Electromagnetic Field Angular Momentum of a Charge at Rest in a Uniform Magnetic Field

Kirk T. McDonald
Joseph Henry Laboratories, Princeton University, Princeton, NJ 08544
(December 21, 2014; updated May 19, 2019)

1 Problem

Discuss the electromagnetic field momentum of a charge at rest in a uniform magnetic field.[1] Comment also on "hidden" angular momentum.

2 Solution

The electromagnetic field momentum can be written for static fields in three equivalent forms for stationary systems,[2]

$$\mathbf{P}_{\rm EM} = \int \mathbf{p}_{\rm EM}\, d{\rm Vol} = \mathbf{P}^{(\rm M)} = \mathbf{P}^{(\rm P)} = \mathbf{P}^{(\rm F)}, \tag{1}$$

where,

$$\mathbf{P}^{(\rm M)} = \int \frac{\varrho \mathbf{A}^{(\rm C)}}{c}\, d{\rm Vol}, \qquad \mathbf{P}^{(\rm P)} = \int \frac{\mathbf{E}\times\mathbf{B}}{4\pi c}\, d{\rm Vol}, \qquad \mathbf{P}^{(\rm F)} = \int \frac{V^{(\rm C)}\mathbf{J}}{c^2}\, d{\rm Vol}, \tag{2}$$

in Gaussian units, where ϱ is the electric charge density, $\mathbf{A}^{(\rm C)}$ is the magnetic vector potential in the Coulomb gauge,[3] $\mathbf{E}$ is the electric field, $\mathbf{B}$ is the magnetic field, $V^{(\rm C)}$ is the electric scalar potential in the Coulomb gauge, and $\mathbf{J}$ is the electric current density.[4] The first form is due to Faraday [7] and Maxwell [8]; the second form is due to Poynting [9] as elaborated upon by J.J. Thomson [10][5,6] and Poincaré [15]; and the third form was introduced by Furry [16].[7]

While the three integrals $\mathbf{P}^{(\rm M)}$, $\mathbf{P}^{(\rm P)}$ and $\mathbf{P}^{(\rm F)}$ are equal for stationary systems, only the integrand $\mathbf{p}^{(\rm P)} = \mathbf{E}\times\mathbf{B}/4\pi c$ (and not $\mathbf{p}^{(\rm M)} = \varrho\mathbf{A}^{(\rm C)}/c$ or $\mathbf{p}^{(\rm F)} = V^{(\rm C)}\mathbf{J}/c^2$) has the interpretation of the volume density of electromagnetic field momentum in a "Maxwellian" field-theory view.

---

[1] Aspects of this problem have been considered in [1]-[3], assuming that $\mathbf{L}^{(\rm P)}$ is physical for an infinite solenoid.

[2] See Appendix A, or [4].

[3] The Coulomb-gauge condition $\boldsymbol{\nabla}\cdot\mathbf{A}^{(\rm C)} = 0$ also holds for static fields in the Lorenz gauge [5], whose gauge condition is $\boldsymbol{\nabla}\cdot\mathbf{A}^{(\rm L)} = -\partial V^{(\rm L)}/\partial ct$.

[4] For discussion of alternative forms of electromagnetic energy, momentum and angular momentum for fields with arbitrary time dependence, see, for example, [6].

[5] Thomson noted the equivalence of $\mathbf{P}^{(\rm M)}$ and $\mathbf{P}^{(\rm P)}$ for a special case in [11, 12], but their general equivalence for stationary systems went unremarked until independent papers by Trammel [13] and Calkin [14].

[6] Note that the field momentum $\mathbf{P}^{(\rm P)}$ is "nonlocal" in the sense that it requires knowledge of the fields $\mathbf{E}$ and $\mathbf{B}$ everywhere, and not just at the positions of charges and currents.

[7] The form of Furry was largely anticipated by Coleman and van Vleck [17], as discussed in Appendix B.

This suggests that the field angular momentum (about the origin) can be computed (for static fields) via the expressions,

$$\mathbf{L}_{\rm EM} = \int \mathbf{r} \times \mathbf{p}_{\rm EM}\, d{\rm Vol} = \mathbf{L}^{\rm (M)} = \mathbf{L}^{\rm (P)} = \mathbf{L}^{\rm (M)}, \tag{3}$$

where,[8]

$$\mathbf{L}^{\rm (M)} = \int \mathbf{r} \times \frac{\varrho \mathbf{A}^{\rm (C)}}{c}\, d{\rm Vol}, \qquad \mathbf{L}^{\rm (P)} = \int \mathbf{r} \times \frac{\mathbf{E} \times \mathbf{B}}{4\pi c}\, d{\rm Vol}, \qquad \mathbf{L}^{\rm (F)} = \int \mathbf{r} \times \frac{V^{\rm (C)} \mathbf{J}}{c^2}\, d{\rm Vol}. \tag{4}$$

Although it is natural in "Maxwellian" field theory to regard $\mathbf{l}^{\rm (P)} = \mathbf{r} \times (\mathbf{E} \times \mathbf{B})/4\pi c$ (and not $\mathbf{l}^{\rm (M)} = \mathbf{r} \times \varrho \mathbf{A}^{\rm (C)}/c$ or $\mathbf{l}^{\rm (F)} = \mathbf{r} \times V^{\rm (C)} \mathbf{J}/c^2$) as the volume density of electromagnetic field angular momentum, we will conclude that, with care, the form $\mathbf{L}^{\rm (M)}$ is the most reliable for computation of field angular momentum, that form $\mathbf{L}^{\rm (F)}$ should not be used, and that the form $\mathbf{L}^{\rm (P)}$ is valid only for fields that fall off sufficiently quickly at large distances (which is not the case for an infinite solenoid).

2.1 Infinite Solenoid

Consider an infinite solenoid of radius R about the z axis with magnetic field $\mathbf{B} = B\,\hat{\mathbf{z}}$ inside and essentially zero field outside. The surface current density is $\mathbf{K} = cB\,\hat{\boldsymbol{\phi}}/4\pi$ in cylindrical coordinates (ρ, ϕ, z), while the surface charge density σ is zero.[9] The Coulomb-gauge vector potential can be written as, using $\mathbf{A}^{\rm (C)}(\mathbf{r}) = \int \mathbf{J}(\mathbf{r}')\, d{\rm Vol}'/c\,|\mathbf{r} - \mathbf{r}'|$,

$$\mathbf{A}^{\rm (C)} = \begin{cases} \frac{B\rho}{2}\,\hat{\boldsymbol{\phi}} = \frac{B}{2}(-y\,\hat{\mathbf{x}} + x\,\hat{\mathbf{y}}) & (\rho < R) \\ \frac{BR^2}{2\rho}\,\hat{\boldsymbol{\phi}} = \frac{BR^2}{2\rho^2}\,(-y\,\hat{\mathbf{x}} + x\,\hat{\mathbf{y}}) & (\rho > R). \end{cases} \tag{5}$$

However, the Coulomb-gauge vector potential is not unique, as the *restricted* gauge transformation $\mathbf{A}'^{\rm (C)} = \mathbf{A}^{\rm (C)} + \boldsymbol{\nabla}\chi$, $V'^{\rm (C)} = V^{\rm (C)} - \partial\chi/\partial ct$ with $\nabla^2\chi = 0$ generates new potentials also in the Coulomb gauge.[10] For example, $\chi = \pm Bxy/2$ leads from eq. (5) to the Coulomb-gauge potentials $\mathbf{A}'^{\rm (C)}$ (often attributed to Landau) for an infinite solenoid,[11]

$$B \begin{cases} x\,\hat{\mathbf{y}}, \\ \frac{y}{2}\left(1 - \frac{R^2}{\rho^2}\right)\hat{\mathbf{x}} + \frac{x}{2}\left(1 + \frac{R^2}{\rho^2}\right)\hat{\mathbf{y}}, \end{cases} \quad \text{and} \quad B \begin{cases} -y\,\hat{\mathbf{x}} & (\rho < R), \\ -\frac{y}{2}\left(1 + \frac{R^2}{\rho^2}\right)\hat{\mathbf{x}} - \frac{x}{2}\left(1 - \frac{R^2}{\rho^2}\right)\hat{\mathbf{y}} & (\rho > R). \end{cases} \tag{6}$$

[8] Note that the field angular momentum $\mathbf{L}^{\rm (P)}$ is "nonlocal" in the sense that it requires knowledge of the fields $\mathbf{E}$ and $\mathbf{B}$ everywhere, and not just at the positions of charges and currents.

[9] If the solenoid is based on electrical currents flowing in a wire of resistivity ϱ there must be an electric field $\mathbf{E} = \varrho\mathbf{K}$ inside the wire, which is shaped by an appropriate nonzero surface charge density along the wire (see, for example, [18]). We avoid such complications here by supposing the solenoid current to be generated by two counterpropagating flows of noninteracting charges of opposite signs inside nonconducting tubes [19, 20], or by two counter-rotating cylinders of fixed, opposite surface charge density [21]. This assumption has the additional advantage that the initial mechanical angular momentum of the solenoid currents is zero.
See Appendix E for discussion of a perfectly conducting solenoid.

[10] See, for example, sec. IIIC of [22].

[11] One might argue that the currents which generate the infinite solenoid are purely azimuthal, so the vector potential should respect this symmetry, as is the case for eq. (5). However, this usage of symmetry is not part of the usual notion of electromagnetic potentials.

2.1.1 A Mechanical Argument for Field Angular Momentum

If we suppose the infinite solenoid to be the limit of a long, but finite solenoid, we could imagine bringing charge q to, say, distance a from the axis at the center of the solenoid by starting it on the axis but well outside the solenoid, moving it along the axis to the center of the solenoid, and finally moving it radially to distance a.[12] It is reasonable to assume that the field angular momentum is initially zero in this case.

