^2} k_0 z} \left[\sin\theta_2 \left(1 - \frac{\sqrt{n_1^2 \sin^2\theta_1 - n_2^2}\, z}{z_{0t}}\right) \hat{\mathbf{x}} \right. \\
&\qquad \left. - \frac{\sin\theta_2 \sqrt{n_1^2 \sin^2\theta_1 - n_2^2}\, x}{z_{0t}}\, \hat{\mathbf{z}}.\right] \\
&\approx \frac{\left|E_{0t}\right|^2}{2\mu_0 c} e^{-2Re(\rho_t^2)}\, e^{-2\sqrt{n_1^2 \sin^2\theta_1 - n_2^2} k_0 z} \sin\theta_1 \left(n_1 \hat{\mathbf{x}} - \frac{n_2 x}{z_{0i}} \sqrt{n_1^2 \sin^2\theta_1 - n_2^2}\, \hat{\mathbf{z}}\right), \qquad (48)
\end{aligned}$$

noting that $z_{0t} = n_1 z_{0i}/n_2$.

Energy flows in the $+x$ direction in medium 2, but this flow is significant only near the origin because of exponential damping factors. Furthermore, energy flows in the $+z$ direction for $x > 0$ and in the $-z$ direction for $x > 0$. That is, energy flows in curves in medium 2, as qualitatively predicted by Newton. Lines of $\langle \mathbf{S} \rangle$ that enter medium 2 from medium 1 at $-x$ return to medium 1 at $+x$.

In particular, the flow of energy between media 1 and 2 across the plane $z = 0$ is described by,

$$\langle S_z \rangle_{2\to 1} = -\frac{\left|E_{0i}\right|^2}{2\mu_0 c} \frac{4 n_1^2 n_2 \cos^2\theta_1}{n_1^2 - n_2^2} \frac{\sin\theta_1 \sqrt{n_1^2 \sin^2\theta_1 - n_2^2}\, x}{z_{0i}} e^{-2(\cos^2\theta_1\, x^2 + y^2)/w_1^2}, \qquad (49)$$

recalling eq. (43). This energy adds to that in the nominal reflected wave at $z = 0$ for $x > 0$,

$$\langle S_z \rangle_r = -\frac{1}{2\mu_0} Re\left(E_{ry}(z=0) B_{rx}^\star(z=0)\right) = -\frac{\left|E_{0i}\right|^2}{2\mu_0 c} n_1 \cos\theta_1\, e^{-2(\cos^2\theta_1\, x^2 + y^2)/w_1^2}, \qquad (50)$$

and subtracts from that for $x < 0$. We define a center of energy $\bar{x}$ for the reflected wave at $z = 0$ according to,

$$\begin{aligned}
\bar{x} &= \frac{\int_{-\infty}^\infty x \left\langle S_z \right\rangle_r\, dx - \int_{-\infty}^0 x \left\langle S_z \right\rangle_{2\to 1}\, dx + \int_0^\infty x \left\langle S_z \right\rangle_{2\to 1}\, dx}{\int_{-\infty}^\infty \left\langle S_z \right\rangle_r\, dx} = \frac{2\int_0^\infty x \left\langle S_z \right\rangle_{2\to 1}\, dx}{\int_{-\infty}^\infty \left\langle S_z \right\rangle_r\, dx} \\
&= \frac{\frac{8 n_1^2 n_2 \cos^2\theta_1}{n_1^2 - n_2^2} \frac{\sin\theta_1 \sqrt{n_1^2 \sin^2\theta_1 - n_2^2}}{z_{0i}} \frac{\sqrt{\pi} w_1^3}{8\sqrt{2}\cos^3\theta_1}}{n_1 \cos\theta_1 \frac{\sqrt{\pi} w_1}{2\sqrt{2}\cos\theta_1}} = \frac{\lambda_0}{\pi} \frac{n_2}{n_1^2 - n_2^2} \tan\theta_1 \sqrt{n_1^2 \sin^2\theta_1 - n_2^2}, \qquad (51)
\end{aligned}$$

recalling that $w_1^2/2z_{0i} = 1/k_1 = \lambda_0/2\pi n_1$. Here, λ_0 is the wavelength in vacuum at angular frequency ω.

