Right and Wrong Use of the Lenz Vector for Non-Newtonian Potentials

Kirk T. McDonald
Joseph Henry Laboratories, Princeton University, Princeton, NJ 08544
Carlos Farina and Alexandre Tort
Instituto de Fisica, Universidado Federal do Rio de Janeiro,
Ilha do Fundao, Rio de Janeiro, CEP 21945, RJ, Brasil
(May 23, 1989)

Abstract

A paper that recently appeared in this Journal [1] propagates an error first published here more than 20 years ago [2]. This concerns a proposed use of the Lenz vector [3, 4] in obtaining the rate of precession of the perihelion for a central-force problem in which the potential has a small departure from the Newtonian form, $V(r) = -k/r$. Three correct solutions to this problem are presented, based on the effective-potential technique, on the method of averages, and on a careful use of the Lenz vector [5].

1 The Effective-Potential Technique

We consider the precession of the orbit of a particle moving under the influence of a central force that departs slightly from the Newtonian form. Exact solutions are not generally available in this case, so we seek approximate methods. In this section we use the effective-potential technique to express the noncircular orbit in the approximate form,

$$r \approx r_0(1 + \epsilon \cos \alpha t), \tag{1}$$

which is an accurate representation only if $\epsilon \ll 1$, due to the neglect of higher-order terms. We believe this technique to be the most reliable in dealing with pertubations from circular orbits, but leave the reader to judge its merits compared to other methods presented in the following sections [6].

For motion under a central force,

$$\mathbf{F} = -G(r)\,\hat{\mathbf{r}}, \tag{2}$$

the total energy E is conserved,

$$E = \frac{1}{2}m\dot{r}^2 + \frac{L^2}{2mr^2} + \int^r G(s)ds, \tag{3}$$

where the dependence on $\dot{\theta}$ has been eliminated via the conserved angular momentum $L = mr^2\dot{\theta}$. This reduces the problem of radial motion to that caused by the effective potential,

$$V_{\text{eff}}(r) = \frac{L^2}{2mr^2} + \int^r G(s)ds. \tag{4}$$

In this approach it is useful to think of the orbit parameters r_0 and ϵ in eq. (1) as dependent on the constants of the motion E and L.

For a specified value of angular momentum L, there is a circular orbit at constant r_0 given by,

$$\frac{dV_{\rm eff}(r_0)}{dr} = -\frac{L^2}{mr_0^3} + G(r_0) = 0. \tag{5}$$

Introducing Ω as the angular velocity of the equilibrium circular orbit, we have,

$$L = mr_0^2\Omega, \tag{6}$$

and,

$$G(r_0) = \frac{L^2}{mr_0^3} = mr_0\Omega^2. \tag{7}$$

On expanding the effective potential about r_0 and supposing that $r - r_0$ is small, the equation of motion becomes,

$$m\ddot{r} = F_{\rm eff}(r) = -\frac{dV_{\rm eff}(r)}{dr} \approx -\frac{d^2V_{\rm eff}(r_0)}{dr^2}(r - r_0). \tag{8}$$

For motion that departs from the circular orbit, we have oscillatory solutions whose radial dependence has the form (1). The associated angular motion is obtained via conservation of angular momentum,

$$\theta \approx \Omega t - 2\epsilon\frac{\Omega}{\alpha}\sin\alpha t, \tag{9}$$

The angular frequency α of the oscillatory motion is found from (8) as,

$$\alpha = \sqrt{\frac{d^2V_{\rm eff}(r_0)}{mdr^2}}\,. \tag{10}$$

On use of eqs. (6) and (7) in eq. (10) we find,

$$\alpha = \Omega\sqrt{3 + \frac{r_0G'(r_0)}{G(r_0)}}\,. \tag{11}$$

For $G(r) \approx k/r^2$, α is approximately equal to Ω and we may expand,

$$\alpha = \Omega\sqrt{1 + \left(2 + \frac{r_0G'(r_0)}{G(r_0)}\right)} \approx \Omega\left(2 + \frac{r_0G'(r_0)}{2G(r_0)}\right). \tag{12}$$

The perihelion of the noncircular orbit, eq. (1), precesses with angular velocity ω given by,

$$\omega = \Omega - \alpha \approx -\Omega\left(1 + \frac{r_0G'(r_0)}{2G(r_0)}\right). \tag{13}$$

The problem at hand is the case of a small departure from the Newtonian force law,

$$\mathbf{F} = -\frac{k}{r^2}\hat{\mathbf{r}} + B(r)\,\hat{\mathbf{r}}, \tag{14}$$

Then eq. (13) leads to the angular velocity of precession as,

$$\omega = \frac{B(r_0)}{mr_0\Omega} + \frac{B'(r_0)}{2m\Omega}\,. \tag{15}$$

2 Erroneous Use of the Lenz Vector

Although the above approach follows the basic line of attack in all problems of small oscillations about equilibrium, it is apparently too lengthy for some people's taste. Greenberg [2] has proposed a much quicker derivation based on the Lenz vector.