The charge q experiences no $\mathbf{v}_q \times \mathbf{B}$ force as it is brought to the center of the solenoid along it axis (and the electrically neutral, azimuthal currents $\mathbf{K}$ in the solenoid also experience no force, $\sigma \mathbf{E}_q + \mathbf{K} \times \mathbf{B}_q / c = 0$), so the total angular momentum remains zero during this process. It seems reasonable to infer that the field angular momentum remains zero so long as the charge is on the axis of the solenoid. But as the charge is moved radially, after having reached the center of the solenoid, at some low velocity $v\,\hat{\boldsymbol{\rho}}$, it experiences a torque,

$$\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F} = \rho\,\hat{\boldsymbol{\rho}} \times \left(\frac{qv\,\hat{\boldsymbol{\rho}}}{c} \times B\,\hat{\mathbf{z}} \right) = -\frac{qB}{c}\rho\frac{d\rho}{dt}\,\hat{\mathbf{z}} = -\frac{qB}{2c}\frac{d\rho^2}{dt}\,\hat{\mathbf{z}}. \tag{7}$$

so long as $\rho < R$, where the magnetic field is nonzero. To keep the charge moving on a straight line to its desired final position, an equal and opposite external torque must be applied, such that,

$$\boldsymbol{\tau}_{\rm ext} = \frac{qB}{2c}\frac{d\rho^2}{dt}\,\hat{\mathbf{z}}. = \frac{d\mathbf{L}}{dt}\,, \tag{8}$$

When the charge ends at rest at its desired position, with zero mechanical angular momentum,[13] the nonzero angular momentum of the system must be stored in the electromagnetic field,[14]

$$\mathbf{L}_{\rm EM} = \int \frac{d\mathbf{L}}{dt}\,dt = \frac{qB}{2c}\,\hat{\mathbf{z}} \int_{\rho<R} \frac{d\rho^2}{dt}\,\hat{\mathbf{z}} = \frac{qB}{2c}\,\hat{\mathbf{z}} \begin{cases} a^2 & (a < R), \\ R^2 & (a > R). \end{cases} \tag{9}$$

[12]This argument follows sec. IV of [23] and p. 35 of [24]. See also [25, 26, 27]. The spirit of the argument can be traced to early experiments by Lodge [28] and by Einstein/de Haas [29], and underlies the Feynman disk paradox [38].

[13]There is no obvious mechanical angular momentum in the final configuration, but in principle there could be "hidden" mechanical angular momentum, as discussed in secs. 2.4.2-2.4.3. The agreement between eqs. (9) and (10) suggests that there is no "hidden" mechanical angular momentum, but the disagreement between eqs. (9) and (12)-(13) and (17) casts doubt on this. We eventually conclude that the field angular momentum is given by eqs. (9)-(10), and so there is no "hidden" mechanical angular momentum in the system for any position of the charge.

[14](Aug. 18, 2021) It is also noteworthy that when the electron is outside the solenoid, its electromagnetic angular momentum is independent of position, $\mathbf{L}_{\rm EM} = q\Phi\,\hat{\mathbf{z}}/2\pi c$, where $\Phi = \pi B R^2$ is the magnetic flux inside the solenoid. Since there is "zero" magnetic field outside the solenoid, this effect appears to be "action at a distance", and can be regarded as a classical "nonlocal" effect, that anticipates to some extent the quantum mechanical Aharonov-Bohm effect [41, 42]. Of course, both $\mathbf{P}_{\rm EM}^{\rm (P)}$ and $\mathbf{L}_{\rm EM}^{\rm (P)}$ associated with an electric charge are "nonlocal" in that they depend on the field at positions other than that of the charge itself.

2.1.2 Maxwell Field Angular Momentum

For an electric charge q at distance a from the solenoid axis, the field angular momentum based on the Maxwell form $\mathbf{P}^{(\mathrm{M})}$ of the field momentum is, using eq. (5),

$$\mathbf{L}^{(\mathrm{M})} = \int \mathbf{r} \times \frac{\varrho \mathbf{A}^{(\mathrm{C})}}{c}\, d\mathrm{Vol} = \frac{qB}{2c}\,\hat{\mathbf{z}} \begin{cases} a^2 & (a < R), \\ R^2 & (a > R), \end{cases} \tag{10}$$

which agrees with (9).[15] However, if we were to use either of the Coulomb-gauge vector potentials of eq. (6), this happy agreement would not hold.

Furthermore, the result (10) is not gauge invariant. As an extreme example, in the so-called Poincaré gauge [33], which depends on the choice of origin of the coordinate system, that origin can be chosen such that the vector potential $\mathbf{A}^{(\mathrm{P})}$ is zero at any particular point outside the solenoid. For a choice that the vector potential be zero at the location of the charge q, the field angular momentum $\mathbf{L}^{(\mathrm{M})}$ would also vanish.

2.1.3 Poynting Field Angular Momentum

For charge q at $(x, y, z) = (a, 0, 0)$ the electric field is $\mathbf{E}(x, y, z) = q\,\mathbf{d}/d^3 = q[(x-a)\,\hat{\mathbf{x}} + y\,\hat{\mathbf{y}} + z\,\hat{\mathbf{z}}]/[(x-a)^2 + y^2 + z^2)^{3/2}$, so the field angular momentum based on the Poynting form $\mathbf{P}^{(\mathrm{P})}$ of the field momentum is (prob. 8.14 of [34]), [16]

$$\begin{aligned}
\mathbf{L}^{(\mathrm{P})} &= \int \mathbf{r} \times \frac{\mathbf{E}\times\mathbf{B}}{4\pi c}\, d\mathrm{Vol} = \frac{qB}{4\pi c}\int (x\,\hat{\mathbf{x}} + y\,\hat{\mathbf{y}} + z\,\hat{\mathbf{z}}) \times \left(\frac{(x-a)\,\hat{\mathbf{x}} + y\,\hat{\mathbf{y}} + z\,\hat{\mathbf{z}}}{[(x-a)^2 + y^2 + z^2]^{3/2}} \times \hat{\mathbf{z}} \right) d\mathrm{Vol} \\
&= -\frac{qB}{4\pi c}\,\hat{\mathbf{z}} \int \frac{x^2 + y^2 - xa}{[(x-a)^2 + y^2 + z^2]^{3/2}}\, dx\, dy\, dz = -\frac{qB}{2\pi c}\,\hat{\mathbf{z}} \int \frac{x^2 + y^2 - xa}{(x-a)^2 + y^2}\, dx\, dy \\
&= -\frac{qB}{2\pi c}\,\hat{\mathbf{z}} \int_0^R \rho\, d\rho \int_0^{2\pi} d\phi\, \frac{\rho^2 - a\rho\cos\phi}{\rho^2 - 2a\rho\cos\phi + a^2} = -\frac{qB}{2\pi c}\,\hat{\mathbf{z}} \int_0^R \rho\, d\rho \int_0^{2\pi} d\phi\, \frac{1 - \frac{a}{\rho}\cos\phi}{1 - 2\frac{a}{\rho}\cos\phi + \frac{a^2}{\rho^2}}\,.
\end{aligned} \tag{11}$$

For $a > R \ge \rho$ the remaining integral is (Dwight 859.113 and 859.122 [36]),

$$\mathbf{L}^{(\mathrm{P})}(a > R) = -\frac{qB}{c}\,\hat{\mathbf{z}} \int_0^R \rho\, d\rho \left(\frac{\rho^2}{a^2 - \rho^2} - \frac{\rho^2}{a^2 - \rho^2} \right) = 0. \tag{12}$$

For $a < R$ the part of the integral for $a > \rho$ vanishes as in eq. (12), and the remaining integral is,

$$\mathbf{L}^{(\mathrm{P})}(a < R) = -\frac{qB}{c}\,\hat{\mathbf{z}} \int_a^R \rho\, d\rho \left(\frac{\rho^2}{\rho^2 - a^2} - \frac{a^2}{\rho^2 - a^2} \right) = -\frac{qB(R^2 - a^2)}{2c}\,\hat{\mathbf{z}}. \tag{13}$$

It is perhaps surprising that the Poynting form (12)-(13) does not agree with the Maxwell form (10) or with the argument leading to eq. (9), in view of general arguments as given, for

[15] The result (10) was also obtained in [30, 31, 32].

[16] The angular momentum $\mathbf{L}_{\mathrm{P}}$ for $a > R$ was also evaluated in sec. J of [35], where it was noted that this does not equal $\mathbf{L}_{\mathrm{M}}$, with the conclusion (in endnote 17) that the latter form must be "in error".

example, in sec. II of [16].[17,18,19]

As a check, for a charge at the origin inside a long solenoid that extends over $|z| < l$,

$$\begin{aligned} \mathbf{L}^{(\mathrm{P})} &= \int \mathbf{r} \times \frac{\mathbf{E}\times\mathbf{B}}{4\pi c}\, d\mathrm{Vol} = \frac{qB}{4\pi c}\int (\rho\,\hat{\boldsymbol{\rho}} + z\,\hat{\mathbf{z}}) \times \left(\frac{\rho\,\hat{\boldsymbol{\rho}} + z\,\hat{\mathbf{z}}}{(\rho^2+z^2)^{3/2}} \times \hat{\mathbf{z}}\right) d\mathrm{Vol} \qquad (14) \\ &\approx -\frac{qB}{c}\int_0^R \rho\, d\rho \int_0^l dz\, \frac{\rho^2}{(\rho^2+z^2)^{3/2}}\,\hat{\mathbf{z}} = -\frac{qBl}{c}\int_0^R \frac{\rho\, d\rho}{(\rho^2+l^2)^{1/2}}\,\hat{\mathbf{z}} = -\frac{qBl(\sqrt{l^2+R^2}-l)}{c}\,\hat{\mathbf{z}} \approx -\frac{qBR^2}{2c}\,\hat{\mathbf{z}}. \end{aligned}$$

(Aug. 19, 2021) As another check, we follow a suggestion in [3] to compute the Poynting angular momentum in a coordinate system with origin at the charge q,

$$\begin{aligned} \mathbf{L}^{(\mathrm{P})} &= \int \mathbf{r} \times \frac{\mathbf{E}\times\mathbf{B}}{4\pi c}\, d\mathrm{Vol} = \frac{qB}{4\pi c}\int_{\mathrm{inside}} \mathbf{r} \times \left(\frac{\mathbf{r}}{r^3}\times\hat{\mathbf{z}}\right) d\mathrm{Vol} = \frac{qB}{4\pi c}\int_{\mathrm{inside}} \frac{z\mathbf{r} - r^2\,\hat{\mathbf{z}}}{r^3}\, d\mathrm{Vol} \\ &= = \frac{qB}{4\pi c}\int_{\mathrm{inside}} \frac{xz\,\hat{\mathbf{x}} + yz\,\hat{\mathbf{z}} - (x^2+y^2)\,\hat{\mathbf{z}}}{r^3}\, d\mathrm{Vol} = -\frac{qB\,\hat{\mathbf{z}}}{4\pi c}\int_{\mathrm{inside}} \frac{x^2+y^2}{(x^2+y^2+z^2)^{3/2}}\, dx\, dy\, dz \\ &= -\frac{qB\,\hat{\mathbf{z}}}{2\pi c}\int_{\mathrm{inside}} dx\, dy = -\frac{q\Phi}{2\pi c}\,\hat{\mathbf{z}}\,, \qquad (15) \end{aligned}$$

where $\Phi = \pi B R^2$ is the magnetic flux inside the solenoid (which is taken to be zero outside the solenoid), and we have used Dwight 200.03. The result (15) applies for the charge either inside or out of the solenoid. However, this does not agree with our results (12)-(13). Apparently, the choice of origin makes a difference to the value of the integral for this system of infinite extent. As remarked at the end of [53], it seems that the field angular momentum $\mathbf{L}^{(\mathrm{P})}$ is ill-defined for an infinte solenoid. [Likewise, the field angular momentum $\mathbf{L}^{(\mathrm{M})}$ is ill-defined, as it depends on the choice of the particular Coulomb gauge for the vector potential $\mathbf{A}^{(\mathrm{C})}$ among the infinite set of possibilities that obey $\boldsymbol{\nabla}\cdot\mathbf{A}^{(\mathrm{C})} = 0$.]