Supposing that the reflected beam is centered on $\bar{x}$ rather than $x = 0$ in the plane $z = 0$, it propagates away from this plane at angle θ_1 with its axis shifted transversely by d with respect to "mirror" reflection, as sketched in the figure on the previous page, where,

$$d = \bar{x}\cos\theta_1 = \frac{\lambda_0}{\pi}\frac{n_2}{n_1^2 - n_2^2}\sin\theta_1\sqrt{n_1^2\sin^2\theta_1 - n_2^2}. \tag{52}$$

The result (52) is not *that quoted in the literature [10]-[23],*

$$d = \frac{\lambda_0}{\pi}\frac{n_1^2}{n_1^2 - n_2^2}\frac{\sin\theta_1\cos^2\theta_1}{\sqrt{n_1^2\sin^2\theta_1 - n_2^2}} \qquad \textit{or} \qquad \frac{\lambda_0}{n_1\pi}\frac{\sin\theta_1}{\sqrt{n_1^2\sin^2\theta_1 - n_2^2}}\,. \tag{53}$$

where different sets of approximations are used. While most experiments have been performed at a single angle θ_1, a recent effort [24] that measured a large range of angles supports eq. (53).

The result (53) is often justified by claiming that the incident wave penetrates into medium 2 at angle θ_1 for a distance $D \approx \lambda_0/\left(2\pi\sqrt{n_1^2\sin^2\theta_1 - n_2^2}\right)$ according to eq. (47), at which depth it is reflected back into medium 1 [18]. If so, then it follows that $d = 2D\sin\theta_1 = \lambda_0\sin\theta_1/\left(\pi\sqrt{n_1^2\sin^2\theta_1 - n_2^2}\right)$. This result diverges at the critical angle, $\sin\theta_1 = n_2/n_1$. However, our argument (52) indicates that the shift of energy from $x < 0$ to $x > 0$ vanishes at the critical angle......

Appendices

A Gaussian Beams with Elliptical Cross Section

Many discussions of Gaussian beams emphasize a single electric field component, such as $E_y = f(r,z)\,e^{i(kz-\omega t)}$, of a cylindrically symmetric beam of angular frequency ω and wave number $k = n\omega/c$ propagating along the z axis in a medium with index of refraction n. Here, we generalize to the case of a beam with an elliptical cross section. Of course, the electric field must satisfy the free-space Maxwell equation $\boldsymbol{\nabla}\cdot\mathbf{E} = 0$. If $f(r,z)$ is not constant and $E_x = 0$, then we must have nonzero E_z. That is, the desired electric field has more than one vector component.

To deduce all components of the electric and magnetic fields of a Gaussian beam from a single scalar wave function, we follow the suggestion of Davis [27] and seek solutions for a vector potential $\mathbf{A}$ that has only a single Cartesian component (such that $(\nabla^2\mathbf{A})_j = \nabla^2 A_j$ [28]). We work in the Lorenz gauge (and SI units), so that the electric scalar potential Φ is related to the vector potential $\mathbf{A}$ by,

$$\boldsymbol{\nabla}\cdot\mathbf{A} = -\frac{n^2}{c^2}\frac{\partial\Phi}{\partial t} = i\frac{n^2\omega}{c^2}\Phi = i\frac{k^2}{\omega}\Phi. \tag{54}$$

The vector potential can therefore have a nonzero divergence, which permits solutions having only a single component.

