The Relation Between Expressions for Time-Dependent Electromagnetic Fields Given by Jefimenko and by Panofsky and Phillips

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(Dec. 5, 1996; updated May 11, 2018)

Abstract

The expressions of Jefimenko for the electromagnetic fields $\mathbf{E}$ and $\mathbf{B}$ in terms of source charge and current densities ρ and $\mathbf{J}$, which have received much recent attention in the American Journal of Physics, appeared previously in sec. 14.3 of the book *Classical Electricity and Magnetism* by Panofsky and Phillips. The latter developed these expressions further into a form that gives greater emphasis to the radiation fields. This Note presents a derivation of the various expressions, and discusses an apparent paradox in applying Panofsky and Phillips' result to static situations.

1 Introduction

A general method of calculation of time-dependent electromagnetic fields was given by Lorenz in 1867 [1], in which the retarded potentials were first introduced.[1] These are,

$$\Phi(\mathbf{x}, t) = \int \frac{[\rho(\mathbf{x}', t')]}{R} \, d^3\mathbf{x}', \qquad \text{and} \qquad \mathbf{A}(\mathbf{x}, t) = \frac{1}{c} \int \frac{[\mathbf{J}(\mathbf{x}', t')]}{R} \, d^3\mathbf{x}', \qquad (1)$$

where Φ and $\mathbf{A}$ are the scalar and vector potentials in Gaussian units,[2] ρ and $\mathbf{J}$ are the charge and current densities, $R = |\mathbf{R}|$ with $\mathbf{R} = \mathbf{x} - \mathbf{x}'$, and a pair of brackets, [ ], implies the quantity within is to be evaluated at the retarded time $t' = t - R/c$ with c being the speed of light in vacuum. Lorenz did not explicitly display the electric field $\mathbf{E}$ and the magnetic field $\mathbf{B}$, although he noted they could be obtained via

$$\mathbf{E} = -\boldsymbol{\nabla}\Phi - \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t}, \qquad \text{and} \qquad \mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A}. \qquad (3)$$

Had Lorenz' work been better received by Maxwell,[3] the expressions discussed below probably would have been well known over a century ago. The retarded potentials came into

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[1]The concept of retarded potentials is due to Riemann [2] (1858), but appeared only in a posthumous publication together with Lorenz' work [1]. Lorenz developed a scalar retarded potential in 1861 when studying waves of elasticity [3].

[2]These potentials obey the Lorenz gauge condition, also first introduced in [1],

$$\boldsymbol{\nabla} \cdot \mathbf{A} + \frac{1}{c}\frac{\partial \Phi}{\partial t} = 0. \qquad (2)$$

[3]Lorenz argued the light was mechanical vibration of electric charge (and that "vacuum" was electrically conductive), while Maxwell considered that light was waves of the electromagnetic field [4]. Maxwell's skepticism may also have been due to a misunderstanding as to the computation of the retarded fields of a uniformly moving charge [5].

general use only after Hertz' experiments on electromagnetic waves (1888) [6] and Thomson's discovery of the electron [7, 8] (1897).[4] At that time basic interest switched from electromagnetic phenomena due to time-dependent charge and current distributions to that due to moving electrons, *i.e.*, point charges. Hence, the Liénard-Wiechert potentials and the corresponding expressions for the electromagnetic fields of a point charge in arbitrary motion [12, 13] form the basis for most subsequent discussions.

For historical perspectives see the books of Whittaker [14] and O'Rahilly [15]. The textbook by Becker [16] contains concise derivations very much in the spirit of the original literature (and is still in print).

Recent interest shown in this Journal in general expressions for time-dependent electromagnetic fields arose from an article by Griffiths and Heald [17] on the conundrum: while time-dependent potentials are "simply" the retarded forms of the static potentials, the time-dependent fields are more than the retarded forms of the Coulomb and the Biot-Savart laws. Of course, it was Maxwell who first expounded the resolution of the conundrum; the something extra is radiation! Hertz' great theoretical paper on electric-dipole radiation (especially the figures) remains the classic example of how time-dependent fields can be thought of as instantaneous static fields close to the source but as radiation fields far from the source [6].

