How Much of Magnetic Energy Is Kinetic Energy?

Kirk T. McDonald
Joseph Henry Laboratories, Princeton University, Princeton, NJ 08544
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1 Problem

How much of the "magnetic" energy stored in a current-carrying inductor is due to the kinetic energy of the moving charges?

Consider the example of a toroidal coil of major radius a and minor radius $b \ll a$ that carries current I in its N turns, which current is due to electrons of charge $-e$ and (rest) mass m.

2 Solution

The "magnetic" energy U stored in an inductor can be written as,

$$U = \frac{1}{2} L I^2 = \int \frac{B^2}{8\pi}\, d\text{Vol}, \tag{1}$$

where L is the self inductance, and the latter form assumes all media have unit (relative) permeability and is expressed in Gaussian units.

For the example of a toroidal inductor, the magnetic field B_0 along its circular axis follows from Ampère's law as,

$$B_0 = \frac{2NI}{ac}\,, \tag{2}$$

where c is the speed of light in vacuum. Thus, the stored "magnetic" energy is,

$$U \approx \frac{B_0^2}{8\pi} 2\pi^2 a b^2 = \frac{N^2 I^2 b^2}{ac^2} \qquad \left(\text{and hence} \quad L \approx \frac{2N^2 b^2}{ac^2}\right). \tag{3}$$

Supposing that all conduction electrons have the same speed v, the current I is related to the number density n of conduction electrons per unit length along the (spiral) conductor according to,

$$I = nev. \tag{4}$$

The total length of the conductor is $2\pi N b$, so the kinetic energy T of the conduction electrons is,

$$T = 2\pi N n b \frac{mv^2}{2} = \pi N b \frac{mI^2}{ne^2} = \frac{\pi N b I^2}{nc^2 r_0}\,, \tag{5}$$

where $r_0 = e^2/mc^2 \approx 3 \times 10^{-13}$ cm is the classical electron radius.

The ratio of the kinetic energy to the "magnetic" energy is,

$$\frac{T}{U} \approx \frac{\pi a}{N n\, b\, r_0}\,. \tag{6}$$

Since only the product nv is determined by eq. (4), the result (6) is ambiguous.

To go further, we suppose that there is one conduction electron per atom in the copper conductor, such that the volume density of conduction electrons $n_e \approx 8 \times 10^{22}/\text{cm}^3$. Then, the linear number density is $n = \pi n_e d^2/4$, where $d \ll b$ is the diameter of the copper conductor. Equation (6) now becomes,

$$\frac{T}{U} \approx \frac{4a}{N n_e\, b\, d^2\, r_0} \approx \frac{a}{6Nb\, d^2} 10^{-9}, \tag{7}$$

for a, b and d measured in cm.

We could also suppose that the N turns are tightly wound on the toroid, such that $Nd = 2\pi a$. Then,

$$\frac{T}{U} \approx \frac{1}{12\pi b\, d} 10^{-9}, \tag{8}$$

As an example, suppose $b = 1$ cm and $d = 1$ mm $= 0.1$ cm, for which $T/U \approx 3 \times 10^{-10}$.

3 Comments

This problem was of interest to Maxwell, who did not have a vision of currents as due to the motion of electrons. In Art. 551 of his *Treatise* [1], he wrote:

It appears, therefore, that a system containing an electric current is a seat of energy of some kind; and since we can form no conception of an electric current except as a kinetic phenomenon, its energy must be kinetic energy, that is to say, the energy which a moving body has in virtue of its motion.

We have already shewn that the electricity in the wire cannot be consider as the moving body in which we are to find the energy, for the energy of a moving body does not depend on anything external to itself, whereas the presence of other bodies near the current alters its energy.

We are therefore led to enquire where there may not be some motion going on in the space outside the wire, which is not occupied by the electric current, but in which the electromagnetic effects of the current are manifested.

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What I now propose to do is to examine the consequences of the assumption that the phenomena of the electric current are those of a moving system, the motion being communicated from one part of the system to another by forces, the nature and laws of which we do not yet even attempt to define, because we can eliminate those forces from the equations of motion by the method given by Lagrange for any connected system.