2.1.4 Furry Field Angular Momentum

For the field angular momentum based on the Furry form $\mathbf{P}^{(\mathrm{F})}$ of the field momentum, we content ourselves with the case of a charge on the solenoid axis at $z = 0$, for a solenoid of length $2l$. Then, the electric scalar potential for charge q is, in the Coulomb gauge,[20]

$$V^{(\mathrm{C})} = \frac{q}{r} = \frac{q}{(\rho^2+z^2)^{1/2}}\,, \qquad (16)$$

so the Furry angular momentum for the solenoid plus charge is,

$$\begin{aligned} \mathbf{L}^{(\mathrm{F})} &= \int \mathbf{r} \times \frac{V\mathbf{J}}{c^2}\, d\mathrm{Vol} = \frac{qB}{4\pi c}\int (R\,\hat{\boldsymbol{\rho}} + z\,\hat{\mathbf{z}}) \times \frac{\hat{\boldsymbol{\phi}}}{(R^2+z^2)^{1/2}}\, d\mathrm{Area} \\ &= \frac{qBR^2}{c}\int_0^l \frac{dz}{\sqrt{R^2+z^2}}\,\hat{\mathbf{z}} = \frac{qBR^2}{c}\sinh^{-1}\frac{l}{R}\,\hat{\mathbf{z}}, \qquad (17) \end{aligned}$$

which diverges as the solenoid length l grows large.

[17]It seems not to have been noticed in [23, 24] that the Poynting form of field angular momentum does not lead to the result (9) found there.

[18]The discrepancy between the Maxwell and Poynting forms for the field angular momentum associated with an infinite solenoid appears in a variant [37] of Feynman's disk paradox [38].

[19]If an infinite solenoid is considered as the limit of a long, finite solenoid, then the angular momentum associated with the return flux outside the solenoid restores agreement between the Maxwell and Poynting forms for the field angular momentum [54].

[20]The form (16) applies everywhere only if the conductor of the solenoid does not "shield" the electric field of the charge. This is consistent with the model of the solenoid currents as due to counter-rotating, opposite charges in nonconducting tubes, or to a pair of counter-rotating, oppositely charged cylinders.

2.1.5 Comments

We are belatedly reminded to review the derivation of eqs. (1) in Appendix A (and [4]), where it was noted near eq. (60) that the various forms for the field momentum are equivalent only if the fields fall off sufficiently fast at large distances. While the magnetic field of the infinite solenoid does not fall off with z, it happens that field momenta $\mathbf{P}^{(\mathrm{M})} = \mathbf{P}^{(\mathrm{P})} = \mathbf{P}^{(\mathrm{F})}$, at least when the charge is on the axis.[21]

Furthermore, it was shown in Appendix A (and sec. 2.2 of [4]) that field angular momentum $\mathbf{L}^{(\mathrm{M})} = \mathbf{L}^{(\mathrm{P})}$ when the fields fall off sufficiently quickly at large distance. It appears that the field of an infinite solenoid does not fall off fast enough for this to hold.[22]

The disagreement between the Furry form $\mathbf{L}^{(\mathrm{F})}$ of field angular momentum and eqs. (9)-(10) is an example of a general result given in Appendix A. There is also disagreement between the computations of angular momentum considered here and the subtle issue of "hidden" angular momentum, as discussed in sec. 2.4 below.

2.2 Magnetic Field Uniform within a Sphere of Radius R

As a calculable example of a system with bounded charge and current distributions, for which the fields falls off at least as fast as $1/r^2$, consider surface currents $\mathbf{K} = 3B\sin\theta\,\boldsymbol{\phi}/8\pi c$ on a sphere of radius R, centered on the origin in a spherical coordinate system (r, θ, ϕ).[23] These generate the magnetic field,[24]

$$\mathbf{B}(r < R) = B\,\hat{\mathbf{z}}, \qquad \mathbf{B}(r > R) = \frac{BR^3}{2r^3}(3\cos\theta\,\hat{\mathbf{r}} - \hat{\mathbf{z}}), \tag{19}$$

which is uniform inside the sphere and the field of a dipole of magnetic moment $BR^3\,\hat{\mathbf{z}}/2$ outside it. The vector potential in the Coulomb gauge can be written as,

$$\mathbf{A}^{(\mathrm{C})}(r < R) = \frac{Br\sin\theta}{2}\,\hat{\boldsymbol{\phi}}, \qquad \mathbf{A}^{(\mathrm{C})}(r > R) = \frac{BR^3\sin\theta}{2r^2}\,\hat{\boldsymbol{\phi}}. \tag{20}$$

For an electric charge q at the origin, where the vector potential is zero, the field angular momentum based on the Maxwell form of the field momentum is,

$$\mathbf{L}^{(\mathrm{M})} = \int \mathbf{r} \times \frac{\varrho\mathbf{A}^{(\mathrm{C})}}{c}\,d\mathrm{Vol} = 0, \tag{21}$$

[21] The field momentum $\mathbf{P}^{(\mathrm{M})}$ is zero for a charge at rest on the solenoid axis, the Poynting form of the field momentum is,

$$\mathbf{P}^{(\mathrm{P})} = \int \frac{\mathbf{E}\times\mathbf{B}}{4\pi c}\,d\mathrm{Vol} = \frac{qB}{4\pi c}\int \frac{\rho\,\hat{\boldsymbol{\rho}} + z\,\hat{\mathbf{z}}}{(\rho^2+z^2)^{3/2}} \times \hat{\mathbf{z}}\,d\mathrm{Vol} = 0, \tag{18}$$

and the Furry form of the field momentum is $\mathbf{P}^{(\mathrm{F})} = \int V^{(\mathrm{C})}\mathbf{J}\,d\mathrm{Vol}/c^2 = 0$ for a charge on the axis.

See Appendix C for discussion of the case when the charge is not on the solenoid axis.

[22] Another view is that for a finite solenoid, the integral of $\mathbf{r} \times \frac{\mathbf{E}\times\mathbf{B}}{4\pi c}$ inside the solenoid is canceled by the integral outside it for the case of a charge on the axis, but for an infinite solenoid the integral outside the solenoid is set to zero, leaving an unphysical, nonzero value (14) for the field angular momentum $\mathbf{L}_{\mathrm{P}}$.

[23] Appendix D shows that the field momenta $\mathbf{P}^{(\mathrm{M})}$ and $\mathbf{P}^{(\mathrm{P})}$ are equal for an example of this type.

[24] See, for example, [26] and sec. II of [39].

The electric field is $\mathbf{E} = q\,\hat{\mathbf{r}}/r^2$, so the field angular momentum based on the Poynting form of the field momentum is,

$$\begin{aligned}
\mathbf{L}^{(\mathrm{P})} &= \int \mathbf{r}\times\frac{\mathbf{E}\times\mathbf{B}}{4\pi c}\,d\mathrm{Vol} \\
&= \frac{qB}{4\pi c}\int_{r<R}\mathbf{r}\times\left(\frac{\hat{\mathbf{r}}}{r^2}\times\hat{\mathbf{z}}\right)\,d\mathrm{Vol} + \frac{qBR^3}{8\pi c}\int_{r>R}\mathbf{r}\times\left(\frac{\hat{\mathbf{r}}}{r^2}\times\frac{3\cos\theta\,\hat{\mathbf{r}}-\hat{\mathbf{z}}}{r^3}\right)\,d\mathrm{Vol} \\
&= -\frac{qB}{2c}\int_0^R r^2\,dr\int_{-1}^1 d\cos\theta\,\frac{\sin^2\theta}{r}\,\hat{\mathbf{z}} + \frac{qBR^3}{4c}\int_R^\infty r^2\,dr\int_{-1}^1 d\cos\theta\,\frac{\sin^2\theta}{r^4}\,\hat{\mathbf{z}} \\
&= -\frac{qBR^2}{3c}\,\hat{\mathbf{z}} + \frac{qBR^2}{3c}\,\hat{\mathbf{z}} = 0. \qquad (22)
\end{aligned}$$

The electric scalar potential is $V^{(\mathrm{C})} = q/r$, so the field angular momentum based on the Furry form of the field momentum is,

$$\mathbf{L}^{(\mathrm{F})} = \int \mathbf{r}\times\frac{V^{(\mathrm{C})}\mathbf{J}}{c^2}\,d\mathrm{Vol} = \frac{3qB}{8\pi c}\int R\,\hat{\mathbf{r}}\times\frac{\sin\theta\,\hat{\boldsymbol{\phi}}}{R}\,d\mathrm{Area} = \frac{3qBR^2}{4c}\int_{-1}^1 d\cos\theta\,\sin^2\theta\,\hat{\mathbf{z}} = \frac{qBR^2}{c}\,\hat{\mathbf{z}}. \qquad (23)$$

The forms $\mathbf{L}^{(\mathrm{M})}$ and $\mathbf{L}^{(\mathrm{P})}$ give the same result, as predicted in Appendix A (and sec. 2.2 of [4]),[25] where no argument was made as to whether the angular momentum computed from the Furry form, $\mathbf{L}^{(\mathrm{F})}$, of the field momentum would agree with the common value based on the Maxwell and Poynting forms (for examples with fields that fall off rapidly enough at large distances).[26]

2.2.1 Alternative Coulomb-Gauge Potentials

In Appendix A (and sec. 2.2 of [4]) it is argued that the electromagnetic field momentum of a static system is correctly computed via both forms $\mathbf{P}^{(\mathrm{P})}$ and $\mathbf{P}^{(\mathrm{M})}$ of the present eq. (2), and that the electromagnetic field angular momentum can by computed via both forms $\mathbf{L}^{(\mathrm{P})}$ and $\mathbf{L}^{(\mathrm{M})}$ of the present eq. (4), using any Coulomb-gauge vector potential $\mathbf{A}^{(\mathrm{C})}$, so long as it falls off sufficiently quickly at large distances.