Of course, the electric and magnetic fields can be deduced from the potentials via,

$$\mathbf{E} = -\boldsymbol{\nabla}\Phi - \frac{\partial \mathbf{A}}{\partial t} = i\frac{\omega}{k^2}\boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\mathbf{A}) + i\omega\mathbf{A}, \tag{55}$$

using the Lorenz condition (54), and,

$$\mathbf{B} = \boldsymbol{\nabla}\times\mathbf{A}. \tag{56}$$

The vector potential satisfies the free-space (Helmholtz) wave equation,

$$\nabla^2\mathbf{A} - \frac{n^2}{c^2}\frac{\partial^2\mathbf{A}}{\partial t^2} = (\nabla^2 + k^2)\mathbf{A} = 0. \tag{57}$$

We seek a solution in which the vector potential is described by a single Cartesian component A_j that propagates in the $+z$ direction with the form,

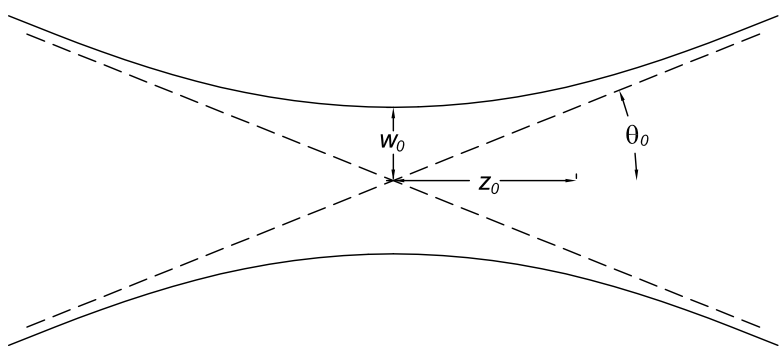
$$A_j(\mathbf{r}) = \psi(\mathbf{r})\, e^{i(kz-\omega t)}. \tag{58}$$

Inserting trial solution (58) into the wave equation (57) we find that,

$$\nabla^2\psi + 2ik\frac{\partial\psi}{\partial z} = 0. \tag{59}$$

In the usual analysis, one now assumes that the beam is cylindrically symmetric about the z axis and can be described in terms of three geometric parameters the diffraction angle θ_0, the waist w_0, and the depth of focus (Rayleigh range) z_0, which are related by,

$$\theta_0 = \frac{w_0}{z_0} = \frac{2}{kw_0}, \qquad \text{and} \qquad z_0 = \frac{kw_0^2}{2} = \frac{2}{k\theta_0^2}. \tag{60}$$



Here, we consider the possibility that the beam has an elliptical cross section, with major and minor axes along the x and y axes. The waist and diffraction angle are different in the x-z and y-z planes, but the Rayleigh range is in common:

$$\theta_{0x} = \frac{w_{0x}}{z_0}, \qquad \text{and} \qquad \theta_{0y} = \frac{w_{0y}}{z_0}. \tag{61}$$

We now convert to the scaled coordinates,

$$\xi = \frac{x}{w_{0x}}, \qquad \upsilon = \frac{y}{w_{0y}}, \qquad \rho^2 = \xi^2 + \upsilon^2, \qquad \text{and} \qquad \varsigma = \frac{z}{z_0}. \tag{62}$$

Changing variables and noting relations (61), the wave equation (59) takes the form,

$$\frac{1}{w_{0x}^2}\frac{\partial^2\psi}{\partial\xi^2} + \frac{1}{w_{0y}^2}\frac{\partial^2\psi}{\partial\upsilon^2} + \frac{1}{z_0^2}\frac{\partial^2\psi}{\partial\varsigma^2} + \frac{2ik}{z_0}\frac{\partial\psi}{\partial\varsigma} = 0. \tag{63}$$

The paraxial approximation is that the term in the relatively small quantity $1/z_0^2$ is neglected, and the resulting paraxial wave equation is,

$$\frac{1}{w_{0x}^2}\frac{\partial^2\psi}{\partial\xi^2} + \frac{1}{w_{0y}^2}\frac{\partial^2\psi}{\partial\upsilon^2} + \frac{2ik}{z_0}\frac{\partial\psi}{\partial\varsigma} \approx 0. \tag{64}$$

