

# Can a Bicycle Speed Up by Leaning?

Kirk T. McDonald

*Joseph Henry Laboratories, Princeton University, Princeton, NJ 08544*

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## 1 Problem

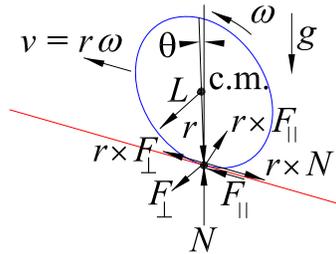
It was claimed in Prob. 6.6 of [1] that a bicycle speeds up when it is leaned (without pedaling or steering). Can this be so?

## 2 Solution

The answer is YES, but a naïve application of Newtonian mechanics suggests that the answer is NO.

For the velocity of the bicycle to increase during leaning of the bicycle to, say, the left, it naïvely seems that there must be a force on it in the direction of the forward velocity.

As the disk leans away from the vertical, a frictional force  $\mathbf{F}$  arises at the point of contact of the disk with the ground, as sketched below supposing the the component  $\mathbf{F}_{\parallel}$  is in the direction of forward motion of the center of mass of the disk.



In this case, the associated torque  $\mathbf{r} \times \mathbf{F}_{\parallel}$  about the center of mass of the disk (of radius  $r$ ) would point to the right (with respect to the forward direction), implying that the angular velocity  $\omega$  of the disk about its center (and hence also the forward velocity,  $r\omega$ , of the center of the disk) would be decreasing, rather than increasing. This contradiction suggests that the forward velocity of the disk cannot increase during the leaning.

However, the above argument is too naïve.

An early, detailed analysis of rolling without slipping of a thin disc on a horizontal surface was given by Routh in Art. 244 of [2], and was discussed by the present author in [3].

A subtle consequence of the condition of rolling without slipping was noted on p. 4 of [3]. Namely, if the disk is falling/leaning to the left ( $\theta$  increasing in the figure above) while the trajectory of the disk curves to the left, then the acceleration of the center of mass along the forward direction can be negative. In this case, the friction force  $\mathbf{F}_{\parallel}$  points backwards (with also a component  $\mathbf{F}_{\perp}$  to the left to push the disk into a left turn), and the torque  $\mathbf{r} \times \mathbf{F}_{\parallel}$  increases the angular velocity  $\omega$  of the disk about its axis, as required for an increasing forward velocity,  $r\omega$ , of the center of the disk.<sup>1</sup>

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<sup>1</sup>See also footnote 3, p. 6 of [3]. Note that the present  $\theta$ ,  $r$  and  $\omega$  are  $\pi/2 - \alpha$ ,  $a$  and  $\omega_1$  in [3].



The contact forces of the two wheels with the ground have horizontal components  $F_i$ , and vertical components  $N_i$  for  $i = 1, 2$ , with directions as sketched in the figure, which anticipates that  $F_1$  points to the right while  $F_2$  points to the left.

We suppose the (massless) drive gear wheel of the bicycle, of radius  $r < R$ , is centered on the center of mass. It is connected by a (massless) chain to a (massless) gear wheel, also of radius  $r$  on the rear wheel, 2, of the bicycle.

The force in the horizontal strut between the center of mass and the center of wheel 1(2) is  $F_{3(4)}$ , and the tension in the upper segment of the chain between the drive gear wheel and the gear wheel on wheel 2 is  $F_5$ .

The force balance at the (massless) drive gear wheel is that,

$$F_3 - F_4 + F_5 = 0. \quad (2)$$

We suppress the details of the mechanism (involving a person) of the drive gear wheel.

The wheels move horizontally, and the normal forces  $N_i$  do not enter into the analysis of the motion of the two wheels of the bicycle.