The discussion of Griffiths and Heald centered on the following expressions for the electromagnetic fields, which they attributed to Jefimenko [18],

$$\mathbf{E} = \int \frac{[\rho]\,\hat{\mathbf{n}}}{R^2}\,d^3\mathbf{x}' + \frac{1}{c}\int \frac{[\dot{\rho}]\,\hat{\mathbf{n}}}{R}\,d^3\mathbf{x}' - \frac{1}{c^2}\int \frac{[\dot{\mathbf{J}}]}{R}\,d^3\mathbf{x}'. \tag{4}$$

where $\dot{\mathbf{J}} = \partial\mathbf{J}/\partial t$, $\hat{\mathbf{n}} = \mathbf{R}/R$ and

$$\mathbf{B} = \frac{1}{c}\int \frac{[\mathbf{J}]\times\hat{\mathbf{n}}}{R^2}\,d^3\mathbf{x}' + \frac{1}{c^2}\int \frac{[\dot{\mathbf{J}}]\times\hat{\mathbf{n}}}{R}\,d^3\mathbf{x}', \tag{5}$$

These expressions indeed contain retarded versions of the Coulomb and Biot-Savart laws as their leading terms, but their relation to radiation is not as manifest as it might be. In particular, eq. (4) seems to suggest that there exist both longitudinal and transverse components of the electric field that fall off as $1/R$. It must be that the second term of eq. (4) cancels the longitudinal component of the third term, although this is not self-evident. Thus, from a pedagogical point of view eq. (4) goes only part way towards resolving the conundrum.

Personally, I found some of the discussion by Griffiths and Heald (and their followers [29]-[31]) surprising in that I imagined it was common knowledge that eq. (4) can be transformed to,

$$\mathbf{E} = \int \frac{[\rho]\,\hat{\mathbf{n}}}{R^2}\,d^3\mathbf{x}' + \frac{1}{c}\int \frac{([\mathbf{J}]\cdot\hat{\mathbf{n}})\,\hat{\mathbf{n}} + ([\mathbf{J}]\times\hat{\mathbf{n}})\times\hat{\mathbf{n}}}{R^2}\,d^3\mathbf{x}' + \frac{1}{c^2}\int \frac{([\dot{\mathbf{J}}]\times\hat{\mathbf{n}})\times\hat{\mathbf{n}}}{R}\,d^3\mathbf{x}'. \tag{6}$$

The combination of eqs. (5) and (6) manifestly displays the mutually transverse character of the radiation fields (those that vary as $1/R$), and to my taste better serves to illustrate the

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[4]FitzGerald seems to have been aware of retarded potentials in his discussion [9, 10] of radiation by a small, oscillating current loop, one year prior to the invention of the Poynting vector [11].

nature of the time-dependent fields. However, after extensive checking the only reference to eq. (6) that I have located is in sec. 14.3 of the 2nd edition of the textbook of Panofsky and Phillips [19, 20].[5,6]

The alert reader may be troubled by the second term in eq. (6), which seems to suggest that static currents give rise to an electric field. One can verify by explicit calculation that this is not so for current in a straight wire or (more tediously) in a circular loop. Indeed, the second term in eq. (6) vanishes whenever both $\nabla \cdot \mathbf{J} = 0$ and $\dot{\mathbf{J}} = 0$ over the whole current distribution, *i.e.*, in the static limit.

In the following section I give a direct derivation of eq. (6) in possible contrast to that of Panofsky and Phillips who used Fourier transforms. Section 3 clarifies why the second term of eq. (6) vanishes in the static limit. Appendix A presents a solution for the fields without use of potentials, and Appendix B considers the Helmholtz decomposition of $\mathbf{E}$ and $\mathbf{B}$.

2   Derivation of the Electric Field

The starting point is, of course, eq. (3) applied to eqs. (1).