An "educated guess" is that the transverse behavior of the wave function ψ has a Gaussian form, but with a width that varies with z. Also, the amplitude of the wave far from its waist should vary as $1/z$. In the scaled coordinates ρ and ς a trial solution is,

$$\psi = h(\varsigma)\, e^{-f(\varsigma)\rho^2}, \tag{65}$$

where the possibly complex functions f and h are defined to obey $f(0) = 1 = h(0)$. Since the transverse coordinate ξ and υ are scaled by the waists w_{0x} and w_{0y}, we see that $Re(f) = w_{0j}^2/w_j^2(\varsigma)$ where $w_j(\varsigma)$ is the beam width in coordinate $j = x$ or y at position ς. From the geometric parameters (62) we see $w_j(\varsigma) \approx \theta_{0j} z = w_{0j}\varsigma$ for large ς. Hence, we expect that $Re(f) \approx 1/\varsigma^2$ for large ς. Also, we expect the amplitude h to obey $|h| \approx 1/\varsigma$ for large ς.

Plugging the trial solution (65) into the paraxial wave equation (64) we find that,

$$-fh\left(\frac{1}{w_{0x}^2} + \frac{1}{w_{0y}^2}\right) + 2f^2h\left(\frac{\xi^2}{w_{0x}^2} + \frac{\upsilon^2}{w_{0y}^2}\right) + \frac{ik}{z_0}(h' - f'h\rho^2) \approx 0, \tag{66}$$

where a $'$ indicates differentiation with respect to ς. We can define the Rayleigh range z_0 and the waists w_{0x} and w_{0y} to be related by,

$$\left(\frac{1}{w_{0x}^2} + \frac{1}{w_{0y}^2}\right) = \frac{k}{z_0}, \tag{67}$$

so eq. (66) can be rewritten as,

$$-fh + ih' + \rho^2 h\left[\frac{2z_0}{k\rho^2}\left(\frac{\xi^2}{w_{0x}^2} + \frac{\upsilon^2}{w_{0y}^2}\right) f^2 - if'\right] \approx 0. \tag{68}$$

If $w_{0x} = w_{0y}$, eq. (68) reduces to the form,

$$-fh + ih' + \rho^2 h(f^2 - if') \approx 0, \tag{69}$$

recalling eq. (60). **The key approximation of this note is that eq. (68) can be written as eq. (69) even when $w_{0x} \neq w_{0y}$.**

Accepting this approximation, we see that for eq. (69) to be true at all values of ρ implies that,

$$\frac{f'}{f^2} = -i, \qquad \text{and} \qquad \frac{h'}{fh} = -i. \tag{70}$$

Thus, $f = h$ is a solution, despite the different physical origin of these two functions as the transverse width and amplitude of the wave. We integrate the first of eq. (70) to obtain,

$$\frac{1}{f} = C + i\varsigma. \tag{71}$$

Our definition $f(0) = 1$ determines that $C = 1$. That is,

$$f = \frac{1}{1 + i\varsigma} = \frac{1 - i\varsigma}{1 + \varsigma^2} = \frac{e^{-i\tan^{-1}\varsigma}}{\sqrt{1 + \varsigma^2}}. \tag{72}$$

Note that $Re(f) = 1/(1 + \varsigma^2) = w_{0j}^2/w_j^2(\varsigma)$, while $|f| = 1/\sqrt{1 + \varsigma^2}$, so that $f = h$ is consistent with the asymptotic expectations discussed above. The longitudinal dependences of the widths of the Gaussian beam are now seen to be,

$$w_j(\varsigma) = w_{0j}\sqrt{1 + \varsigma^2}. \tag{73}$$

The lowest-order wave function is,

$$\psi_0 = f\, e^{-f\rho^2} = \frac{e^{-i\tan^{-1}\varsigma}}{\sqrt{1 + \varsigma^2}}\, e^{-\rho^2/(1+\varsigma^2)}\, e^{i\varsigma\rho^2/(1+\varsigma^2)}. \tag{74}$$