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I have chosen this method because I wish to shew that there are other ways of viewing the phenomena which appear to be more satisfactory, and at the same time more consistent ... than those which proceed on the hypothesis of direct action at a distance.

It appears that although Maxwell did not make a calculation like that of eq. (8) he correctly understood that the (ordinary) kinetic energy of "electricity" in the wire of an inductor can account for only a small part of the "magnetic" energy of that inductor. This reinforced his vision of some kind of active æther in and through which electromagnetic effects are transmitted. He also had come to realize that an abstract, mathematical characterization of that æther suffices, and that the earlier mechanical models of an æther are of little/no enduring utility.

Maxwell returned to the question of what portion of the "magnetic" energy is actually ordinary kinetic energy in Arts. 574-577 of the *Treatise* and considered several experiments, which at the time produced null results.[1,2,3,4]

There remains an issue of terminology. Maxwell's definition seems very reasonable, that kinetic energy is *the energy which a moving body has in virtue of its motion* (Art. 551). Moving charges are associated with a magnetic field, and energy is associated with this field. Can/should we therefore say that this energy is "kinetic" because it is due to the motion of the charges? Only a tiny fraction of "magnetic" energy is associated with the "ordinary kinetic energy" $\sum mv^2/2$ of the moving charges, so we would have to invent a new category of "extraordinary kinetic energy", which Maxwell called *electrokinetic energy* (Art. 573). This usage is not common, and I do not advocate it. On the whole, the common term "magnetic energy" seems descriptive enough, although it draws attention away from the tiny component of ordinary kinetic energy associated with magnetic phenomena.[5,6]

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[1] See also [2].

[2] This theme was reviewed by Cullwick in [3] and in chap. 18 [4]. He noted that with time all of the experiments considered by Maxwell did provide evidence for small amounts of ordinary momentum, energy and angular momentum associated with currents in circuits. However, Cullwick ended his chap. 18 by making the striking (and unsupportable, as known even to Maxwell) hypothesis that all "magnetic" energy is due to the ordinary kinetic energy of the conduction electrons.

Cullwick may have confused the random Fermi velocity $\mathbf{v}_\mathrm{F}$ of conduction electrons with their drift velocity $\mathbf{v}_d$ due to an applied electric field. While $v_d \ll v_\mathrm{F} \approx \alpha c$, on summing over conduction electrons, $\sum \mathbf{v}_\mathrm{F} = 0$, such that their kinetic energy sums to $\sum m(\mathbf{v}_\mathrm{F} + \mathbf{v}_d)^2/2 = \sum m v_\mathrm{F}^2/2 + \sum m v_d^2/2$, with the first term being independent of the electric current, and should not be considered as a "magnetic" energy.

[3] *May 30, 2020.* In [5] it was argued that conduction electrons have a large "effective mass" such that the magnetic energy is entirely the kinetic energy of the conduction electrons. In an example considered there, a current of 1 A flowed in a copper wire of radius 0.5 mm that formed a circular loop of circumference 1 m. Then, $U/T = 3 \times 10^9$ for electrons with their nominal mass $m_0 \approx 10^{-30}$ kg. The suggestion of [5] was that the conduction electrons have "effective mass" of $3 \times 10^9 m_0 \approx 3 \times 10^{-21}$ kg in this example. Now, the number of conduction electrons in this loop is $\approx 10^{23}$, so their total "effective mass" would be ≈ 300 kg. However, the observed weight of current-carrying wires has negligible dependence on the strength of the current, so the hypothesis in [5] as to an "effective mass" is untenable.

On p. 261 of [4], the issue of a large "effective mass" of conduction electrons was avoided by the claim that somehow most conduction electron don't participate in electric currents, while those few that do have drift speeds close to that of light.

These views are vestiges of the notion that magnetic effects are "mechanical", and that the magnetic field is a mathematical crutch, rather than a physical entity, distinct from electric charge, with nonzero energy.

[4] De Gennes [6] used a classical model of a superconductor in which magnetic field energy is considered to be separate from the kinetic energy of conduction electrons to deduce the London equation that governs the Meissner effect (that a magnetic field is expelled from a metal when it makes a transition from normal- to super-conductivity).