For wheel 1,  $\mathbf{F} = m\mathbf{a}$  is simply,

$$F_3 - F_1 = ma, \quad (3)$$

and the torque equation about the center of mass of wheel 1 is,

$$\tau_1 = RF_1 = I_1\alpha = mR^2\frac{a}{R} = mRa, \quad (4)$$

and hence, recalling eq. (2),

$$F_1 = ma, \quad F_3 = 2ma = F_5 - F_4. \quad (5)$$

For wheel 2,  $\mathbf{F} = m\mathbf{a}$  tells us that,

$$F_2 - F_4 + F_5 = ma, \quad (6)$$

and the torque equation about the center of mass of wheel 2 is,

$$\tau_2 = rF_5 - RF_2 = I_2\alpha = mR^2\frac{a}{R} = mRa, \quad (7)$$

and hence, recalling eq. (5),

$$F_2 = ma + F_5 - F_4 = 3ma, \quad F_5 = 4ma\frac{R}{r}, \quad F_4 = F_2 + F_5 - ma = 2ma\left(1 + \frac{2R}{r}\right). \quad (8)$$

This confirms that  $F_1$  points to the right while  $F_2 = 3F_1$  points to the right.

That is, the drive mechanism causes the rear wheel to rotate counterclockwise, leading to the reaction force  $F_2$  that points to the left. The only torque on the front wheel is due to the frictional force  $F_1$ , which must point to the right for the angular velocity of this wheel to increase as the bicycle accelerates to the left.

### A.2.2 Analysis of the Bicycle as a Whole

Considering the bicycle as a whole, the only relevant (external) forces are  $F_1$ ,  $F_2$ ,  $N_1$ ,  $N_2$  and the downward force of gravity,  $2mg$ .

The horizontal force equation is,

$$2ma = F_2 - F_1, \quad (9)$$

which is consistent with eqs. (5) and (8).

The vertical equation of motion is simply that,

$$N_1 + N_2 = 2mg. \quad (10)$$

The torque equation about the center of mass of the system is,

$$\tau_{\text{cm}} = \frac{dL_{\text{cm}}}{dt} = 2I\alpha = 2mR^2 \frac{a}{R} = 2mRa = (F_1 - F_2)R + (N_2 - N_1)d, \quad (11)$$

$$N_2 - N_1 = \frac{1}{d}[2mRa + (F_2 - F_1)R] = \frac{4mRa}{d}, \quad (12)$$

$$N_1 = mg - \frac{2maR}{d}, \quad N_2 = mg + \frac{2maR}{d}. \quad (13)$$

The normal force on the front wheel is less than that on the rear, as familiar with rapidly accelerating cars, for which the front end tends to rise.<sup>2</sup> See, for example, sec. 2.2 of [4].

*This problem was suggested by Ralph Wang. Thanks to Jason Moore for e-discussions of this issue.*

## References

- [1] M. Levi, *Why Cats Land on Their Feet, and 76 Other Physical Paradoxes and Puzzles*, (Princeton U. Press, 2012), [http://kirkmcd.princeton.edu/examples/mechanics/levi\\_12.pdf](http://kirkmcd.princeton.edu/examples/mechanics/levi_12.pdf)
- [2] E.J. Routh, *The Advanced Part of a Treatise on the Dynamics of a System of Rigid Bodies*, 6<sup>th</sup> ed. (Macmillan, 1905; reprinted by Dover Publications, 1955), [http://kirkmcd.princeton.edu/examples/mechanics/routh\\_advanced\\_rigid\\_dynamics.pdf](http://kirkmcd.princeton.edu/examples/mechanics/routh_advanced_rigid_dynamics.pdf)
- [3] A.J. McDonald and K.T. McDonald, *The Rolling Motion of a Disk on a Horizontal Plane* (Mar. 8, 2001), <http://kirkmcd.princeton.edu/examples/rollingdisk.pdf>
- [4] K.T. McDonald, *Rocket Car* (Oct. 1, 2012), <http://kirkmcd.princeton.edu/examples/rocketcar.pdf>

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<sup>2</sup>Conversely, for rapidly decelerating cars (and bicycles) the rear end tends to rise.