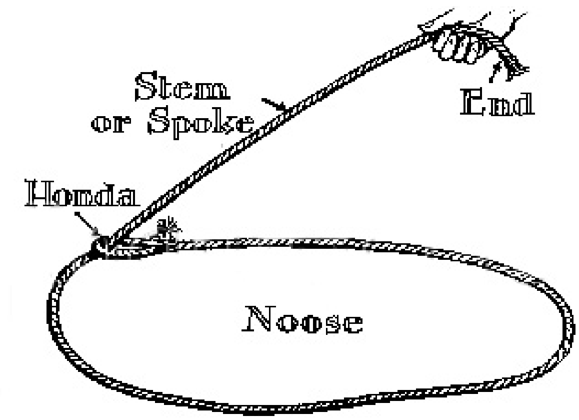
Spinning Lasso

Chris G. Tully and Kirk T. McDonald
Joseph Henry Laboratories, Princeton University, Princeton, NJ 08544
(December 11, 2009)

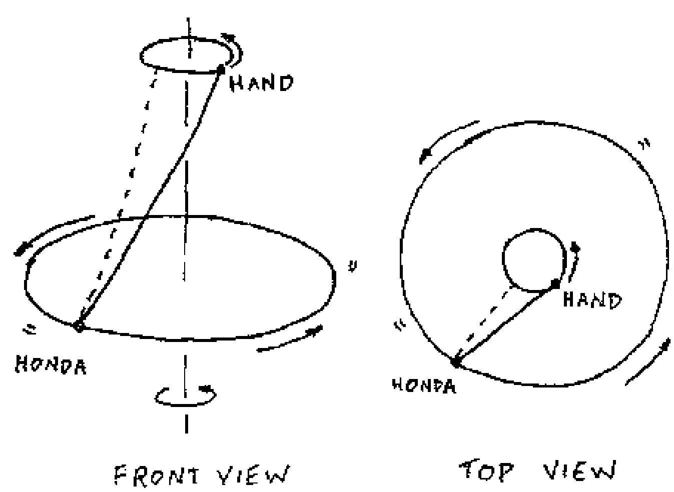
1 Problem

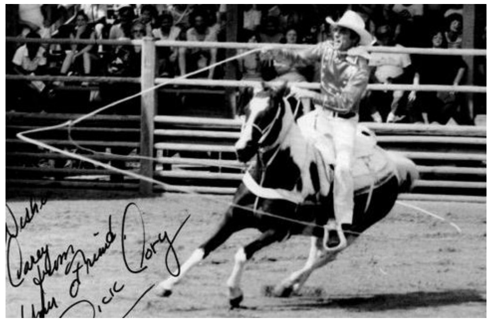
A lasso is a rope of linear mass density ρ that ends in a loop/honda; the other end of the rope is feed through the honda to create a large loop (the noose). The remaining length of rope is called the spoke. See, for example, [1].



1. Consider the case that a circular noose of radius r is spinning a (nearly) horizontal plane with angular velocity ω. The spoke is supported above the center of the noose and makes angle θ to the vertical.[1] There is no friction between the rope and the honda. What is the tension T in the noose? What should the mass m_h of the honda be such that the force of the spoke on the honda does not perturb the shape of the noose, as seen in the figure on the next page?

[1]In practice the point of support of the spoke is driven in a small circle. See, for example, sec. 1.4 of [1].





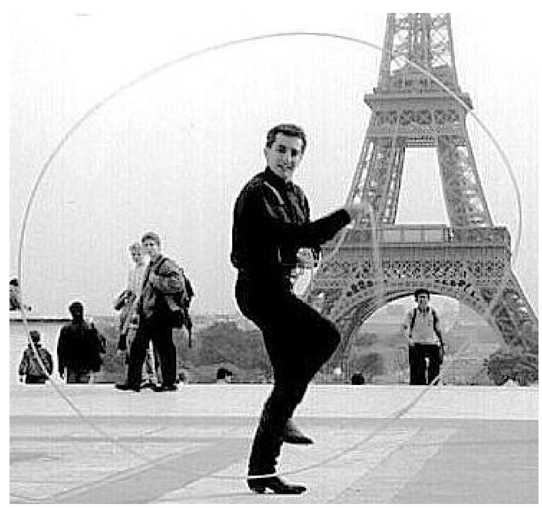
2. Suppose the noose of part 1 is subject to a transverse disturbance that results in a wave which propagates opposite (in the frame of the noose) to the sense of rotation of the noose (in the lab frame). What is the angular velocity of propagation of this disturbance in the lab frame? What does this imply about the shape of the perturbed, spinning noose in the lab frame?