We recall that the Lenz vector,

$$\mathbf{A} = \hat{\mathbf{r}} + \frac{\mathbf{L} \times \mathbf{p}}{km}\,, \tag{16}$$

where $\mathbf{L}$ and $\mathbf{p}$ are the angular and linear momentum, respectively, is a constant of the motion in the case of the central force,

$$\mathbf{F} = -\frac{k}{r^2}\,\hat{\mathbf{r}}. \tag{17}$$

As noted by Greenberg [2] the constancy of $\mathbf{A}$ is readily verified with the useful relation that,

$$\frac{d\hat{\mathbf{r}}}{dt} = \frac{\mathbf{L}}{mr^2} \times \hat{\mathbf{r}}. \tag{18}$$

We now consider the effect of a small correction, $B(r)$, to the Newtonian force, as in eq. (14). We again desire an expression for $\boldsymbol{\omega}$, the rate of precession of the perihelion of the orbit. The vector $\boldsymbol{\omega}$ is of course parallel to the angular momentum $\mathbf{L}$.

The hypothesis of Greenberg is that $\boldsymbol{\omega}$ can be found by supposing the vector $\mathbf{A}$ rotates with angular velocity $\boldsymbol{\omega}$. While this hypothesis is sound, the method used by Greenberg to evaluate $\boldsymbol{\omega}$ is inaccurate.

Here we follow the derivation as given by Greenberg to compare with the standard result (15). We look for a time dependence of $\mathbf{A}$ of the form,

$$\frac{d\mathbf{A}}{dt} = \boldsymbol{\omega} \times \mathbf{A}. \tag{19}$$

Direct differentiation of eq. (16), combined with eqs. (14) and (18), leads to,

$$\frac{d\mathbf{A}}{dt} = \frac{B(r)}{km}\mathbf{L} \times \hat{\mathbf{r}}, \tag{20}$$

while we also find by direct substitution into eq. (16) that,

$$\boldsymbol{\omega} \times \mathbf{A} = \boldsymbol{\omega} \times \hat{\mathbf{r}} - \frac{\boldsymbol{\omega} \cdot \mathbf{L}}{km}\,\mathbf{p}. \tag{21}$$

The argument of Greenberg is that $\mathbf{p}$ averages to zero over a closed orbit, and that $B(r)$ in eq. (20) may be averaged to $B(r_0)$, leading to,

$$\boldsymbol{\omega} = \frac{B(r_0)}{km}\,\mathbf{L}, \qquad \text{and} \qquad \omega = \frac{B(r_0)}{mr_0\Omega}\,. \tag{22}$$

We see that the result (22) bears some resemblance to the correct form (15), but the dependence of ω on the derivative of the force field is missing in eq. (22). The authors of [1, 2] may have been misled by the coincidence that for the case $B(r) = C/r^4$, the absolute values of ω found with eqs. (15) and (22) are the same, although the signs are opposite. Thus, the second term in eq. (15) is actually the larger for some cases of interest, and its absence cannot be excused as neglect of a small correction.

3 Correct Use of the Lenz Vector

A correct use of the Lenz vector for non-Newtonian potentials has been presented in this Journal by Sivardière [5], without mention of the attempt by Greenberg [2] to solve the same problem. Here we recast the argument of Sivardière into the notation used above.

First, it is useful to note that for the Newtonian potential, $V = -k/r$, the (constant) Lenz vector $\mathbf{A}$ of eq. (16) is oriented along the major axis of the elliptical orbit, and has magnitude equal to the eccentricity, ϵ, of the orbit. To see this, we introduce θ as the angle between $\mathbf{A}$ and radius vector $\mathbf{r}$, and evaluate the scalar product,

$$\mathbf{A} \cdot \mathbf{r} = Ar\cos\theta = r + \frac{\mathbf{L} \times \mathbf{p} \cdot \mathbf{r}}{km} = r - \frac{L^2}{km}, \tag{23}$$

using eq. (16). On solving for r, we find,

$$r = \frac{L^2}{km(1 - A\cos\theta)}, \tag{24}$$

which verifies the interpretation of $\mathbf{A}$ as the "eccentricity vector", with the magnitude A being equal to the eccentricity ϵ of the orbit.