The factor $e^{-i\tan^{-1}\varsigma}$ in ψ_0 is the so-called Gouy phase shift [29], which changes from 0 to $\pi/2$ as z varies from 0 to ∞, with the most rapid change near the z_0. For large z the phase factor $e^{i\varsigma\rho^2/(1+\varsigma^2)}$ can be written as $e^{i(z_0/z)(x^2/w_{0x}^2 + y^2/w_{0y}^2)} \approx e^{ikr_\perp^2/(2z)}$, recalling eqs. (62) and (67). When this is combined with the traveling wave factor $e^{i(kz-\omega t)}$ we have,

$$e^{i[kz(1+r_\perp^2/2z^2)-\omega t]} \approx e^{i(kr-\omega t)}, \tag{75}$$

where $r = \sqrt{z^2 + r_\perp^2}$. Thus, the wave function ψ_0 is a modulated spherical wave for large z, but is a modulated plane wave near its waist.

To obtain the electric and magnetic fields of a Gaussian beam that is polarized in the y direction we take the vector potential to be,

$$A_x = 0, \qquad A_y = \frac{E_0}{i\omega}\psi_0\, e^{i(kz-\omega t)} = \frac{E_0}{i\omega} f\, e^{-f\rho^2} e^{i(kz-\omega t)}, \qquad A_z = 0. \tag{76}$$

Then,

$$\boldsymbol{\nabla} \cdot \mathbf{A} = -\frac{2fy}{w_{0y}^2} A_y. \tag{77}$$

and the electric field follows from eq. (55) as,

$$E_x \approx 0, \qquad E_y \approx E_0 f\, e^{-f\rho^2} e^{i(kz-\omega t)}, \qquad E_z \approx -\frac{2iyf}{kw_{0y}^2} E_y, \tag{78}$$

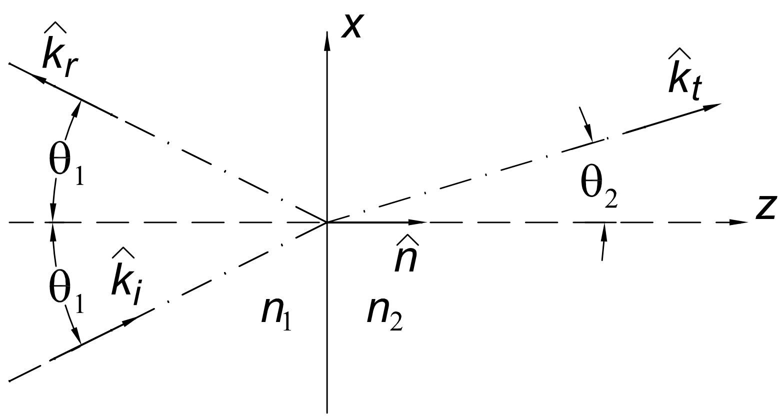
where we neglect terms of order $1/z_0^2$. Similarly, the magnetic field follows from eq. (56) as,

$$B_x = -\sqrt{\epsilon\mu} E_y = -\frac{n}{c} E_y, \qquad B_y = 0, \qquad B_z = \frac{2ixf}{kw_{0x}^2}\frac{n}{c} E_y. \tag{79}$$

B Fresnel Relations for Plane Waves

For reference we present a derivation of the well-known Fresnel relations for reflection and refraction of a plane wave at a planar interface.

The geometry and notation are again as shown in the figure below, with the plane of incidence being the x-z plane.