3. The plane of the noose in parts 1 and 2 is not actually horizontal. For a better approximation to the motion of a lasso spun "horizontally", suppose a noose of radius r and mass $m_n = 2\pi r\rho$, as found in part 1, is spun with angular velocity ω about the vertical while supported by a spoke of length $L = r/\sin\theta$ and mass $m_s = L\rho$ such that the center of the hoop is constrained to be directly below the upper end of the spoke. Deduce an expression for the (small) angle α of the plane of the hoop to the horizontal.

4. Suppose now the noose is spun in a vertical plane that also contains the spoke, such that the angular velocity ω of the (circular) noose is constant. In this case the center of mass of the noose cannot be at rest. Show that the desired motion of the notion is possible if the end of the spoke is forced to move in a small vertical circle of radius a at constant angular velocity ω.

   Consider the case that the radius r of the noose equals the length L of the spoke, and $a \ll r$. How must the cowboy adjust the tension in the spoke (close to the honda) as a function of its angle? What is the minimum value of ω, below which the noose would collapse? What should the mass m_h of the honda be such that the force of the spoke on the honda does not perturb the shape of the noose?

The author of [1] performing the "Texas skip":



2 Solution

1. Let T be the tension of the rope in the noose, and therefore also in the spoke just on the other side of the honda. An angular segment $d\phi$ of the spinning noose has mass $dm = \rho r\ d\phi$, so the inward radial force $T\ d\phi$ on this segment obeys,

$$T\ d\phi = dm\omega^2 r = \rho\omega^2 r^2\ d\phi, \qquad i.e., \qquad T = \rho\omega^2 r^2. \tag{1}$$

   The tension T in the spoke just above the honda exerts an inward radial force of $T\sin\theta$.

   If the honda has no mass other than the linear mass density ρ of the rope, the horizontal component $T\sin\theta$ of tension in the spoke would deform the noose inwards at the honda. However, if the honda has mass m_h (due, say, to wrapping the honda with a heavy metal wire), it will execute uniform circular motion in the horizontal plane of the noose (and transmit no horizontal force to the noose) provided,

$$m_h\omega^2 r = T\sin\theta = \rho\omega^2 r^2 \sin\theta, \qquad i.e., \qquad m_h = \rho r\sin\theta. \tag{2}$$

   The weight $2\pi\rho r g$ of the noose plus the added weight $\rho r g\sin\theta$ of the honda is supported by the vertical component of the tension $T\cos\theta = \rho\omega^2 r^2\cos\theta$ in the spoke just above that honda. That is,

$$\rho\omega^2 r^2\cos\theta = (2\pi + \sin\theta)\rho r g. \tag{3}$$

   and so,

$$r = \frac{(2\pi + \sin\theta)g}{\omega^2\cos\theta}. \tag{4}$$

   Thus, the size of the noose depends on the angular velocity ω and the angle θ of the spoke.

2. The speed c of transverse waves on the noose is, recalling eq. (3),

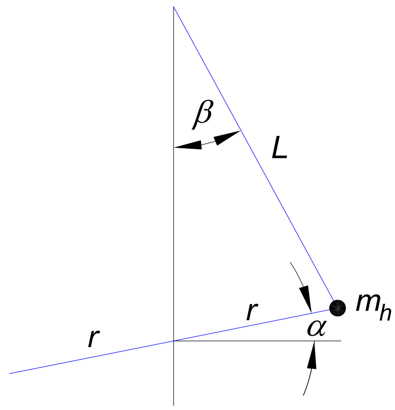
$$c = \sqrt{\frac{T}{\rho}} = \omega r. \tag{5}$$

   Thus, the angular velocity ω_p with respect to the lab frame of a radial perturbation that moves opposite to the sense of rotation of the noose is,

$$\omega_p = \frac{c}{r} - \omega = 0. \tag{6}$$

   Hence, to an observer in the lab frame, the shape of the perturbed noose does not change with time, although the noose is "rotating through the perturbation".