We now re-examine the argument of sec. 2 beginning at eq. (20). Greater care is required in calculating the time average of this expression. First, we decompose,

$$\hat{\mathbf{r}} = \hat{\boldsymbol{\epsilon}}\cos\theta + \hat{\boldsymbol{\delta}}\sin\theta, \tag{25}$$

where $\hat{\boldsymbol{\epsilon}}$ is a unit vector along the intial direction of $\mathbf{A}$, and $\hat{\boldsymbol{\delta}} = \hat{\mathbf{L}} \times \hat{\boldsymbol{\epsilon}}$ is a unit vector orthogonal to $\hat{\boldsymbol{\epsilon}}$ and $\mathbf{L}$. Then, the time average of eq. (20) is,

$$\left\langle \frac{d\mathbf{A}}{dt} \right\rangle = \frac{\mathbf{L}}{km} \times \hat{\boldsymbol{\epsilon}} \left\langle B(r)\cos\theta \right\rangle + \frac{\mathbf{L}}{km} \times \hat{\boldsymbol{\delta}} \left\langle B(r)\sin\theta \right\rangle. \tag{26}$$

We will evaluate the time average of a function $f(t)$ as,

$$\langle f(t) \rangle = \frac{1}{T}\int_0^T f(t)\,dt = \frac{1}{T}\int_0^{2\pi} f(\theta)\frac{dt}{d\theta}\,d\theta = \frac{\Omega m}{2\pi L}\int_0^{2\pi} r^2 f\,d\theta, \tag{27}$$

where $T = 2\pi/\Omega$ is the period of the orbit. The use of 2π as the period of the orbit in angle θ is not strictly correct for the precessing orbits under consideration which limits the accuracy of the method of this section independent of the approximations (28)-(29) used below.

Following the approximation of eq. (1), we write,

$$r \approx r_0(1 + \epsilon\cos\theta), \tag{28}$$

and so,

$$B(r) \approx B(r_0) + (r - r_0)B'(r_0) \approx B(r_0) + \epsilon r_0\cos\theta B'(r_0). \tag{29}$$

On inserting eqs. (27)-(29) into eq. (26), only the integrals in $\cos^2\theta$ will be nonzero, leading to,

$$\left\langle \frac{d\mathbf{A}}{dt} \right\rangle = \frac{r_0^2\Omega}{2k}\left[2B(r_0) + r_0B'(r_0)\right]\hat{\mathbf{L}} \times \boldsymbol{\epsilon} = \boldsymbol{\epsilon} \times \mathbf{A}, \tag{30}$$

with,

$$\omega = \frac{r_0^2 \Omega}{2k} \left[2B(r_0) + r_0 B'(r_0) \right]. \tag{31}$$

Noting that $k \approx m\Omega^2 r_0^3$, we obtain the result (15) of sec. 1.

This vindicates the insight of Greenberg [2] that the Lenz vector **A** points along the precessing major axis of the orbit of a particle in a non-Newtonian potential. Sivardière [5] notes that the Lenz-vector technique may be used for orbits with arbitrary eccentricity ϵ, although the method is accurate only if the perturbation $B(r)$ is small. (This remark holds for the method of sec. 4 as well.) Sivardière also gives examples of the use of the Lenz vector for orbits perturbed by noncentral forces. It remains that considerable care is required when using the Lenz vector.

4 The Method of Averages

The precession of orbits offers an instructive illustration of the so-called method of averages of Krylov and Bogoliubov [7]. This technique arose from the problem of characterizing the solutions of nonlinear differential equations. When applied to the Kepler problem it proves to be closely related to the averaging of the Lenz vector discussed in sec. 3 above.

The starting point in the present case is the "orbit equation" obtained in the usual way from $\mathbf{F} = m\mathbf{a}$ via the substitution,

$$u = \frac{1}{r}, \tag{32}$$

and after elimination of the time t in favor of polar angle θ,

$$\frac{d^2 u}{d\theta^2} + u = \frac{km}{L^2} - \frac{mB(u)}{L^2 u^2}. \tag{33}$$

Here we have assumed a central force of the form (14). If the small correction B is actually zero we immediately find the standard elliptical orbit,

$$u = \frac{1}{r} = \frac{1}{r_0}(1 - \epsilon \cos\theta), \tag{34}$$

where $r_0 = L^2/km$ is the average radius.