However, it seems unlikely that if a Coulomb-gauge vector potential different from $\mathbf{A}^{(\mathrm{C})}(\mathbf{r}) = \int \mathbf{J}(\mathbf{r}')\,d\mathrm{Vol}'/c\,|\mathbf{r}-\mathbf{r}'|$, were used in the Maxwell forms that the result would be the same. This would be consistent with the argument in Appendix A if the alternative vector potential did not go to zero at large distances.

We recall (sec. IIIC of [22]) that an alternative Coulomb-gauge vector potential would have the form $\mathbf{A}^{(\mathrm{C})} + \boldsymbol{\nabla}\chi$ where the *restricted* gauge function χ obeys Laplace's equation $\nabla^2\chi = 0$ everywhere. Now, the "trivial" case of $\chi = 0$ has $\boldsymbol{\nabla}\chi = 0$ everywhere, which includes large distances. Then, by the uniqueness theorem for solutions to Laplace's equation with Neumann boundary conditions (see, for example sec. 1.9 of [40]), this is the only solution

[25] The equivalence of $\mathbf{L}^{(\mathrm{M})}$ and $\mathbf{L}^{(\mathrm{P})}$ for well behaved fields was first argued by Trammel at the end of [13].

[26] Due to the additional factor of $\mathbf{r}$ in the integrands for field angular momentum, compared to those of field momentum, the requirements on the falloff of the fields at larger distances are greater in the case of field angular momentum for the equivalence of different forms to hold. In particular, an infinite solenoid is well enough behaved for computations of field momentum via various forms, but not for computations of field angular momentum. For the latter, the Maxwell form $\mathbf{L}^{(\mathrm{M})}$ appears to be the most robust.

to Laplace's equation for which $\boldsymbol{\nabla}\chi = 0$ everywhere at large distances. Hence, all possible alternative, Coulomb-gauge vector potentials do not go to zero everywhere at large distances, and so none of these would lead to a correct computation of the field momentum (or angular momentum) using the Maxwell forms.[27,28]

2.3 Charge Not on the Axis of a Uniform Magnetic Field

This example encourages us to suppose that it is consistent to compute field angular momentum in static fields that fall off quickly enough with distance according to either of the forms,

$$\mathbf{L}^{(\mathrm{M})} = \int \mathbf{r} \times \frac{\varrho \mathbf{A}^{(\mathrm{C})}}{c}\, d\mathrm{Vol}, \qquad \text{or} \qquad \mathbf{L}^{(\mathrm{P})} = \int \mathbf{r} \times \frac{\mathbf{E}\times\mathbf{B}}{4\pi c}\, d\mathrm{Vol}, \tag{24}$$

while the Furry form should not be used. However, the Furry form does play a role, discussed in sec. 2.4.

It is tempting to favor the Maxwell form, $\mathbf{L}^{(\mathrm{M})}$, which appears simple to evaluate, as only the vector potential is needed. However, the Coulomb-gauge vector potential in a region of uniform magnetic field is not unique: $\mathbf{A}^{(\mathrm{C})} = (a_x + by)\,\hat{\mathbf{x}} + [a_y + (B+b)x]\,\hat{\mathbf{y}}$ leads to $\mathbf{B} = B\,\hat{\mathbf{z}}$ for any values of the constants a_x, a_y and b.[29] To make a computation of angular momentum via the vector potential is seems advisable to calculate the latter from the (static) current density $\mathbf{J}$,

$$\mathbf{A}^{(\mathrm{C})}(\mathbf{r}) = \int \frac{\mathbf{J}(\mathbf{r}')}{c\,|\mathbf{r}-\mathbf{r}'|}\, d\mathrm{Vol}, \tag{25}$$

rather than using other forms that corresponds to the same magnetic field $\mathbf{B} = \boldsymbol{\nabla}\times\mathbf{A}^{(\mathrm{C})}$. Then, the vector potential of a finite solenoid vanishes on the solenoid axis, and we infer that the angular momentum is zero for a charge at rest on the axis of a finite solenoid. Hence, use of the Poynting form (11) is unreliable if one uses the field of an infinite solenoid as an approximation for that of a finite solenoid.

The vector potential for a long solenoid is approximately given by eq. (5), so it appears that the (Maxwell) field angular momentum of charge q at rest at distance a from the solenoid axis (and $|z_q| \ll l$, where solenoid extends over the range $-l/2 < z < l/2$), is approximately that given in eq. (10),

$$\mathbf{L}^{(\mathrm{M})}(a < R) \approx \frac{qBa^2}{2c}\,\hat{\mathbf{z}}, \qquad \mathbf{L}^{(\mathrm{M})}(a > R) \approx \frac{qBR^2}{2c}\,\hat{\mathbf{z}}. \tag{26}$$

The field angular momentum is nonzero even when the charge is outside the solenoid where the magnetic field is very weak. This indicates that there exists some interaction between the charge and the magnetic field outside the solenoid.[30]

---

[27] Thanks to D. Griffiths for this argument.

[28] Note that the "Landau" forms (6), based on gauge functions $\chi = \pm Bxy/2$, do not vanish at infinity.

[29] The vector potential $\mathbf{A}^{(\mathrm{C})} = Br_\perp\,\hat{\boldsymbol{\phi}}/2$ corresponds to $a_x = 0 = a_y$, $b = -B/2$.

[30] Such effects were considered by Lodge in 1889 [28]. For example, if there is initially zero current in the solenoid, an azimuthal electric field is induced outside the solenoid as the current rises, which imparts a mechanical angular momentum to a charge outside the solenoid with sign opposite to the field angular momentum of eq. (26), such that the total angular momentum remains zero.

2.4 "Hidden" Momentum and Angular Momentum

2.4.1 Thomson's Momentum Paradox

As part of his considerations of momentum in the electromagnetic field, J.J. Thomson (1904) discussed the example of an electric charge external to a short solenoid that can be represented as a magnetic dipole $\mathbf{m}$ [11, 12]. He deduced that this system has electromagnetic field momentum,

$$\mathbf{P}_{\text{EM}} = \frac{\mathbf{E} \times \mathbf{m}}{c}, \tag{27}$$

where $\mathbf{E}$ is the (approximately constant) electric field of the charge at the location of the solenoid, by computing both $\mathbf{P}^{(\text{M})}$ and $\mathbf{P}^{(\text{P})}$ (and approximating $\mathbf{B}$ as zero outside the solenoid). He then argued on p. 348 of [11] that if the solenoid were demagnetized the stored electromagnetic momentum (27) should appear as mechanical momentum, but the form $\mathbf{P}^{(\text{M})}$ suggests that the momentum is associated with the solenoid, which should therefore move, while the form $\mathbf{P}^{(\text{P})}$ suggests that the momentum is associated with the charge, which should therefore move.

Thomson left this paradox unresolved.

2.4.2 "Hidden" Mechanical Momentum

A resolution to this paradox emerged in 1967 when Shockley introduced the notion of "hidden" mechanical momentum [21], a consequence of which is that Thomson's charge plus solenoid contains an unnoticed ("hidden") amount of mechanical momentum that happens to be equal and opposite to the field momentum (27). So, if the current in the solenoid drops to zero, no mechanical momentum appears in the system and nothing moves.

An explanation of "hidden" mechanical momentum was given by Coleman and van Vleck [17] who argued (see Appendix B) that for stationary systems,

$$\mathbf{P}_{\text{hidden}} = -\int \frac{V^{(\text{C})}\mathbf{J}}{c^2}\, d\text{Vol} = -\mathbf{P}^{(\text{F})} = -\mathbf{P}_{\text{EM}}, \tag{28}$$

such that $\mathbf{P}_{\text{total}} = \mathbf{P}_{\text{mech}} + \mathbf{P}_{\text{EM}} = \mathbf{P}_{\text{hidden}} + \mathbf{P}_{\text{EM}} = 0$.

2.4.3 The Density of "Hidden" Mechanical Momentum

The success of eq. (28) in resolving Thomson's paradox suggests that we identify the density of "hidden" mechanical momentum as,

$$\mathbf{p}_{\text{hidden}} \stackrel{?}{=} -\frac{V^{(\text{C})}\mathbf{J}}{c^2}. \tag{29}$$

The existence of this classical interaction between a long solenoid and a charge outside should make it less surprising that there is also an interesting quantum interaction in this case, as noted by Aharonov and Bohm [41, 42].

which is an aspect of the moving charges that constitute the current density $\mathbf{J}$. Then, classical mechanics implies that associated with the density (29) a density of mechanical angular momentum with respect to the origin,

$$\mathbf{l}_{\rm hidden} \stackrel{?}{=} -\,\mathbf{r}\times\frac{V^{(\rm C)}\mathbf{J}}{c^2}\,, \tag{30}$$

such that the total "hidden" mechanical angular momentum is,

$$\mathbf{L}_{\rm hidden} \stackrel{?}{=} \int \mathbf{l}_{\rm hidden}\, d{\rm Vol} = -\int \mathbf{r}\times\frac{V^{(\rm C)}\mathbf{J}}{c^2}\, d{\rm Vol} = -\mathbf{L}^{(\rm F)}, \tag{31}$$

recalling eq. (4).

However, we encounter a difficulty with this prescription for the case of a charge plus solenoid (all "at rest") in that sec. 2.1, particularly footnote 7, leads to the conclusion that there is no "hidden" mechanical angular momentum in this system.[31] This difficulty suggests that eq. (29) is not actually the density of "hidden" mechanical momentum.

We recall that Aharonov, Peale and Vaidman proposed a relation between "hidden" mechanical momentum in stationary systems and the stress tensor, in eqs. (10)-(12) of [43]. A generalization of this proposal that applies to nonstationary systems has been given in [44], where a subsystem with a specified volume can interact with the rest of the system via bulk forces as well as contact forces and/or transfer of mass/energy across its surface (which can be in motion),

$$\mathbf{P}_{\rm hidden} \equiv -\int \frac{f^0}{c}(\mathbf{r}-\mathbf{r}_{\rm cm})\, d{\rm Vol} = \mathbf{P} - M\mathbf{v}_{\rm cm} - \oint_{\rm boundary} (\mathbf{r}-\mathbf{r}_{\rm cm})\,(\mathbf{p}-\rho\mathbf{v}_b)\cdot d\mathbf{Area}, \tag{33}$$

where,[32]

$$f^\mu = \frac{\partial T^{\mu\nu}}{\partial x^\nu} = \partial_\nu T^{\mu\nu}\,, \qquad \partial_\nu = \left(\frac{\partial}{\partial ct}, \boldsymbol{\nabla}\right), \tag{34}$$

is the 4-force density exerted on the subsystem by the rest of the system, with $T_{\mu\nu}$ being the stress-energy-momentum 4-tensor of the subsystem, $\mathbf{P}$ is the total momentum of the subsystem, $M = U/c^2$ is its total "mass", U is its total energy, $\mathbf{r}_{\rm cm}$ is its center of mass/energy, $\mathbf{v}_{\rm cm} = d\mathbf{r}_{\rm cm}/dt$, $\mathbf{p}$ is its momentum density, $\rho = u/c^2$ is its "mass" density, u is its energy density, $\mathbf{v}_b$ is the velocity (field) of its boundary.[33]

[31] In sec. 2.1 we found $\mathbf{L}^{(\rm F)}$ to be nonzero while the field angular momentum $\mathbf{L}_{\rm EM} = \mathbf{L}^{(\rm P)} = \mathbf{L}^{(\rm M)}$ is zero. Then, we have,

$$\mathbf{L}_{\rm total} = \mathbf{L}_{\rm EM} + \mathbf{L}_{\rm hidden} \neq 0, \tag{32}$$

in contradiction to the argument of sec. 2.1.1 that $\mathbf{L}_{\rm total} = 0$ when the charge is on the solenoid axis.