3. A vertical section of a lasso with an almost horizontal noose is sketched below.



Assuming that the center of the noose is under point of support of the spoke, and that the length L of the spoke is related to the radius r of the noose by $L = r/\sin\theta$, the angles α and $\beta \approx \theta$ are related by,

$$L\sin\beta = r\cos\alpha, \quad \sin\beta = \frac{r}{L}\cos\alpha = \sin\theta\cos\alpha, \quad \cos\beta = \sqrt{1-\sin^2\theta\cos^2\alpha}, \tag{7}$$

$$\dot\beta = -\frac{\sin\theta\sin\alpha}{\sqrt{1-\sin^2\theta\cos^2\alpha}}\dot\alpha, \tag{8}$$

and the angle α can be taken as the single coordinate of the system. The equation of motion for the system can be gotten from Lagrange's equation,

$$\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot\alpha} = \frac{\partial\mathcal{L}}{\partial\alpha}, \qquad \text{where} \qquad \mathcal{L} = \text{KE} - \text{PE}. \tag{9}$$

The potential energy of the system, relative to the point of support of the spoke is,

$$\begin{aligned} \text{PE} &= -\frac{m_s g L\cos\beta}{2} - m_n g(L\cos\beta + r\sin\alpha) - m_h g L\cos\beta \\ &= -\left(\frac{m_s}{2} + m_n + m_h\right)\frac{gr}{\sin\theta}\sqrt{1-\sin^2\theta\cos^2\alpha} - m_n g r\sin\alpha. \end{aligned} \tag{10}$$

The kinetic energy of the spoke, assuming the lasso rotates about the vertical with fixed angular velocity ω, is,

$$\text{KE}_s = \frac{m_s(L\sin\beta)^2\omega^2/3}{2} + \frac{m_s L^2\dot\beta^2/3}{2} = \frac{m_s r^2\omega^2\cos^2\alpha}{6} + \frac{m_s r^2\sin^2\alpha\,\dot\alpha^2}{6(1-\sin^2\theta\cos^2\alpha)}, \tag{11}$$

the kinetic energy of the honda is,

$$\text{KE}_h = \frac{m_h(L\sin\beta)^2\omega^2}{2} + \frac{m_h L^2\dot\beta^2}{2} = \frac{m_h r^2\omega^2\cos^2\alpha}{2} + \frac{m_h r^2\sin^2\alpha\,\dot\alpha^2}{2(1-\sin^2\theta\cos^2\alpha)}, \tag{12}$$

and the kinetic energy of the noose is,

$$\begin{aligned} \text{KE}_n &= \frac{m_n(r\cos\alpha\,\dot\alpha - L\sin\beta\,\dot\beta)^2}{2} + \frac{\boldsymbol{\omega}\cdot\mathsf{I}_n\cdot\boldsymbol{\omega}}{2} \\ &= \frac{m_n r^2\cos^2\alpha\,\dot\alpha^2}{2}\left(1 + \frac{\sin\theta\sin\alpha}{\sqrt{1-\sin^2\theta\cos^2\alpha}}\right)^2 + \frac{\boldsymbol{\omega}\cdot\mathsf{I}_n\cdot\boldsymbol{\omega}}{2}, \end{aligned} \tag{13}$$

where the inertia tensor I_n is diagonal with respect to the body axes of the noose, whose symmetry axis we label as 3. That is, $I_{11} = I_{22} = m_n r^2/2$ while $I_{33} = m_n r^2$. The angular velocity vector with respect to the body axes is $\boldsymbol{\omega} = (\dot{\alpha} + \omega \sin\alpha, 0, \omega\cos\alpha)$, and so,

$$\begin{aligned} \mathrm{KE}_n &= \frac{m_n r^2}{2}\left\{\frac{\dot{\alpha}^2 + 2\dot{\alpha}\omega\sin\alpha + \omega^2\sin^2\alpha}{4}\right. \\ &\quad \left. + \cos^2\alpha\left[\omega^2 + \dot{\alpha}^2\left(1 + \frac{\sin\theta\sin\alpha}{\sqrt{1-\sin^2\theta\cos^2\alpha}}\right)^2\right]\right\} \end{aligned} \tag{14}$$