The approach of Krylov and Bogoliubov is to seek a solution when B is nonzero of the form,

$$u = \frac{1}{r_0} \left[1 - \epsilon(\theta) \cos\phi(\theta) \right]. \tag{35}$$

This introduces two unknown functions, ϵ and ϕ, so eq. (33) alone will be insufficient to determine them. For the needed second condition, we ask that the derivative of u with respect to θ have the same form as when B is zero,

$$\frac{du}{d\theta} = \frac{\epsilon}{r_0} \sin\phi. \tag{36}$$

On substituting eq. (36) into eqs. (33), and eq. (35) into eq. (36) we can solve for the derivatives of ϵ and ϕ with respect to θ,

$$\frac{d\epsilon}{d\theta} = -\frac{B(u)}{ku^2} \sin\phi, \tag{37}$$

$$\frac{d\phi}{d\theta} = 1 - \frac{B(u)}{\epsilon k u^2} \cos\phi. \tag{38}$$

The method of averages consists in approximating the right-hand sides of eqs. (37)-(38) by their averages over one period in ϕ.

To implement this, we need a further approximation for the factor B/u^2,

$$\frac{1}{u^2} = \frac{r_0^2}{(1-\epsilon\cos\phi)^2} \approx r_0^2(1+2\epsilon\cos\phi), \tag{39}$$

$$B(u) \approx B(r_0) + (r - r_0)B'(r_0) \approx B(r_0) + \epsilon r_0 \cos\phi B'(r_0), \tag{40}$$

$$\frac{B(u)}{u^2} \approx r_0^2 \left\{B(r_0) + \epsilon\left[2B(r_0) + r_0 B'(r_0)\right]\cos\phi\right\}. \tag{41}$$

From eqs. (37) and (41) we see that ϵ is constant on average over many orbits, as expected. On combining eqs. (38) and (41), and averaging $\cos^2\phi$ to 1/2 we find,

$$\frac{d\phi}{d\theta} \approx 1 - \frac{r_0^2}{2k}\left[2B(r_0) + r_0 B'(r_0)\right]. \tag{42}$$

In the notation of eqs. (1) and (10), we have $\phi = \alpha t$ and $\theta \approx \Omega t$, so $d\phi/d\theta \approx \alpha/\Omega$, and the angular velocity of precession is,

$$\omega = \Omega - \alpha \approx \frac{r_0^2\Omega}{2k}\left[2B(r_0) + r_0 B'(r_0)\right]. \tag{43}$$

Noting that $k \approx m\Omega^2 r_0^3$, eq. (43) reduces to our previous result (15).

References

[1] Carlos Farina and Marcos Machado, *The Rutherford cross section and the perihelion shift of Mercury with the Runge-Lenz vector*, Am. J. Phys. **55**, 921 (1987),
http://kirkmcd.princeton.edu/examples/mechanics/farina_ajp_55_921_87.pdf

[2] David F. Greenberg, *Accidental Degeneracy*, Am. J. Phys. **34**, 1101 (1966),
http://kirkmcd.princeton.edu/examples/mechanics/greenberg_ajp_34_1101_66.pdf

[3] Our use of the term "Lenz vector" is in honor of that designation in its one historically important application in physics: W. Pauli, Jr. Über das Wasserstoffspektrum von Standpunkt der neuen Quantenmechanik, Z. Phys. **36**, 336 (1926),
http://kirkmcd.princeton.edu/examples/QM/pauli_zp_36_336_26.pdf
translated as *On the Hydrogen Spectrum from the Standpoint of the New Quantum Mechanics*, in *Sources of Quantum Mechanics*, ed. by B.L van der Waerden (North-Holland, 1967), http://kirkmcd.princeton.edu/examples/QM/vanderwaerden_67.pdf

[4] The history of multiple discovery of the Lenz vector has been traced by Herbert Goldstein, *Prehistory of the "Runge-Lenz" vector*, Am. J. Phys. **43**, 735 (1975), http://kirkmcd.princeton.edu/examples/mechanics/goldstein_ajp_43_737_75.pdf
More on the prehistory of the Laplace or Runge-Lenz vector, Am. J. Phys. **44**, 1123 (1976), http://kirkmcd.princeton.edu/examples/mechanics/goldstein_ajp_44_1123_76.pdf

[5] Jean Sivardière, *Precession of elliptic orbits*, Am. J. Phys. **52**, 909 (1984), http://kirkmcd.princeton.edu/examples/mechanics/sivardiere_ajp_52_909_84.pdf

[6] The effective-potential technique has been correctly applied to the present problem by two of us in, Carlos Farina and Alexandre Tort, *A simple way of evaluating the speed of precession of orbits*, Am. J. Phys. **56**, 761 (1988), http://kirkmcd.princeton.edu/examples/mechanics/farina_ajp_56_761_88.pdf
However, we did not point out the discrepancy between the result of this paper and that of [1].

[7] N.N. Krylov and N.N. Bogoliubov, *Introduction to Nonlinear Mechanics* (Princeton U. Press, 1943).