[5] A separate issue is the terminology for the tiny net amount of ordinary momentum associated with currents in circuits. In some situations the term "hidden mechanical momentum" is used for this. See, for example, [7].

[6] For a discussion of the relation of the drift kinetic energy, $\sum m v_d^2/2$, of conduction electrons to the imaginary part of the conductivity in the Drude model, see [8].

In his Art. 551, Maxwell was wrestling with the question of how the older, mechanical notion of the kinetic energy $mv^2/2$ of a moving mass should be extended into the larger domain of electrodynamics. Similar issues arose after 1905 in the context of Einstein's theory of special relativity, which considered masses with high velocity, and also in the context of the his concept of photons/particles of light.

The attitude favored by this author is that the term "kinetic energy" should be restricted to its original meaning of $mv^2/2$, rather than following Maxwell's suggestion that any energy of a body/system "in virtue of its motion" be called "kinetic energy". In this view, a photon does not possess kinetic energy simply because it has no (rest) mass. Similarly, in this view a fast-moving mass with total energy $mc^2/\sqrt{1-v^2/c^2} = \gamma mc^2 = mc^2 + mv^2/2 + mv^4/8c^2 + \cdots$ still has "kinetic energy" $mv^2/2$, although one is tempted to speak of its "relativistic kinetic energy" as $(\gamma - 1)mc^2 = mv^2/2 + mv^4/8c^2 + \cdots$. And, in this view, the magnetic energy of a circuit at rest is to be considered as part of the rest energy of that system, rather than as a "kinetic energy".[7,8]

References

[1] J.C. Maxwell, *A Treatise on Electricity and Magnetism*, Vol. 2 (Clarendon Press, 1873), http://kirkmcd.princeton.edu/examples/EM/maxwell_treatise_v2_73.pdf
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[2] K.T. McDonald, *Does Centrifugal Force Affect Electric Currents?* (June 25, 2019), http://kirkmcd.princeton.edu/examples/tolman.pdf

[3] E.G. Cullwick, *Electromagnetic Momentum and Electron Inertia in a Current Circuit* Proc. IEE **103**, 159 (1956), http://kirkmcd.princeton.edu/examples/EM/cullwick_piee_103_159_56.pdf

[4] E.G. Cullwick, *Electromagnetism and Relativity* (Longmans, 1959), http://kirkmcd.princeton.edu/examples/EM/cullwick_em_rel.pdf

[5] A.K.T. Assis and J.A. Hernandes, *Magnetic Energy and Effective Inertial Mass of the Conduction Electrons in Circuit Theory*, Elec. Phenom. **6**, 29 (2006), http://kirkmcd.princeton.edu/examples/EM/assis_ep_6_29_06.pdf

[6] P.G. de Gennes, *Superconductivity of Metals and Alloys* (CRC Press, 1999), eqs. 1-6 and 1-7, http://kirkmcd.princeton.edu/examples/QM/deGennes_99.pdf
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[7] In sec. 2 of [9], Maxwell allowed that energy stored in electromagnetic fields might be either "actual" (kinetic) or "potential". It is consistent to say that a magnetostatic field stores only "potential" energy.

[8] In nonrelativistic quantum theory, analysis is based on a Hamiltonian consisting of mechanical kinetic energy and interaction energy (without any consideration of electromagnetic field-only energy), where the latter is mainly due to electromagnetic interactions. Here, it is "natural" to keep the electromagnetic-interaction energy separate from mechanical kinetic energy, but this leaves open the issue of whether the electromagnetic field energy is partly "kinetic".

[7] K.T. McDonald, *On the Definition of "Hidden" Momentum* (July 9, 2012), http://kirkmcd.princeton.edu/examples/hiddendef.pdf

[8] K.T. McDonald, *Kinetic Energy of Conduction Electrons in the Drude Model* (Oct. 10, 2009), http://kirkmcd.princeton.edu/examples/drude.pdf

[9] J.C. Maxwell, *A Dynamical Theory of the Electromagnetic Field*, Phil. Trans. Roy. Soc. London **155**, 459 (1865), http://kirkmcd.princeton.edu/examples/EM/maxwell_ptrsl_155_459_65.pdf