For steady motion (with $\dot{\alpha} = 0 = \ddot{\alpha}$), eq. (9) reduces to,

$$\frac{\partial \mathcal{L}}{\partial \alpha} = \frac{\partial \mathrm{KE}(\dot{\alpha}=0)}{\partial \alpha} - \frac{\partial \mathrm{PE}}{\partial \alpha} = 0. \tag{15}$$

From eqs. (11)-(16) and (14) we have that,

$$\begin{aligned} \mathrm{KE}(\dot{\alpha}=0) &= \frac{(m_h + m_s/3) r^2\omega^2\cos^2\alpha}{2} + \frac{m_n r^2\omega^2}{2}\left(\frac{\sin^2\alpha}{4} + \cos^2\alpha\right) \\ &= \frac{(m_h + m_s/3) r^2\omega^2\cos^2\alpha}{2} + \frac{m_n r^2\omega^2}{8}(1 + 3\cos^2\alpha). \end{aligned} \tag{16}$$

Using eqs. (10) and (16) in eq. (15), we find,

$$\begin{aligned} &-(m_h + m_s/3) r^2\omega^2\cos\alpha\sin\alpha - \frac{3 m_n r^2\omega^2\cos\alpha\sin\alpha}{4} \\ &= -\left(\frac{m_s}{2} + m_n + m_h\right)\frac{gr}{\sin\theta}\frac{\sin^2\theta\cos\alpha\sin\alpha}{\sqrt{1-\sin^2\theta\cos^2\alpha}} - m_n g r\cos\alpha. \end{aligned} \tag{17}$$

This has the trivial solution that $\cos\alpha = 0$, in which case the noose rotates in a vertical plane. The nontrivial solution (for small angle α) is, recalling eq. (4) and that $m_s = m_n/2\pi\sin\theta$ and $m_h = \sin\theta\, m_n/2\pi$,

$$\begin{aligned} \alpha &\approx \frac{m_n g}{\omega^2 r(3m_n/4 + m_h + m_s/3) - (m_n + m_h + m_s/2) g\tan\theta} \\ &= \frac{\cos\theta}{(2\pi + \sin\theta)(3/4 + \sin\theta/2\pi + 1/6\pi\sin\theta) - \sin\theta - \sin^2\theta/2\pi - 1/4\pi}. \end{aligned} \tag{18}$$

Example: For $\theta = 45^\circ$, $\alpha \approx 7^\circ$.

An analysis of oscillations about the equilibrium value of α for small θ is given in [2].

4. We first assume that the end of the spoke is driven in a very small vertical circle, and that the radius r of the noose equals the length L of the spoke, such that the center of the noose is essentially at the end of the spoke, which is nearly at rest.

   If the noose is to move in a vertical circle of radius r at angular velocity ω, then the tension in the noose must vary with angle ϕ, defined to be $\pi/2$ at the top of the noose.

Assuming that the center of the noose is at rest, the inward radial force on a segment of the noose of angular width $d\phi$ must be $dm\,\omega^2 r = \rho\omega^2 r^2\,d\phi$, such that,

$$T\,d\phi + \rho g r \sin\phi\,d\phi = \rho\omega^2 r^2\,d\phi, \qquad i.e., \qquad T(\phi) = \rho\omega^2 r^2 - \rho g r \sin\phi. \tag{19}$$

Since the tension is positive everywhere, we must have that,

$$\omega > \sqrt{\frac{g}{r}}\,, \tag{20}$$

i.e., the angular velocity must be greater than that of a simple pendulum of length r.

The preceding argument appears to conflict with the following: The tangential force on a segment of angular width $d\phi$ must be zero if the angular velocity ω is to be constant,

$$T'\,d\phi - \rho g r \cos\phi\,d\phi = 0, \qquad i.e., \qquad T(\phi) = T_0 + \rho g r \sin\phi. \tag{21}$$

The spoke exerts an inward force $T(\phi)$ on the noose at the position of the honda, taken to be angle ϕ. If the honda has no mass other than the linear mass density ρ of the rope, the tension of the spoke would deform the noose inwards at the honda. However, if the honda has mass m_h, it will execute uniform circular motion (and transmit no force to the noose) provided,