The time derivative $\partial/\partial t$ acts only on $[\mathbf{J}]$ because of the relation $t' = t - R/c$. Thus,

$$-\frac{1}{c}\frac{\partial \mathbf{A}}{\partial t} = -\frac{1}{c^2}\int \frac{[\dot{\mathbf{J}}]}{R}\, d^3\mathbf{x}'. \tag{7}$$

Also,

$$-\boldsymbol{\nabla}\Phi = -\int \boldsymbol{\nabla}\left(\frac{[\rho]}{R}\right) d^3\mathbf{x}' = -\int [\rho]\,\boldsymbol{\nabla}\left(\frac{1}{R}\right) d^3\mathbf{x}' - \int \frac{\boldsymbol{\nabla}[\rho]}{R}\, d^3\mathbf{x}'. \tag{8}$$

But,

$$\boldsymbol{\nabla}\left(\frac{1}{R}\right) = -\frac{\hat{\mathbf{n}}}{R^2}, \qquad \text{and} \qquad \boldsymbol{\nabla}[\rho] = \boldsymbol{\nabla}\rho(\mathbf{x}', t - R/c) = [\dot{\rho}]\left(-\frac{\boldsymbol{\nabla}R}{c}\right) = -\frac{[\dot{\rho}]\,\hat{\mathbf{n}}}{c}. \tag{9}$$

Equations (7)-(9) combine to give eq. (4).

It is now desired to transform the second term of eq. (4), and the continuity equation,

$$\boldsymbol{\nabla}\cdot\mathbf{J} = -\dot{\rho}, \tag{10}$$

suggests itself for this purpose. Some care is needed to apply this at the retarded coordinates $\mathbf{x}'$ and $t' = t - R/c$ because of the implicit dependence of the current density on $\mathbf{x}'$ through

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[5]The Fourier transforms of eqs. (4)-(5) which appear as eqs. (14-33) and (14-36) in the book of Panofsky and Phillips also appear as eqs. (18)-(19) of [21], as eq. (16) of [22] and eq. (21), p. 466 of [23].

A form of eq. (4) can be recognized in eq. (2a), p. 505, of [24]; see also [25]. There, the context is Kirchhoff integrals for vector fields in terms of source charges and current plus relevant knowledge of the fields on some bounding surface. Since, the expressions (4)-(5) (or their Fourier transforms) are valid without use of information from a bounding surface, we learn that the integrals based on that surface information are actually zero if all sources are contained within that surface. This last point is, however, omitted from most discussions of diffraction theory.

[6]The attribution of eqs. (4)-(6) to Jefimenko now includes sec. 8.2, p. 261 of [26], sec. 10.2.2, p. 427 of [27], and sec. 6.5, p. 246 of [28].

R. Introducing $\boldsymbol{\nabla}' = \partial/\partial\mathbf{x}'$, then,

$$\boldsymbol{\nabla}' \cdot [\mathbf{J}] = [\boldsymbol{\nabla}' \cdot \mathbf{J}] + [\dot{\mathbf{J}}] \cdot \left(-\frac{\boldsymbol{\nabla}'R}{c} \right) = -[\dot{\rho}] + \frac{[\dot{\mathbf{J}}] \cdot \hat{\mathbf{n}}}{c}. \tag{11}$$

Thus,

$$\frac{1}{c} \int \frac{[\dot{\rho}]\,\hat{\mathbf{n}}}{R}\, d^3\mathbf{x}' = -\frac{1}{c} \int \frac{(\boldsymbol{\nabla}' \cdot [\mathbf{J}])\hat{\mathbf{n}}}{R}\, d^3\mathbf{x}' + \frac{1}{c^2} \int \frac{([\dot{\mathbf{J}}] \cdot \hat{\mathbf{n}})\hat{\mathbf{n}}}{R}\, d^3\mathbf{x}'. \tag{12}$$

If the first term on the righthand side of eq (12) actually varies as $1/R^2$ then the radiation field within eq. (4) will have the form given in eq. (6), since the last term in eq. (12) is the negative of the longitudinal component of the last term in eq. (4).