[32] We use the conventions of the Appendix to [45].

[33] Important qualifications are that the values of $\mathbf{p}$ and ρ at the boundary are the limit of those just inside the boundary, and that the volume integral of f^0 does not include possible delta-functions at the boundary.

If a system is partitioned into subsystems that are "mechanical" and "electromagnetic, these subsystems have no boundary (except at infinity), so that for a system with all matter within a finite region, the "hidden" momenta of the two subsytems each has the form,[34]

$$\mathbf{P}_{\rm hidden} = \mathbf{P} - M\mathbf{v}_{\rm cm}. \tag{35}$$

Note that the electromagnetic subsystem can have "hidden" momentum according to eq. (34), and for a stationary system, where $\mathbf{v}_{\rm cm,EM} = 0$, the field momentum is to be called a "hidden" momentum. In contrast, the field momentum of a plane electromagnetic wave is not a "hidden" momentum.[35]

For an isolated, stationary system the center of mass/energy is at rest, $\mathbf{r}_{\rm cm}$ is constant, and the total momentum is zero [17]. According to eq. (33), such a system (as a whole) has no "hidden" momentum. But if the system is partitioned into (stationary) mechanical and electromagnetic subsystems, each can have nonzero "hidden" momentum, with the constraint that,

$$\mathbf{P}_{\rm total} = 0 = \mathbf{P}_{\rm hidden}^{\rm mech} + \mathbf{P}_{\rm hidden}^{\rm EM} \qquad \text{(stationary)}. \tag{36}$$

According to eqs. (33)-(34),

$$\mathbf{P}_{\rm hidden}^{\rm EM} = \mathbf{P}_{\rm EM} = \int \frac{\mathbf{E}\times\mathbf{B}}{4\pi c}\, d{\rm Vol} \qquad \text{(stationary)}, \tag{37}$$

but the equivalent statement for the mechanical subsystem, $\mathbf{P}_{\rm hidden,mech} = \mathbf{P}_{\rm mech}$, is not immediately informative.[36] Instead, we consider the first form of eq. (33),

$$\mathbf{P}_{\rm hidden}^{\rm mech} = -\frac{1}{c}\int f_{\rm mech}^{0}\left(\mathbf{r} - \mathbf{r}_{\rm cm}^{\rm mech}\right) d{\rm Vol} \qquad \text{(stationary)}. \tag{38}$$

As noted in eq. (10) of [43], for a mechanical system (stationary or not) that interacts with an electromagnetic field described by the tensor $\mathcal{F}^{\mu\nu}$, the 4-force density on the mechanical subsystem due to the electromagnetic subsystem is the Lorentz force,

$$f_{\rm mech}^{\mu}\ (= \partial_\nu T_{\rm mech}^{\mu\nu}) = \mathcal{F}^{\mu\nu}J_\nu = \left(\frac{\mathbf{J}\cdot\mathbf{E}}{c}, \varrho\mathbf{E} + \frac{\mathbf{J}}{c}\times\mathbf{B}\right), \qquad f_0^{\rm mech} = \frac{\mathbf{J}\cdot\mathbf{E}}{c}, \tag{39}$$

where the current density 4-vector is $J_\mu = (c\varrho, -\mathbf{J})$. Thus the "hidden" momentum of the mechanical subsystem is,[37]

$$\mathbf{P}_{\rm hidden}^{\rm mech} = -\int \frac{\mathbf{J}\cdot\mathbf{E}}{c^2}\left(\mathbf{r} - \mathbf{r}_{\rm cm}^{\rm mech}\right) d{\rm Vol}. \tag{40}$$

[34]The form (35) was proposed for the "hidden" momentum of a nonstationary mechanical subsystem in eq. (78) of [39], but field momentum was taken to be "overt" there.

[35]See sec. 3.2 of [44].

[36]Many people appear to assume that $\mathbf{P}_{\rm mech} = 0$ for a stationary mechanical subsystem.

[37]The "hidden" momentum for the electromagnetic subsystem (stationary or not) follows from $f_{\rm EM}^0 = \partial_\mu T_{\rm EM}^{0\mu}$, where $T_{\rm EM}^{0\mu} = (u_{\rm EM}, c\,\mathbf{p}_{\rm EM}) = (u_{\rm EM}, \mathbf{S}/c)$, and $\mathbf{S}$ is the Poynting vector. Then, $f_{\rm EM}^0 = \partial u_{\rm EM}/\partial ct + \boldsymbol{\nabla}\cdot\mathbf{S}/c = \partial u_{\rm EM}/\partial ct - \partial u_{\rm EM}/\partial ct - \mathbf{J}\cdot\mathbf{E}/c = -\mathbf{J}\cdot\mathbf{E}/c$, and $\mathbf{P}_{\rm hidden}^{\rm EM} = \int \mathbf{J}\cdot\mathbf{E}\left(\mathbf{r} - \mathbf{r}_{\rm cm}^{\rm EM}\right) d{\rm Vol}/c^2$ for any isolated system that is partitioned into electromagnetic and mechanical subsystems. If $\int \mathbf{J}\cdot\mathbf{E}\, d{\rm Vol} = 0$ (as will be seen below to hold for stationary systems), or $\mathbf{r}_{\rm cm}^{\rm EM} = \mathbf{r}_{\rm cm}^{\rm mech}$, then $\mathbf{P}_{\rm hidden}^{\rm EM} = -\mathbf{P}_{\rm hidden}^{\rm mech}$.

Note that for "free" fields, where $\mathbf{J} = 0$ everywhere, the "hidden" electromagnetic momentum vanishes. That is, in general $\mathbf{P}_{\rm hidden}^{\rm EM} \neq \mathbf{P}_{\rm EM}$ (for example, see [46]), although they are equal for stationary systems.

For a stationary system, $\mathbf{E} = -\boldsymbol{\nabla} V^{(\mathrm{C})}$ and $\boldsymbol{\nabla} \cdot \mathbf{J} = \partial^j J_j = 0$, so,

$$-\mathbf{J} \cdot \mathbf{E} = J_j\, \partial^j V^{(\mathrm{C})} = \partial^j (J_j V^{(\mathrm{C})}) - V^{(\mathrm{C})} \partial^j\, J_j = \partial^j (J_j V^{(\mathrm{C})}), \tag{41}$$

$$\begin{aligned} -\mathbf{J} \cdot \mathbf{E}\, r_i &= r_i J_j\, \partial^j V^{(\mathrm{C})} = \partial^j (r_i J_j V^{(\mathrm{C})}) - V^{(\mathrm{C})} \partial^j (r_i J_j) \\ &= \partial^j (r_i J_j V^{(\mathrm{C})}) - V^{(\mathrm{C})} J_j\, \partial^j r_i - V^{(\mathrm{C})} r_i\, \partial^j\, J_j = \partial^j (r_i J_j V^{(\mathrm{C})}) - V^{(\mathrm{C})} J_i, \end{aligned} \tag{42}$$

and,

$$P^{\mathrm{mech}}_{\mathrm{hidden},i} = -\int \frac{\mathbf{J} \cdot \mathbf{E}}{c^2} \left(r_i - r^{\mathrm{mech}}_{\mathrm{cm},i} \right) d\mathrm{Vol} = -\int \frac{V^{(\mathrm{C})} J_i}{c^2}\, d\mathrm{Vol} + \frac{1}{c^2} \int \partial^j \left[\left(r_i - r^{\mathrm{mech}}_{\mathrm{cm},i} \right) J_j V^{(\mathrm{C})} \right] d\mathrm{Vol}. \tag{43}$$

The latter volume integral becomes a surface integral at infinity, so is zero for any bounded current density. Then,

$$\mathbf{P}^{\mathrm{mech}}_{\mathrm{hidden}} = -\int \frac{\mathbf{J} \cdot \mathbf{E}}{c^2} \left(\mathbf{r} - \mathbf{r}^{\mathrm{mech}}_{\mathrm{cm}} \right) d\mathrm{Vol} = -\int \frac{V^{(\mathrm{C})} \mathbf{J}}{c^2}\, d\mathrm{Vol} = -\mathbf{P}_{\mathrm{EM}} = -\mathbf{P}^{\mathrm{EM}}_{\mathrm{hidden}} \quad \text{(stationary)}. \tag{44}$$

Thus, Thomson's paradox of sec. 2.4.1 is resolved by the existence of "hidden" mechanical momentum according to eq. (33).