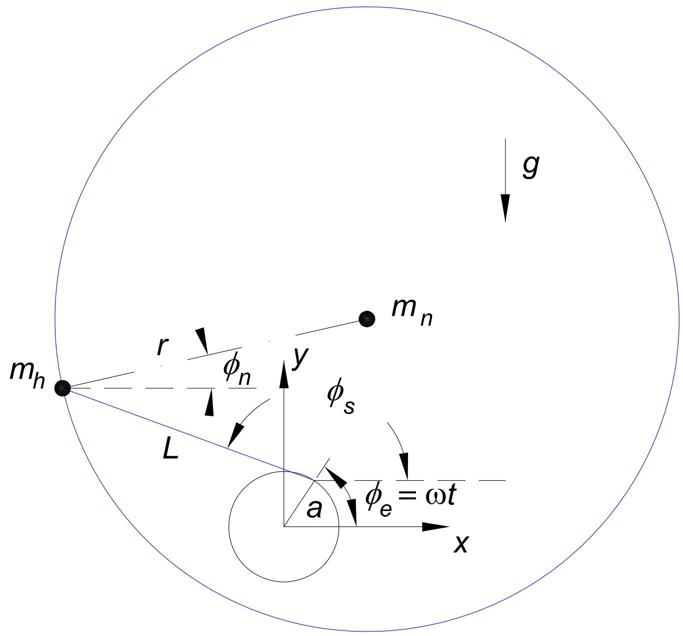
$$m_h\omega^2 r = T(\phi) + m_h g\cos\phi = \rho\omega^2 r^2 - \rho g r\cos\phi + m_h g\cos\phi, \tag{22}$$

which implies that we must have,

$$m_h = \rho r. \tag{23}$$

The weight of the honda appropriate for a vertical loop is somewhat larger than that in eq. (2) for a horizontal loop of the same radius.

We now give a more detailed analysis of the motion of the lasso in a vertical plane using a coordinate system with origin at the center of the circle of radius a around which the end of the spoke is driven at constant angular velocity ω. The line from the origin to the end of the spoke makes angle $\phi = \omega t$ now measured with respect to the horizontal x-axis.



The coordinates of the end of the spoke are,

$$x_e = a\cos\omega t, \qquad y_e = a\sin\omega t. \tag{24}$$

The coordinates and velocity of the center of the spoke are,

$$x_s = x_e + \frac{L}{2}\cos\phi_s, \qquad y_s = y_e + \frac{L}{2}\sin\phi_s, \tag{25}$$

$$v_s^2 = \dot{x}_s^2 + \dot{y}_s^2 = a^2\omega^2 + \frac{L^2\dot{\phi}_s^2}{4} + aL\omega\dot{\phi}_s\cos(\phi_s - \omega t)\,, \tag{26}$$

where ϕ_s is the angle of the spoke with respect to the horizontal. Similarly, the coordinates and velocity of the honda are,

$$x_h = x_e + L\cos\phi_s, \qquad y_h = y_e + L\sin\phi_s, \tag{27}$$

$$v_h^2 = \dot{x}_h^2 + \dot{y}_h^2 = a^2\omega^2 + L^2\dot{\phi}_s^2 + 2aL\omega\dot{\phi}_s\cos(\phi_s - \omega t). \tag{28}$$

Also, the coordinates and velocity of the center of the (circular) noose are,

$$x_n = x_h + r\cos\phi_n, \qquad y_n = y_h + r\sin\phi_n, \tag{29}$$

$$v_n^2 = a^2\omega^2 + L^2\dot{\phi}_s^2 + r^2\dot{\phi}_n^2 + 2aL\omega\dot{\phi}_s\cos(\phi_s - \omega t) + 2ar\omega\dot{\phi}_n\cos(\phi_n - \omega t) + 2rL\dot{\phi}_n\dot{\phi}_s\cos(\phi_n - \phi_s). \tag{30}$$

where ϕ_n is the angle of the line from the honda to the center of the noose with respect to the horizontal.