The integral involving $\boldsymbol{\nabla}' \cdot [\mathbf{J}]$ can be transformed further by examining the components of the integrand,

$$\frac{(\boldsymbol{\nabla}' \cdot [\mathbf{J}])\hat{\mathbf{n}}_i}{R} = \frac{\partial\,[\mathbf{J}]_j}{\partial x'_j}\frac{R_i}{R^2} = \frac{\partial}{\partial x'_j}\left(\frac{[\mathbf{J}]_j\,R_i}{R^2}\right) - [\mathbf{J}]_j\frac{\partial}{\partial x'_j}\left(\frac{R_i}{R^2}\right) = \frac{\partial}{\partial x'_j}\left(\frac{[\mathbf{J}]_j\,R_i}{R^2}\right) + \frac{[\mathbf{J}]_i - 2([\mathbf{J}] \cdot \hat{\mathbf{n}})\hat{\mathbf{n}}_i}{R^2}, \tag{13}$$

where summation is implied over index j. The volume integral of the first term becomes a surface integral with the aid of Gauss' theorem, and hence vanishes assuming the currents are contained within a bounded volume,

$$\int_V \frac{\partial}{\partial x'_j}\left(\frac{[\mathbf{J}]_j\,R_i}{R^2}\right)\, d^3\mathbf{x}' = \int_S d\mathbf{S}' \cdot \left(\frac{[\mathbf{J}]\,R_i}{R^2}\right) = 0. \tag{14}$$

The remaining term can be summarized as,

$$-\frac{1}{c} \int \frac{(\boldsymbol{\nabla}' \cdot [\mathbf{J}])\hat{\mathbf{n}}}{R}\, d^3\mathbf{x}' = \frac{1}{c} \int \frac{2([\mathbf{J}] \cdot \hat{\mathbf{n}})\hat{\mathbf{n}} - [\mathbf{J}]}{R^2}\, d^3\mathbf{x}' = \frac{1}{c} \int \frac{([\mathbf{J}] \cdot \hat{\mathbf{n}})\hat{\mathbf{n}} + ([\mathbf{J}] \times \hat{\mathbf{n}}) \times \hat{\mathbf{n}}}{R^2}\, d^3\mathbf{x}'. \tag{15}$$

Finally, equations (4), (12) and (15) combine to yield eq. (6).

3 The Static Limit

To ascertain the behavior of the second term of eq. (6) in the static limit, refer to eq. (15) which indicates its relation to $\boldsymbol{\nabla}' \cdot [\mathbf{J}]$. The latter is expanded in eq. (11), and accordingly vanishes if both $\boldsymbol{\nabla} \cdot \mathbf{J} = 0$ and $\dot{\mathbf{J}} = 0$. Since $\boldsymbol{\nabla} \cdot \mathbf{J} = -\dot{\rho}$, the second term of eq. (6) vanishes in the static limit (*i.e.*, when both $\dot{\rho}$ and $\dot{\mathbf{J}}$ vanish), as claimed in the Introduction.

It remains that expression (6) may be more cumbersome than expression (4) for explicit calculation of the electric field in time-dependent situations where radiation is not the dominant concern. This point has been illustrated in the calculation of the fields of a moving charge [17, 29, 30, 31, 32] and related examples [33, 34, 35, 36, 37, 38]. In another Note [39] I discuss how eqs. (4) and (5) can be used to clarify a subtle issue regarding the fields outside an infinite solenoid with a time-dependent current.

A   Appendix: Solution for the Fields without Use of Potentials

Maxwell's equations for the fields $\mathbf{E}$ and $\mathbf{B}$ can be combined into wave equations of the form,

$$\nabla^2\mathbf{E} - \frac{1}{c^2}\frac{\partial^2\mathbf{E}}{\partial t^2} = 4\pi\boldsymbol{\nabla}\rho + \frac{4\pi}{c^2}\frac{\partial\mathbf{J}}{\partial t}, \qquad \nabla^2\mathbf{B} - \frac{1}{c^2}\frac{\partial^2\mathbf{B}}{\partial t^2} = -\frac{4\pi}{c}\boldsymbol{\nabla}\times\mathbf{J}, \tag{16}$$

in terms of source charge and current densities ρ and $\mathbf{J}$.[7] Each of these six scalar component equations can be solved by the method of Riemann and Lorenz in terms of retarded source quantities,

$$\mathbf{E}(\mathbf{x},t) = -\int\frac{[\boldsymbol{\nabla}'\rho]}{R}\,d^3\mathbf{x}' - \frac{1}{c^2}\int\frac{[\dot{\mathbf{J}}]}{R}\,d^3\mathbf{x}', \qquad \mathbf{B}(\mathbf{x},t) = \frac{1}{c}\int\frac{[\boldsymbol{\nabla}'\times\mathbf{J}]}{R}\,d^3\mathbf{x}'. \tag{17}$$