The incident, reflected and transmitted waves can be written as,

$$\mathbf{E}_i = \mathbf{E}_{0i}\, e^{i(k_{ix}x+k_{iz}z-\omega t)}, \qquad \mathbf{E}_r = \mathbf{E}_{0r}\, e^{i(k_{rx}x+k_{rz}z-\omega t)}, \qquad \mathbf{E}_t = \mathbf{E}_{0t}\, e^{i(k_{tx}x+k_{tz}z-\omega t)}, \tag{80}$$

$$\mathbf{B}_i = \frac{\mathbf{k}_i}{\omega}\times\mathbf{E}_{0i}\, e^{i(k_{ix}x+k_{iz}z-\omega t)}, \qquad \mathbf{B}_r = \frac{\mathbf{k}_r}{\omega}\times\mathbf{E}_{0r}\, e^{i(k_{rx}x+k_{rz}z-\omega t)}, \qquad \mathbf{B}_t = \frac{\mathbf{k}_t}{\omega}\times\mathbf{E}_{0t}\, e^{i(k_{tx}x+k_{tz}z-\omega t)}, \tag{81}$$

where we have used the plane-wave Maxwell equation $i\mathbf{k}\times\mathbf{E} = \boldsymbol{\nabla}\times\mathbf{E} = -\partial\mathbf{B}/\partial t = i\omega\mathbf{B}$, and we note that the wave equation $\nabla^2\mathbf{E} = \partial^2\mathbf{E}/\partial t^2$ requires that,

$$k_i^2 = k_{ix}^2 + k_{iz}^2 = \frac{n_1^2\omega^2}{c^2}, \qquad k_r^2 = k_{rx}^2 + k_{rz}^2 = \frac{n_1^2\omega^2}{c^2}, \qquad k_t^2 = k_{tx}^2 + k_{tz}^2 = \frac{n_2^2\omega^2}{c^2}, \tag{82}$$

so that,

$$k_i = k_r = \frac{n_1}{n_2}k_t. \tag{83}$$

Continuity of the tangential component of $\mathbf{E}$ at the interface requires that the argument of the exponential factors all be equal there, and hence,

$$k_{ix} = k_{rx} = k_{tx}, \tag{84}$$

$$\theta_i = \theta_r = \theta_1, \qquad k_{rz} = -k_{iz}, \tag{85}$$

and,

$$n_1 \sin\theta_1 = n_2 \sin\theta_2 \quad (= n_2 \sin\theta_t). \tag{86}$$

Then, the continuity of the x- and y-components of $\mathbf{E} = \mu\omega\mathbf{H}\times\mathbf{k}/k^2$ at the interface implies,

$$H_{0iy} - H_{0ry} = \frac{\mu_2}{\mu_1}\frac{n_1^2}{n_2^2}\frac{k_{tz}}{k_{iz}}H_{0ty} = \frac{\mu_2}{\mu_1}\frac{n_1\cos\theta_2}{n_2\cos\theta_1}H_{0ty}, \qquad \text{and} \qquad E_{0iy} + E_{0ry} = E_{0ty}, \tag{87}$$

noting that $E_x \propto \mu k_z H_y/n^2$. Similarly, continuity of the tangential component of $\mathbf{H} = \mathbf{k} \times \mathbf{E}/\mu\omega$ at the interface implies,

$$E_{0iy} - E_{0ry} = \frac{\mu_1}{\mu_2}\frac{k_{tz}}{k_{iz}} E_{0ty} = \frac{\mu_1}{\mu_2}\frac{n_2 \cos\theta_2}{n_1 \cos\theta_1} E_{0ty}, \qquad \text{and} \qquad H_{0iy} + H_{0ry} = H_{0ty}. \tag{88}$$