The potential energy of the system is,

$$\begin{aligned} \mathrm{PE} &= g(m_s y_s + m_h y_h + m_n y_n) \qquad (31) \\ &= g\{(m_h + \rho L + 2\pi\rho r)a\sin\omega t + (2m_h + \rho L + 4\pi\rho r)\frac{L}{2}\sin\phi_s + 2\pi\rho r^2\sin\phi_n\}, \end{aligned}$$

noting that the mass of the noose is $m_n = 2\pi\rho r$ and that of the spoke is $m_s = \rho L$. The kinetic energy of the system is,

$$\begin{aligned}
\mathrm{KE} &= \frac{m_s v_s^2}{2} + \frac{I_s \dot{\phi}_s^2}{2} + \frac{m_h v_h^2}{2} + \frac{m_n v_n^2}{2} + \frac{I_n \dot{\phi}_n^2}{2} \\
&= \frac{\rho a^2 L \omega^2}{2} + \frac{\rho L^3 \dot{\phi}_s^2}{6} + \frac{\rho a L^2 \omega}{2} \dot{\phi}_s \cos(\phi_s - \omega t) \\
&\quad + \frac{m_h a^2 \omega^2}{2} + \frac{m_h L^2 \dot{\phi}_s^2}{2} + m_h a L \omega \dot{\phi}_s \cos(\phi_s - \omega t) \\
&\quad + \pi\rho a^2 r \omega^2 + \pi\rho r L^2 \dot{\phi}_s^2 + 2\pi\rho r^3 \dot{\phi}_n^2 + 2\pi\rho a r L^2 \omega \dot{\phi}_s \cos(\phi_s - \omega t) \\
&\quad + 2\pi\rho a r^2 \omega \dot{\phi}_n \cos(\phi_n - \omega t) + 2\pi\rho r^2 L \omega \dot{\phi}_n \cos(\phi_n - \phi_s), \qquad (32)
\end{aligned}$$

noting that the relevant moment of inertia of the spoke is $I_s = m_s L^2/12 = \rho L^3/12$, and that of the noose is $I_n = m_n r^2 = 2\pi\rho r^3$.

We seek a solution for steady motion, $\ddot{\phi}_n = \ddot{\phi}_s = 0$, $\dot{\phi}_n = \dot{\phi}_s = \omega$, in which case the terms $d/dt(\partial\mathcal{L}/\partial\dot{\phi}_n)$ and $d/dt(\partial\mathcal{L}/\partial\dot{\phi}_s)$ vanish in Lagrange's equations for coordinates ϕ_n and ϕ_s. We write $\phi_n = \omega t + \phi_{0n}$ and $\phi_s = \omega t + \phi_{0s}$, such that kinetic energy simplifies to,

$$\begin{aligned}
\mathrm{KE}_{\mathrm{steady}} &= \left(\frac{\rho L}{2} + m_h + 2\pi\rho r\right) a L \omega^2 \cos\phi_{0s} + 2\pi\rho a r^2 \omega^2 \cos\phi_{0n} \\
&\quad + 2\pi\rho r^2 L \omega^2 \cos(\phi_{0n} - \phi_{0s}) + \text{constant}. \qquad (33)
\end{aligned}$$

The system is now described by the two "coordinates" ϕ_{0n} and ϕ_{0s}, for which Lagrange's equations of motion are,

$$\begin{aligned}
\frac{d}{dt}\frac{\partial \mathrm{KE}}{\partial \dot{\phi}_{0n}} &= 0 = \frac{\partial(\mathrm{KE}_{\mathrm{steady}} - \mathrm{PE})}{\partial \phi_{0n}} \\
&= -2\pi\rho a r^2 \omega^2 \sin\phi_{0n} - 2\pi\rho a r^2 \omega^2 \sin(\phi_{0n} - \phi_{0s}) - 2\pi\rho r^2 g \cos\phi_{0n}, \qquad (34)
\end{aligned}$$