Note that the argument leading to eq. (44) indicates that for stationary systems the "hidden" mechanical momentum does not depend on the location of $\mathbf{r}^{\mathrm{mech}}_{\mathrm{cm}}$ *(which could be varied arbitrarily by changing the charge of mass ratio of the various particles in the system),*

$$\mathbf{P}^{\mathrm{mech}}_{\mathrm{hidden}} = -\int \frac{\mathbf{J} \cdot \mathbf{E}}{c^2}\, \mathbf{r}\, d\mathrm{Vol} = -\int \frac{V^{(\mathrm{C})} \mathbf{J}}{c^2}\, d\mathrm{Vol} = -\mathbf{P}_{\mathrm{EM}} = -\mathbf{P}^{\mathrm{EM}}_{\mathrm{hidden}} \quad \textit{(stationary)}. \tag{45}$$

Hence, we now have a fourth form for the field momentum of a finite, stationary system, which we call the Aharonov form,

$$\mathbf{P}_{\mathrm{EM}} = \mathbf{P}^{(\mathrm{A})} = \int \frac{\mathbf{J} \cdot \mathbf{E}}{c^2}\, \mathbf{r}\, d\mathrm{Vol} \qquad \textit{(stationary)}. \tag{46}$$

The success of the form (44) for the "hidden" momentum of the mechanical subsystem suggests that we consider the merits of defining the density of "hidden" mechanical momentum in a stationary system to be,

$$\mathbf{p}^{\mathrm{mech}}_{\mathrm{hidden}} = -\frac{\mathbf{J} \cdot \mathbf{E}}{c^2} \left(\mathbf{r} - \mathbf{r}^{\mathrm{mech}}_{\mathrm{cm}} \right) \qquad \text{(stationary)}. \tag{47}$$

2.4.4 "Hidden" Angular Momentum

According to eq. (47), the density of "hidden" mechanical angular momentum for the mechanical subsystem of a stationary system is,[38]

$$\begin{aligned} \mathbf{l}^{\mathrm{mech}}_{\mathrm{hidden}} &= \mathbf{r} \times \mathbf{p}^{\mathrm{mech}}_{\mathrm{hidden}} = -\frac{\mathbf{J} \cdot \mathbf{E}}{c^2}\, \mathbf{r} \times \left(\mathbf{r} - \mathbf{r}^{\mathrm{mech}}_{\mathrm{cm}} \right) = \frac{\mathbf{J} \cdot \mathbf{E}}{c^2}\, \mathbf{r} \times \mathbf{r}^{\mathrm{mech}}_{\mathrm{cm}} \\ &= \frac{\mathbf{J} \cdot \mathbf{E}}{c^2} \left(\mathbf{r} - \mathbf{r}^{\mathrm{mech}}_{\mathrm{cm}} \right) \times \mathbf{r}^{\mathrm{mech}}_{\mathrm{cm}} = \mathbf{r}^{\mathrm{mech}}_{\mathrm{cm}} \times \mathbf{p}^{\mathrm{mech}}_{\mathrm{hidden}} \qquad \text{(stationary)}. \end{aligned} \tag{48}$$

[38] If we inferred from eq. (45) that the density of "hidden" mechanical angular momentum were $\mathbf{r} \times -(\mathbf{J} \cdot \mathbf{E})\, \mathbf{r}/c^2 = 0$, there would be no "hidden" mechanical angular momentum in any stationary system.

In this view, the total "hidden" mechanical angular momentum for a stationary system is,

$$\mathbf{L}_{\text{hidden}}^{\text{mech}} = \int \mathbf{l}_{\text{hidden}}^{\text{mech}}\, d\text{Vol} = \mathbf{r}_{\text{cm}}^{\text{mech}} \times \int \mathbf{p}_{\text{hidden}}^{\text{mech}}\, d\text{Vol} = \mathbf{r}_{\text{cm}}^{\text{mech}} \times \mathbf{P}_{\text{hidden}}^{\text{mech}}, \tag{49}$$

where $\mathbf{r}_{\text{cm}}^{\text{mech}}$ is the center of mass/energy of the mechanical subsystem.[39,40,41]

For the example of a solenoid plus charge at rest, where the mass of the solenoid is much larger than the mass of the charge, $\mathbf{r}_{\text{cm}}^{\text{mech}}$ is approximately at the center of the solenoid, which we take to be the origin. That is, $\mathbf{r}_{\text{cm}}^{\text{mech}} \approx 0$, and so $\mathbf{L}_{\text{hidden}}^{\text{mech}} \approx 0$, according to eq. (49), independent of the location of the charge. This agrees with the inference in footnote 7 of sec. 2.1.1 that the "hidden" mechanical angular momentum of a charge + solenoid is zero for any location of the charge.[42]

Hence, it appears that the definition (33) of "hidden" momentum of a (sub)system leads to a satisfactory identification of its density as,

$$\mathbf{p}_{\text{hidden}}^{\text{subsystem}} = -\frac{f_{\text{subsystem}}^{0}}{c}\left(\mathbf{r} - \mathbf{r}_{\text{cm}}^{\text{subsystem}}\right) = -\frac{\partial_\mu T_{\text{subsystem}}^{0\mu}}{c}\left(\mathbf{r} - \mathbf{r}_{\text{cm}}^{\text{subsystem}}\right), \tag{51}$$

which can be written as eq. (47) for the mechanical subsystem of a stationary system.[43]

2.5 Quasistatic Systems

The above results were established using three conditions of stationary electromagnetic systems, $\mathbf{E} = -\boldsymbol{\nabla} V^{(\text{C})}$, $\boldsymbol{\nabla} \times \mathbf{B} = 4\pi \mathbf{J}/c$ and $\boldsymbol{\nabla} \cdot \mathbf{J} = 0$. Then, the electromagnetic momentum, which is equal and opposite to the "hidden" mechanical momentum, is of order $1/c^2$, which is very small compared to typical mechanical momenta. Only in isolated, stationary systems, for which $\mathbf{P}_{\text{total}} = 0$, are these tiny momenta relatively prominent.

In nonstationary systems, the electromagnetic and "hidden" mechanical momentum can have terms of order $1/c^3$ (and higher). For quasistatic systems with all velocities small compared to c (and all accelerations small compared to c^2/R where R is the characteristic

---

[39]The result (49) is similar to that of eq. (26) of [25]. As the result is zero when measured about the center of mass/energy of the mechanical subsystem, we could say that the "hidden" mechanical angular momentum is "orbital" rather than "intrinsic", where the latter is with respect to the center of mass/energy.

[40]The existence of nonzero "hidden" mechanical angular momentum when $\mathbf{r}_{\text{cm}}^{\text{mech}}$ is not at the origin is the key to the resolution of Mansuripur's paradox [47].

[41]For example, the "hidden" mechanical angular momentum of a charge q of mass M_q at distance $\mathbf{r}_q$ from a point, Ampèrian magnetic dipole $\mathbf{m}$ of mass M_m, all at rest, is, about the location of moment $\mathbf{m}$,

$$\mathbf{L}_{\text{hidden}}^{\text{mech}} = \frac{M_q}{M_q + M_m}\mathbf{r}_q \times \left(\frac{\mathbf{m}}{c} \times \frac{-q\mathbf{r}_q}{r_q^3}\right), \qquad \text{as [16]}, \qquad \mathbf{P}_{\text{hidden}}^{\text{mech}} = -\mathbf{P}_{\text{EM}} = \frac{\mathbf{m} \times \mathbf{E}_q}{c}. \tag{50}$$

[42]It remains that the charge-to-mass ratio of the particles in a stationary system could be altered to change the location of $\mathbf{r}_{\text{cm}}^{\text{mech}}$ without any change to the field momentum, the "hidden" mechanical momentum or the field angular momentum, but the "hidden" mechanical angular momentum would change (even in models of electrical-current filaments as due to counterflows of opposite charges where the charge-to-mass ratio is the same along any particular filament.

[43]It is noteworthy that eq. (51) indicates that the local density of "hidden" momentum in any subsystem is directed radially from the center of mass/energy of that subsystem.

size of the system) these higher-order terms will be negligible, and it is a good approximation to consider the electromagnetic and "hidden" mechanical momentum only at order $1/c^2$, for which the results of secs. 2.1-2.4 can be used.[44,45,46]

While the field momentum (= "hidden" field momentum) of a quasistatic system can be computed using any of four volume densities, it turns out that the field angular momentum can only be computed in general using two of those densities, of which $\mathbf{r} \times (\mathbf{E} \times \mathbf{B})/4\pi c$ is often considered to be "the" density of field angular momentum,

$$\mathbf{L}_{\rm EM} = \int \mathbf{r} \times \frac{\mathbf{E} \times \mathbf{B}}{4\pi c}\, d{\rm Vol} = \int \mathbf{r} \times \frac{\varrho \mathbf{A}^{\rm (C)}}{c}\, d{\rm Vol} \neq \int \mathbf{r} \times \frac{V^{\rm (C)}\mathbf{J}}{c^2}\, d{\rm Vol} \neq \mathbf{r}_{\rm cm}^{\rm EM} \times \mathbf{P}_{\rm EM} = \mathbf{L}_{\rm hidden}^{\rm EM}. \qquad (52)$$

The equality of the first two forms in eq. (52) was first demonstrated in [13]. That these forms differ from $\int \mathbf{r} \times \frac{V^{\rm (C)}\mathbf{J}}{c^2}\, d{\rm Vol}$ is shown in Appendix A. For a solenoid + single charge, the fourth form, $\mathbf{L}_{\rm hidden}^{\rm EM}$, equals the first two forms only when the charge is on the solenoid axis.[47]

Appendix A: General Relations between $\mathbf{L}^{\rm (P)}$, $\mathbf{L}^{\rm (F)}$ and $\mathbf{L}^{\rm (M)}$

This Appendix is due to V. Hnizdo.

For stationary systems where $\mathbf{E} = -\boldsymbol{\nabla} V^{\rm (C)}$, *i.e.*, $E_i = -\partial_i V^{\rm (C)}$, and $\boldsymbol{\nabla} \times \mathbf{B} = 4\pi \mathbf{J}/c$,

$$\begin{aligned}
[\mathbf{r} \times (\mathbf{E} \times \mathbf{B})]_i &= \epsilon_{ijk} r_j \epsilon_{klm}(-\partial_l V) B_m = -\epsilon_{ijk}\epsilon_{klm} r_j [\partial_l (V B_m) - V^{\rm (C)} \partial_l B_m] \\
&= -\epsilon_{ijk}\epsilon_{klm}[\partial_l (r_j V^{\rm (C)} B_m) - V^{\rm (C)} B_m \partial_l r_j] + \epsilon_{ijk} V^{\rm (C)} r_j (\boldsymbol{\nabla} \times \mathbf{B})_k \\
&\to \epsilon_{ijk}\epsilon_{klm} V^{\rm (C)} B_m \delta_{jl} + \frac{4\pi}{c}\epsilon_{ijk} V^{\rm (C)} r_j J_k = -\epsilon_{ijk}\epsilon_{mjk} V^{\rm (C)} B_m + \frac{4\pi V^{\rm (C)}}{c}(\mathbf{r} \times \mathbf{J})_i \\
&= -\delta_{ij}^{mj} V^{\rm (C)} B_m + \frac{4\pi}{c}(\mathbf{r} \times V^{\rm (C)}\mathbf{J})_i = -2\delta_{im} V^{\rm (C)} B_m + \frac{4\pi}{c}(\mathbf{r} \times V^{\rm (C)}\mathbf{J})_i \\
&= -2V^{\rm (C)} B_i + \frac{4\pi}{c}(\mathbf{r} \times V^{\rm (C)}\mathbf{J})_i, \qquad (53)
\end{aligned}$$

where in the third line we drop the full-differential term $\partial_l (r_j V^{\rm (C)} B_m)$, anticipating that $\int \partial_l (r_j V^{\rm (C)} B_m)\, d{\rm Vol} \to \oint r_j V^{\rm (C)} B_m\, d{\rm Area} \to 0$ for fields that fall off sufficiently quickly at

[44] The departure of the electric field $\mathbf{E} = -\boldsymbol{\nabla} V^{\rm (C)} - \partial \mathbf{A}^{\rm (C)}/\partial ct$ from the form $\mathbf{E} = -\boldsymbol{\nabla} V^{\rm (C)}$ for stationary systems is of order v/c, which has negligible effect on the electromagnetic and "hidden" mechanical momentum in the quasistatic approximation.