noting that $H_x \propto k_z H_y/\mu$. Combining eqs. (87) and (88) we find,

$$\frac{E_{0ry}}{E_{0iy}} = \frac{n_1 \cos\theta_1 - \frac{\mu_1}{\mu_2} n_2 \cos\theta_2}{n_1 \cos\theta_1 + \frac{\mu_1}{\mu_2} n_2 \cos\theta_2}, \qquad \frac{E_{0ty}}{E_{0iy}} = \frac{2n_1 \cos\theta_1}{n_1 \cos\theta_1 + \frac{\mu_1}{\mu_2} n_2 \cos\theta_2}, \tag{89}$$

$$\frac{H_{0ry}}{H_{0iy}} = \frac{n_2 \cos\theta_1 - \frac{\mu_2}{\mu_1} n_1 \cos\theta_2}{n_2 \cos\theta_1 + \frac{\mu_2}{\mu_1} n_1 \cos\theta_2}, \qquad \frac{H_{0ty}}{H_{0iy}} = \frac{2n_2 \cos\theta_1}{n_2 \cos\theta_1 + \frac{\mu_2}{\mu_1} n_1 \cos\theta_2}. \tag{90}$$

For the special case of the electric field polarized perpendicular to the plane of incidence, $\mathbf{H}$ has no y-component, and we write,

$$\frac{E_{0r\perp}}{E_{0i\perp}} = \frac{n_1 \cos\theta_1 - \frac{\mu_1}{\mu_2} n_2 \cos\theta_2}{n_1 \cos\theta_1 + \frac{\mu_1}{\mu_2} n_2 \cos\theta_2}, \qquad \frac{E_{0t\perp}}{E_{0i\perp}} = \frac{2n_1 \cos\theta_1}{n_1 \cos\theta_1 + \frac{\mu_1}{\mu_2} n_2 \cos\theta_2}, \tag{91}$$

Similarly, for the special case of the electric field polarized parallel to the plane of incidence, $\mathbf{E}$ has no y-component, and for each of the three waves, $H_y \propto nE_\parallel/\mu$, so we can write,

$$\frac{E_{0r\parallel}}{E_{0i\parallel}} = \frac{\frac{\mu_1}{\mu_2} n_2 \cos\theta_1 - n_1 \cos\theta_2}{\frac{\mu_1}{\mu_2} n_2 \cos\theta_1 + n_1 \cos\theta_2}, \qquad \frac{E_{0t\parallel}}{E_{0i\parallel}} = \frac{2n_1 \cos\theta_1}{\frac{\mu_1}{\mu_2} n_2 \cos\theta_1 + n_1 \cos\theta_2}, \tag{92}$$

As usual, we note that the case of internal reflection when $\sin\theta_2 = (n_1/n_2)\sin\theta_1 > 1$ can be accommodated in the above formalism by writing,

$$\cos\theta_2 = \sqrt{1 - \sin^2\theta_2} = \frac{\sqrt{n_2^2 - n_1^2 \sin^2\theta_1}}{n_2} = i\frac{\sqrt{n_1^2 \sin^2\theta_1 - n_2^2}}{n_2}. \tag{93}$$

In this case, we also write,

$$k_{tz} = ik_0 \frac{\sqrt{n_1^2 \sin^2\theta_1 - n_2^2}}{n_2}, \tag{94}$$

such that the transmitted wave is damped in z according to,

$$e^{ik_{tz}z} = e^{-\left(\sqrt{n_1^2 \sin^2\theta_1 - n_2^2}/n_2\right)k_0 z}. \tag{95}$$

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Book Three, Part 1, Question 29: "Are not the Rays of Light very small Bodies emitted from shining Substances? For such Bodies will pass through uniform Media in Right Lines without bending into the Shadow, which is the Nature of Rays of Light. ..... The Rays of Light in going out of Glass into a Vacuum, are bent towards the Glass; and if they fall too obliquely on the Vacuum, ther are bent backwards into the Glass, and totally reflected; and this Reflexion cannot be ascribed to the Resistance of an absolute Vacuum, but must be caused the the Power of the Glass attracting the Rays at their going out of it into the Vacuum, and bringing them back."

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