Now, for charge density that falls off sufficiently quickly at large distances,

$$-\int\frac{\boldsymbol{\nabla}'\rho}{R}\,d^3\mathbf{x}' = \int\rho\boldsymbol{\nabla}'\left(\frac{1}{R}\right)d^3\mathbf{x}' = \int\frac{\rho\,\hat{\mathbf{n}}}{R^2}\,d^3\mathbf{x}', \tag{18}$$

as follows on integration by parts, so for $[\rho] = \rho(t - R/c)$,

$$-\int\frac{[\boldsymbol{\nabla}'\rho]}{R}\,d^3\mathbf{x}' = \int\frac{[\rho\,\hat{\mathbf{n}}]}{R^2}\,d^3\mathbf{x}' + \int\frac{[\dot{\rho}]\,\boldsymbol{\nabla}'(1/R)}{cR}\,d^3\mathbf{x}' = \int\frac{[\rho]\,\hat{\mathbf{n}}}{R^2}\,d^3\mathbf{x}' + \frac{1}{c}\int\frac{[\dot{\rho}]\,\hat{\mathbf{n}}}{R}\,d^3\mathbf{x}'. \tag{19}$$

Using this in eq. (17) yields eq. (4) for $\mathbf{E}$, from which eq. (6) follows as before. Similarly, integrating eq. (17) for $\mathbf{B}$ by parts, and using the chain rule for the derivative of the time dependence on $t' = t - R/c$ leads to eq. (5).[8]

B   Appendix: Helmholtz Decomposition of the Fields

This section was added on May 11, 2018, at the suggestion of Joseph Subotnik.

Recall that Helmholtz showed in 1858 [41, 42] (in a hydrodynamic context) that any vector field, say $\mathbf{E}$, which vanishes suitably quickly at infinity can be decomposed as,[9,10]

$$\mathbf{E} = \mathbf{E}_{\mathrm{irr}} + \mathbf{E}_{\mathrm{rot}}, \tag{20}$$

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[7]The wave equation for $\mathbf{E}$ was given by Lorenz [1] for a medium of electrical conductivity σ, but with $\mathbf{E}$ replaced by $\mathbf{J}$ according to Ohm's law, $\mathbf{J} = \sigma\mathbf{E}$. Lorenz noted that a solution to this equation exists via the method of retarded potentials, but his did not explicitly display this for $\mathbf{J}$. Had he done so, he would have arrived at eq. (17). See also the Appendix to [5].

[8]Jefimenko's equations (4)-(5) have been deduced from the wave equations (16) by use of a retarded Green function in [30], in sec. 6.5 of [28] and in sec. II of [40].

[9]The essence of this decomposition was anticipated by Stokes (1849) in secs. 5-6 of [43].

[10]Radiation fields, which fall off as $1/r$ at large distance r from their (bounded) source, do fall off sufficiently quickly for Helmholtz' decomposition to apply, as reviewed in [44]. Doubts as to this were expressed in [45], but see [46].

where the irrotational and rotational components $\mathbf{E}_{\text{irr}}$ and $\mathbf{E}_{\text{rot}}$ obey,[11]

$$\boldsymbol{\nabla} \times \mathbf{E}_{\text{irr}} = 0, \qquad \text{and} \qquad \boldsymbol{\nabla} \cdot \mathbf{E}_{\text{rot}} = 0. \tag{21}$$

As noted in [42], the Helmholtz decomposition of electromagnetic fields is closely related to use of the Coulomb gauge for the electromagnetic potentials $\mathbf{A}^{(C)}$ and $\Phi^{(C)}$, from which the fields $\mathbf{E}$ and $\mathbf{B}$ can be deduced according to,

$$\mathbf{E} = -\boldsymbol{\nabla}\Phi^{(C)} - \frac{1}{c}\frac{\partial \mathbf{A}^{(C)}}{\partial t}, \qquad \mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A}^{(C)}. \tag{22}$$