Similarly, motion of charges leads to nonzero time dependence of the electric field, $\partial \mathbf{E}/\partial t$, and of the charge density, $\partial \varrho/\partial t$, at order v/c. The resulting "corrections" to the Maxwell equation $\boldsymbol{\nabla} \times \mathbf{B} = 4\pi \mathbf{J}/c + \partial \mathbf{E}/\partial ct$ and $\boldsymbol{\nabla} \cdot \mathbf{J} = -\partial \varrho/\partial t$ due to the time derivatives are at one order higher in $1/c$, and so negligible effect on the electromagnetic and "hidden" mechanical momentum in the quasistatic approximation.

[45] See also [48].

[46] The quasistatic approximation consider here is looser than the so-called Darwin approximation [49] in which terms of order $1/c^2$ in the electric field are considered, leading to terms of order $1/c^4$ in field momentum and "hidden" momentum. The present quasistatic approximation encompasses the approximation of the fields by their instantaneous values, as well as effects of induction and of the displacement current for low frequencies.

[47] In computing $\mathbf{r}_{\rm cm}^{\rm EM} = \int U^{\rm EM}\, \mathbf{r}\, d{\rm Vol} / \int U^{\rm EM}\, d{\rm Vol}$, the field energy $U^{\rm EM}$ does not include the electromagnetic self energy of charges. For the case of a solenoid + single charge (at rest) the field energy is just the magnetic energy, whose center is the center of the solenoid, *i.e.*, the origin, so $\mathbf{r}_{\rm cm}^{\rm EM} = 0$ and $\mathbf{L}_{\rm hidden}^{\rm EM} = 0$ for any location of the charge.

large distances. Then, for bounded, stationary systems, where $\mathbf{L}^{(\mathrm{P})} = \mathbf{L}^{(\mathrm{M})}$,

$$\mathbf{L}^{(\mathrm{P})} = \int \mathbf{r} \times \frac{\mathbf{E} \times \mathbf{B}}{4\pi c}\, d\mathrm{Vol} = -\int \frac{V^{(\mathrm{C})}\mathbf{B}}{2\pi c}\, d\mathrm{Vol} + \int \mathbf{r} \times \frac{V^{(\mathrm{C})}\mathbf{J}}{c^2}\, d\mathrm{Vol} = \mathbf{L}^{(\mathrm{F})} - \int \frac{V^{(\mathrm{C})}\mathbf{B}}{2\pi c}\, d\mathrm{Vol}. \quad (54)$$

That is, the Poynting and Furry forms of field angular momentum are not equal, in general.[48]

Following [13], we can also establish a relation between $\mathbf{L}_{\mathrm{P}}$ and $\mathbf{L}_{\mathrm{M}}$ by starting with the vector-calculus identity,

$$\boldsymbol{\nabla}(\mathbf{E} \cdot \mathbf{A}^{(\mathrm{C})}) = (\mathbf{E} \cdot \boldsymbol{\nabla})\mathbf{A}^{(\mathrm{C})} + (\mathbf{A}^{(\mathrm{C})} \cdot \boldsymbol{\nabla})\mathbf{E} + \mathbf{E} \times (\boldsymbol{\nabla} \times \mathbf{A}^{(\mathrm{C})}) + \mathbf{A}^{(\mathrm{C})} \times (\boldsymbol{\nabla} \times \mathbf{E}), \qquad (55)$$

where the last term vanishes for stationary systems, which obey $\boldsymbol{\nabla} \times \mathbf{E} = 0$. Then, since $\mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A}^{(\mathrm{C})}$, we have that,

$$\mathbf{E} \times \mathbf{B} = \mathbf{E} \times (\boldsymbol{\nabla} \times \mathbf{A}^{(\mathrm{C})}) = \boldsymbol{\nabla}(\mathbf{E} \cdot \mathbf{A}^{(\mathrm{C})}) - (\mathbf{E} \cdot \boldsymbol{\nabla})\mathbf{A}^{(\mathrm{C})} - (\mathbf{A}^{(\mathrm{C})} \cdot \boldsymbol{\nabla})\mathbf{E}. \qquad (56)$$

Now,

$$(\mathbf{E} \cdot \boldsymbol{\nabla})A_i = E_j \partial_j A_i^{(\mathrm{C})} = \partial_j(E_j A_i^{(\mathrm{C})}) - A_i^{(\mathrm{C})} \partial_j E_j = \partial_j(E_j A_i^{(\mathrm{C})}) - 4\pi\rho A_i^{(\mathrm{C})}, \qquad (57)$$

using the first Maxwell equation. Similarly, in the Coulomb gauge, where $\boldsymbol{\nabla} \cdot \mathbf{A}^{(\mathrm{C})} = 0$,

$$(\mathbf{A}^{(\mathrm{C})} \cdot \boldsymbol{\nabla})E_i = A_j^{(\mathrm{C})} \partial_j E_i = \partial_j(A_j^{(\mathrm{C})} E_i) - E_i \partial_j A_j^{(\mathrm{C})} = \partial_j(A_j^{(\mathrm{C})} E_j). \qquad (58)$$

Thus,

$$(\mathbf{E} \times \mathbf{B})_i = 4\pi\rho A_i^{(\mathrm{C})} + \partial_i(\mathbf{E} \cdot \mathbf{A}^{(\mathrm{C})}) - \partial_j(E_j A_i^{(\mathrm{C})}) - \partial_j(A_j^{(\mathrm{C})} E_j). \qquad (59)$$

When integrating eq. (59) over all space, the terms beginning with derivatives can be integrated parts, and vanish for fields that fall off suitably quickly. Hence,

$$\mathbf{P}^{(\mathrm{P})} = \int \frac{\mathbf{E} \times \mathbf{B}}{4\pi c}\, d\mathrm{Vol} = \int \frac{\rho \mathbf{A}^{(\mathrm{C})}}{c}\, d\mathrm{Vol} = \mathbf{P}^{(\mathrm{M})}, \qquad (60)$$

Likewise, in the integral of $\mathbf{r} \times (\mathbf{E} \times \mathbf{B})$, we use eq. (59), and again, integration by parts of the terms involving derivatives indicates that these terms vanish (for fields that fall off suitably quickly), and,

$$\mathbf{L}^{(\mathrm{P})} = \int \mathbf{r} \times \frac{\mathbf{E} \times \mathbf{B}}{4\pi c}\, d\mathrm{Vol} = \int \mathbf{r} \times \frac{\rho \mathbf{A}^{(\mathrm{C})}}{c}\, d\mathrm{Vol} = \mathbf{L}^{(\mathrm{M})}. \qquad (61)$$

[48] For stationary fields that fall off sufficiently quickly at large distance, $\int V^{(\mathrm{C})}\mathbf{B}\, d\mathrm{Vol} = \int \mathbf{E} \times \mathbf{A}^{(\mathrm{C})}\, d\mathrm{Vol}$.

Appendix B: The Argument of Coleman and Van Vleck

An explanation of "hidden" mechanical momentum was given by Coleman and van Vleck [17], who remarked in footnote 9 that an electric charge q of rest mass m and velocity $\mathbf{v}$ that moves in an electric field $\mathbf{E} = -\boldsymbol{\nabla} V^{(\mathrm{C})} - \partial \mathbf{A}^{(\mathrm{C})}/\partial ct$ has the conserved energy,[49]

$$U = \gamma mc^2 + qV^{(\mathrm{C})}, \tag{62}$$

where $\gamma = 1/\sqrt{1 - v^2/c^2}$. If in addition the moving charge is part of a current density $\mathbf{J}$ that is stationary (which implies that the "conductor" which supports the current density is "at rest"), then $\mathbf{A}^{(\mathrm{C})}$ is constant in time, $\mathbf{E} = -\boldsymbol{\nabla} V^{(\mathrm{C})}$, and,

$$\int \mathbf{J}\, d\mathrm{Vol} = 0, \tag{63}$$

and the current density at a point $\mathbf{x}$ can be written as,

$$\mathbf{J}(\mathbf{x}) = n(\mathbf{x}) q \mathbf{v}(\mathbf{x}), \tag{64}$$

where $n(\mathbf{x})$ is the volume number density of like charges q in a small region around $\mathbf{x}$. The mechanical momentum associated with the current density $\mathbf{J}$ is,

$$\mathbf{P}_{\mathrm{mech}} = \int n(\mathbf{x})\gamma(\mathbf{x}) m\mathbf{v}(\mathbf{x})\, d\mathrm{Vol} = \frac{m}{q}\int n(\mathbf{x})\gamma(\mathbf{x}) q\mathbf{v}(\mathbf{x})\, d\mathrm{Vol} = \frac{m}{q}\int \gamma(\mathbf{x})\mathbf{J}(\mathbf{x})\, d\mathrm{Vol} \tag{65}$$

If the electric scalar potential $V^{(\mathrm{C})}$ were zero everywhere, then $\mathbf{E} = 0$, the speed v of the charges could not vary with position, $\gamma(\mathbf{x}) = \gamma_0$, a constant, and the energy of each charge is just $U = \gamma_0 mc^2$. Thus,

$$\mathbf{P}_{\mathrm{mech}}(V^{(\mathrm{C})} = 0) = \frac{\gamma_0 m}{q}\int \mathbf{J}(\mathbf{x})\, d\mathrm{Vol} = 0. \tag{66}$$

When a (stationary) electric field is present, the scalar potential varies with position and energy conservation (62) implies that,

$$\gamma_0 mc^2 = \gamma(\mathbf{x}) mc^2 + qV^{(\mathrm{C})}(\mathbf{x}), \tag{67}$$

such that,

$$\begin{aligned}\mathbf{P}_{\mathrm{mech}} &= \frac{m}{q}\int n(\mathbf{x})\gamma(\mathbf{x}) q\mathbf{v}(\mathbf{x})\, d\mathrm{Vol} = \frac{m}{q}\int n(\mathbf{x})\left(\gamma_0 - \frac{qV}{mc^2}\right) q\mathbf{v}(\mathbf{x})\, d\mathrm{Vol} \\ &= \frac{\gamma_0 m}{q}\int \mathbf{J}\, d\mathrm{Vol} - \int \frac{V^{(\mathrm{C})}\mathbf{J}}{c^2}\, d\mathrm{Vol} = -\int \frac{V^{(\mathrm{C})}\mathbf{J}}{c^2}\, d\mathrm{Vol} \equiv \mathbf{P}_{\mathrm{hidden}}. \end{aligned} \tag{68}$$

[49]The energy conservation (62) ignores gravity, and supposes that the only force that can act on an electric charge is the Lorentz force $q(\mathbf{E} + \mathbf{v}/c \times \mathbf{B})$. Furthermore, the scalar potential V is the microscopic potential, which is not necessarily the same as the potential due to charges external to the current density.