and,

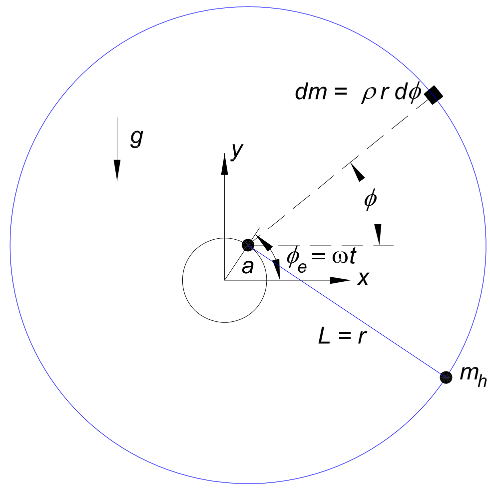
$$\begin{aligned}
\frac{d}{dt}\frac{\partial \mathrm{KE}}{\partial \dot{\phi}_{0s}} &= 0 = \frac{\partial(\mathrm{KE}_{\mathrm{steady}} - \mathrm{PE})}{\partial \phi_{0s}} \\
&= -\left(\frac{\rho L}{2} + m_h + 2\pi\rho r\right) a L \omega^2 \sin\phi_{0s} + 2\pi\rho a r^2 \omega^2 \sin(\phi_{0n} - \phi_{0s}) \\
&\quad -(2m_h + \rho L + 4\pi\rho r)\frac{gL}{2}\cos(\phi_{0s}). \qquad (35)
\end{aligned}$$

A particularly interesting solution would be $L = r$ and $\phi_{0n} = \phi_{0s} + \pi$, for which the center of the noose coincides with the end of the spoke, $(x_n, y_n) = (x_e, y_e)$. Then, eqs. (34)-(35) imply that,

$$\tan\phi_{0n} = -\frac{g}{2a\omega^2} = \tan\phi_{0s} = -\frac{g}{a\omega^2}\,. \qquad (36)$$

This cannot be true in general, but if $a \ll g/\omega^2$ we obtain a solution with $r = L$, $\phi_{0s} \approx -\pi/2$ and $\phi_{0n} \approx \pi/2$.

We return to the issue of the apparent conflict between eqs. (19) and (21), and consider the case that $r = L$, $\phi_{0s} \approx \pi/2 \approx -\phi_{0n}$, as shown below.



An element $dm = \rho\, r\, d\phi$ on the noose at angle $\phi = \omega t + \phi_0$ has coordinates,

$$x = a\cos\omega t + r\cos\phi, \qquad y = a\sin\omega t + r\sin\phi, \tag{37}$$

and acceleration

$$\ddot{x} = -\omega^2 x = -\omega^2(a\cos\omega t + r\cos\phi), \qquad \ddot{y} = -\omega^2 y = -\omega^2(a\sin\omega t + r\sin\phi). \tag{38}$$

The inward and tangential components of the acceleration (with respect to the center of the noose) are,

$$a_{\rm in} = -\ddot{x}\cos\phi - \ddot{y}\sin\phi = \omega^2[r + a\cos(\phi - \omega t)], \tag{39}$$
$$a_\phi = -\ddot{x}\sin\phi + \ddot{y}\cos\phi = \omega^2 a\sin(\phi - \omega t). \tag{40}$$

The equations of motion for the element dm are,

$$\rho r\, d\phi\, a_{\rm in} = \rho r\, d\phi\, \omega^2[r + a\cos(\phi - \omega t)] = T\, d\phi + \rho g\, d\phi\, \sin\phi, \tag{41}$$
$$\rho r\, d\phi\, a_\phi = \rho r\, d\phi\, \omega^2 a\sin(\phi - \omega t) = T'\, d\phi - \rho g\, d\phi\, \cos\phi. \tag{42}$$

The radial equation of motion (41) leads to,

$$T(\phi) = \rho r\omega^2[r + a\cos(\phi - \omega t)] - \rho g\sin\phi, \tag{43}$$

which differs only slightly from eq. (19) for $a \ll r$. The azimuthal equation of motion (42) integrates to,

$$T(\phi) = T_0 - \rho r\omega^2 a\cos(\phi - \omega t)] + \rho g\sin\phi, \tag{44}$$

so we again have trouble?????

References

[1] C. Bunks, *The Lasso* (July 8, 1996),
http://kirkmcd.princeton.edu/examples/mechanics/bunks_lasso.pdf

[2] F. Moisy, *Supercritical bifurcation of a spinning hoop*, Am. J. Phys. **71**, 999 (2003),
http://kirkmcd.princeton.edu/examples/mechanics/moisy_ajp_71_999_03.pdf

[3] P.-T. Brun, N. Ribe and B. Audoly, *An introduction to the mechanics of the lasso*, Proc. Roy. Soc. London A **470**, 20140512 (2014),
http://kirkmcd.princeton.edu/examples/mechanics/brun_prsla_470_20140512.pdf