In the Coulomb gauge, the vector potential obeys,

$$\boldsymbol{\nabla} \cdot \mathbf{A}^{(C)} = 0, \qquad \Rightarrow \qquad \boldsymbol{\nabla} \cdot \left(-\frac{1}{c}\frac{\partial \mathbf{A}^{(C)}}{\partial t}\right) = 0, \tag{23}$$

and the scalar potential is the instantaneous Coulomb potential,

$$\Phi^{(C)} = \int \frac{\rho}{R} d^3\mathbf{x}', \tag{24}$$

such that the corresponding instantaneous Coulomb electric field obeys,

$$\mathbf{E}_{\text{instantaneous}} = -\boldsymbol{\nabla}\Phi^{(C)} = \int \frac{\rho\,\hat{\mathbf{n}}}{R^2} d^3\mathbf{x}', \qquad \Rightarrow \qquad \boldsymbol{\nabla} \times \mathbf{E}_{\text{instantaneous}} = 0. \tag{25}$$

Hence, the Helmholtz decomposition of the electric field can be written as,

$$\mathbf{E}_{\text{irr}} \;=\; \mathbf{E}_{\text{instantaneous}} = -\boldsymbol{\nabla}\Phi^{(C)} = \int \frac{\rho\,\hat{\mathbf{n}}}{R^2} d^3\mathbf{x}', \tag{26}$$

$$\mathbf{E}_{\text{rot}} \;=\; -\frac{1}{c}\frac{\partial \mathbf{A}^{(C)}}{\partial t} = \mathbf{E} - \mathbf{E}_{\text{irr}} \tag{27}$$

$$\;=\; \int \frac{([\rho]-\rho)\,\hat{\mathbf{n}}}{R^2} d^3\mathbf{x}' + \frac{1}{c}\int \frac{([\mathbf{J}]\cdot\hat{\mathbf{n}})\,\hat{\mathbf{n}} + ([\mathbf{J}]\times\hat{\mathbf{n}})\times\hat{\mathbf{n}}}{R^2} d^3\mathbf{x}' + \frac{1}{c^2}\int \frac{([\dot{\mathbf{J}}]\times\hat{\mathbf{n}})\times\hat{\mathbf{n}}}{R} d^3\mathbf{x}',$$

recalling eq. (6).[12] Of course, since $\boldsymbol{\nabla} \cdot \mathbf{B} = 0$,

$$\mathbf{B}_{\text{irr}} = \mathbf{B}. \tag{28}$$

B.1 Are the Radiation Fields Rotational?

The "radiation" fields are those terms of eqs. (5)-(6) whose integrands are proportional to $1/R$,

$$\mathbf{B}_{\text{rad}} = \frac{1}{c^2}\int \frac{[\dot{\mathbf{J}}]\times\hat{\mathbf{n}}}{R} d^3\mathbf{x}', \qquad \mathbf{E}_{\text{rad}} = \frac{1}{c^2}\int \frac{([\dot{\mathbf{J}}]\times\hat{\mathbf{n}})\times\hat{\mathbf{n}}}{R} d^3\mathbf{x}'. \tag{29}$$

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[11]The irrotational component is sometimes labeled "longitudinal" or "parallel", and the rotational component is sometimes labeled "solenoidal" or "transverse".

[12]The last term in eq. (27), the "radiation" electric field (if nonzero), has sources at which $\boldsymbol{\nabla} \cdot \mathbf{E}_{\text{rad}}$ is nonzero, so this term is not irrotational. In contrast, idealized, source-free plane electromagnetic waves have zero divergence. And, since $\boldsymbol{\nabla} \times \mathbf{E}_{\text{rad}} = -\partial\mathbf{B}_{\text{rad}}/\partial ct$, the radiation electric field is not rotational either.

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Are these radiation fields rotational (*i.e.*, with zero divergence)?

For example, the radiation fields of an idealized "point," (Hertzian) oscillating electric dipole $\mathbf{p} = \mathbf{p}_0 \, e^{\omega t}$, are (eq.(9.19) of [47]),

$$\mathbf{B}_{\text{rad}} = k^2 \, \hat{\mathbf{n}} \times \mathbf{p} \frac{e^{ikr}}{r} = k^2 \, \mathbf{r} \times \mathbf{p} \frac{e^{ikr}}{r^2} \,, \qquad \mathbf{E}_{\text{rad}} = \mathbf{B}_{\text{rad}} \times \hat{\mathbf{n}} = k^2 (\hat{\mathbf{n}} \times \mathbf{p}) \times \hat{\mathbf{n}} \frac{e^{ikr}}{r} \,, \qquad (30)$$

where $k = \omega/c$ and $\hat{\mathbf{n}} = \mathbf{r}/r$. The divergences of the radiation fields are,

$$
\begin{aligned}
\frac{1}{k^2} \boldsymbol{\nabla} \cdot \mathbf{B}_{\text{rad}} &= \boldsymbol{\nabla} \cdot \left(\mathbf{r} \times \mathbf{p} \frac{e^{ikr}}{r^2} \right) = \mathbf{p} \frac{e^{ikr}}{r^2} \cdot \boldsymbol{\nabla} \times \mathbf{r} - \mathbf{r} \cdot \boldsymbol{\nabla} \times \mathbf{p} \frac{e^{ikr}}{r^2} = \mathbf{r} \cdot \left(\boldsymbol{\nabla} \frac{e^{ikr}}{r^2} \right) \times \mathbf{p} \\
&= \mathbf{p} \cdot \mathbf{r} \times \boldsymbol{\nabla} \frac{e^{ikr}}{r^2} = \mathbf{p} \cdot \mathbf{r} \times \mathbf{r} \left(ik - \frac{2}{r} \right) \frac{e^{ikr}}{r^3} = 0, \qquad (31)
\end{aligned}
$$

$$
\begin{aligned}
\frac{1}{k^2} \boldsymbol{\nabla} \cdot \mathbf{E}_{\text{rad}} &= \boldsymbol{\nabla} \cdot \left(\mathbf{p} \frac{e^{ikr}}{r} - (\mathbf{p} \cdot \mathbf{r}) \mathbf{r} \frac{e^{ikr}}{r^3} \right) = \mathbf{p} \cdot \boldsymbol{\nabla} \frac{e^{ikr}}{r} - (\mathbf{p} \cdot \mathbf{r}) \boldsymbol{\nabla} \cdot \left(\mathbf{r} \frac{e^{ikr}}{r^3} \right) - \mathbf{r} \frac{e^{ikr}}{r^3} \cdot \boldsymbol{\nabla} (\mathbf{p} \cdot \mathbf{r}) \\
&= (\mathbf{p} \cdot \mathbf{r}) \left(ik - \frac{1}{r} \right) \frac{e^{ikr}}{r^2} - (\mathbf{p} \cdot \mathbf{r}) \left(\mathbf{r} \cdot \boldsymbol{\nabla} \frac{e^{ikr}}{r^3} + \frac{e^{ikr}}{r^3} \boldsymbol{\nabla} \cdot \mathbf{r} \right) \\
&\quad - \mathbf{r} \frac{e^{ikr}}{r^3} \cdot \{ (\mathbf{p} \cdot \boldsymbol{\nabla}) \mathbf{r} + \mathbf{p} \times (\boldsymbol{\nabla} \times \mathbf{r}) \} \\
&= (\mathbf{p} \cdot \mathbf{r}) \frac{e^{ikr}}{r^2} \left(ik - \frac{1}{r} - ik + \frac{3}{r} - \frac{3}{r} - \frac{1}{r} \right) = -2 (\mathbf{p} \cdot \mathbf{r}) \frac{e^{ikr}}{r^3} \,. \qquad (32)
\end{aligned}
$$

Thus, while $\mathbf{B}_{\text{rad}}$ is rotational, $\mathbf{E}_{\text{rad}}$ is not (and is not irrotational either).[13]

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