The arguments of this section apply to the model of the current density in the solenoid as due to counterflowing, noninteracting, opposite charges with potential V due to external charges, but for the model of counter-rotating cylinders of charge, the potential includes a microscopic electric potential that keeps the charges fixed with respect to the rotating cylinders in the presence of the external electric field. This issue was noted in [43], where it was remarked that there is a "hidden" mechanical momentum due to the stresses in the rotating cylinders.

The result (68) is the "hidden" mechanical momentum postulated by Shockley.

Furthermore, comparing with eq. (1), which applies for stationary systems, we have that in such systems,[50]

$$\mathbf{P}_{\rm EM} = -\mathbf{P}_{\rm hidden}, \qquad \mathbf{P}_{\rm total} = \mathbf{P}_{\rm EM} + \mathbf{P}_{\rm hidden} = 0. \tag{69}$$

Appendix C: Field Momentum for an Infinite Solenoid plus Charge

Although there is a general theorem that $\mathbf{P}^{(\rm M)} = \mathbf{P}^{(\rm P)} = \mathbf{P}^{(\rm F)}$ for bounded, stationary systems, eq. (60) above, we illustrate this equality happens to hold for an infinite solenoid plus charge at rest, again supposing the current that generates the magnetic field is due to counter-rotating rings with fixed, opposite charge distributions.

For an electric charge q at distance a from the axis of an infinite solenoid of radius R, the field momentum based on the Maxwell form $\mathbf{P}^{(\rm M)}$ is, recalling eq. (5),

$$\mathbf{P}^{(\rm M)} = \int \frac{\varrho \mathbf{A}^{(\rm C)}}{c}\, d{\rm Vol} = \frac{qB}{2c}\,\hat{\boldsymbol{\phi}} \begin{cases} a & (a < R), \\ \frac{R^2}{a} & (a > R). \end{cases} \tag{70}$$

For charge q at $(x, y, z) = (a, 0, 0)$ the electric field is $\mathbf{E}(x, y, z) = q\,\mathbf{d}/d^3 = q[(x - a)\,\hat{\mathbf{x}} + y\,\hat{\mathbf{y}} + z\,\hat{\mathbf{z}}]/[(x - a)^2 + y^2 + z^2)^{3/2}$, so the field momentum based on the Poynting form $\mathbf{P}^{(\rm P)}$ is (prob. 8.14 of [34]),

$$\begin{aligned} \mathbf{P}^{(\rm P)} &= \int \frac{\mathbf{E}\times\mathbf{B}}{4\pi c}\, d{\rm Vol} = \frac{qB}{4\pi c}\int \frac{(x-a)\,\hat{\mathbf{x}} + y\,\hat{\mathbf{y}} + z\,\hat{\mathbf{z}}}{[(x-a)^2 + y^2 + z^2]^{3/2}} \times \hat{\mathbf{z}}\, d{\rm Vol} \\ &= \frac{qB}{4\pi c}\int \frac{y\,\hat{\mathbf{x}} - (x-a)\,\hat{\mathbf{y}}}{[(x-a)^2 + y^2 + z^2]^{3/2}}\, dx\, dy\, dz = \frac{qB}{2\pi c}\,\hat{\mathbf{y}} \int \frac{a - x}{(x-a)^2 + y^2}\, dx\, dy \\ &= \frac{qB}{2\pi c}\,\hat{\mathbf{y}} \int_0^R \rho\, d\rho \int_0^{2\pi} d\phi\, \frac{a - \rho\cos\phi}{\rho^2 - 2a\rho\cos\phi + a^2} = \frac{qB}{2\pi c}\,\hat{\mathbf{y}} \int_0^R d\rho \int_0^{2\pi} d\phi\, \frac{\frac{a}{\rho} - \cos\phi}{1 - 2\frac{a}{\rho}\cos\phi + \frac{a^2}{\rho^2}}\,. \end{aligned} \tag{71}$$

For $a > R \geq \rho$ the remaining integral is (Dwight 859.113, 859.122, 141.1 and 143.1),

$$\mathbf{P}_{\rm P}(a > R) = \frac{qB}{ac}\,\hat{\mathbf{y}} \int_0^R d\rho\, \frac{a^2\rho - \rho^3}{a^2 - \rho^2} = \frac{qBR^2}{2ac}\,\hat{\mathbf{y}}. \tag{72}$$

For $a < R$ the part of the integral for $a > \rho$ has the form of eq. (72) but with $R \to a$, and the remaining integral is,

$$\frac{qB}{c}\,\hat{\mathbf{y}} \int_a^R d\rho \left(\frac{a\rho}{\rho^2 - a^2} - \frac{a\rho}{\rho^2 - a^2}\right) = 0. \tag{73}$$

[50]The Furry form $\mathbf{P}^{(\rm F)}$ is identical to the "hidden" mechanical momentum (68) except for the overall sign. The result of Furry was obtained by transforming the Maxwell form $\mathbf{P}^{(\rm F)}$ of field momentum, without any consideration of mechanical momentum. Furry's paper [16] was published one year after the paper of Coleman and van Vleck [17], and although Furry referenced the latter paper he did not notice that his argument starting on p. 634 was anticipated in footnote 9 of [17].

That is,

$$\mathbf{P}_{\rm P}(a < R) = \frac{qBa}{2c}\,\hat{\mathbf{y}}, \tag{74}$$

and since $\hat{\mathbf{y}} = \hat{\boldsymbol{\phi}}$ at $(a, 0, 0)$, $\mathbf{P}_{\rm P} = \mathbf{P}_{\rm M}$ for an infinite solenoid (although $\mathbf{L}_{\rm P} \neq \mathbf{L}_{\rm M}$).

For the field momentum based on the Furry form $\mathbf{P}^{\rm (F)}$ we consider a solenoid of half length l. Then, the Coulomb-gauge electric scalar potential for charge q at $(a, 0, 0)$ is,[51]

$$V^{\rm (C)} = \frac{q}{r} = \frac{q}{(\rho^2 - 2a\rho\cos\phi + a^2 + z^2)^{1/2}}\,, \tag{75}$$

so the Furry momentum for the solenoid plus charge is,

$$\begin{aligned}
\mathbf{P}^{\rm (F)} &= \int \frac{V^{\rm (C)}\mathbf{J}}{c^2}\,d{\rm Vol} = \frac{qB}{4\pi c}\int \frac{1}{(R^2+z^2)^{1/2}}\,d{\rm Area}\,\hat{\boldsymbol{\phi}} = \frac{qB}{4\pi c}\int_0^{2\pi} d\phi \int_0^l dz\,\frac{-\hat{\mathbf{x}}\sin\phi + \hat{\mathbf{y}}\cos\phi}{\sqrt{R^2 - 2aR\cos\phi + a^2 + z^2}} \\
&= \frac{qB}{4\pi c}\hat{\mathbf{y}}\int_0^{2\pi} d\phi \int_0^l dz\,\frac{\cos\phi}{\sqrt{R^2 - 2aR\cos\phi + a^2 + z^2}} \\
&= \frac{qB}{2\pi c}\hat{\mathbf{y}}\int_0^{2\pi} d\phi\,\cos\phi(R^2 - 2aR\cos\phi + a^2)\sinh^{-1}\frac{l}{\sqrt{R^2 - 2aR\cos\phi + a^2}}\,.
\end{aligned} \tag{76}$$

While it may seem unlikely that this result is finite and independent of the half length l of the solenoid as this grows large, there is a general theorem that $\mathbf{P}^{\rm (M)} = \mathbf{P}^{\rm (P)} = \mathbf{P}^{\rm (F)}$ for bounded, stationary systems. Then, since the result (70) holds for the charge at radius a small compared to the half length l of the solenoid, we infer that $\mathbf{P}^{\rm (M)} = \mathbf{P}^{\rm (P)} = \mathbf{P}^{\rm (F)}$ also holds for an infinite solenoid.[52]

C.1. "Hidden" Mechanical Momentum *(June 2020)*

The example of a long solenoid plus electric charge, all at rest, has zero total momentum but nonzero field momentum (70). The latter is compensated by an equal and opposite "hidden" mechanical momentum in the electric current of the solenoid. If the current in the solenoid somehow decays to zero, both the field momentum and the "hidden" momentum disappear. The decaying magnetic field is associated with an azimuthal electric field, such that the charge takes on a nonzero "overt" momentum. This is compensated by an equal an opposite "overt" momentum of the solenoid, arising from conversion of its initial "hidden" momentum as the latter decays, such that the final, total momentum of the system is zero [52].

[51]The form (75) applies everywhere only if the conductor of the solenoid does not "shield" the electric field of the charge. This is consistent with the model of the solenoid currents as due to counter-rotating, opposite charges in nonconducting tubes, or to a pair of counterrotating, oppositely charged cylinders.

[52]For $a = 0$, *i.e.*, for the charge on the axis,

$$\mathbf{P}^{\rm (F)}(a = 0) = \frac{qBR^2}{2\pi c}\sinh^{-1}\frac{l}{R}\,\hat{\mathbf{y}}\int_0^{2\pi} d\phi\,\cos\phi = 0, \tag{77}$$

which does equal $\mathbf{P}^{\rm (M)}$ and $\mathbf{P}^{\rm (P)}$ for any length l in this special case.

Appendix D: Field Momentum of Shells of Charge and Current

Following Romer [26],[53] we consider a spherical shell of radius a with surface-charge density,

$$\sigma(r=a) = \frac{3p\cos\theta}{4\pi a^3}\,, \tag{78}$$

proportional to $\cos\theta$ (with respect to the z-axis), such that the charge distribution has electric dipole moment $\mathbf{p} = p\,\hat{\mathbf{z}}$, and the electric field $\mathbf{E}$ has the form,

$$\mathbf{E} = \begin{cases} -\frac{\mathbf{p}}{a^3} & (r < a), \\ \frac{3(\mathbf{p}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}}-\mathbf{p}}{r^3} & (r > a), \end{cases} \tag{79}$$

for which the tangential component of $\mathbf{E}$ is continuous across $r = a$.The system also includes an electrically neutral spherical shell of radius b with surface currents proportional to $\sin\theta'$ (with respect to the z'-axis), such that the current distribution has magnetic dipole moment $\mathbf{m} = m\,\hat{\mathbf{z}}'$, and the magnetic field $\mathbf{B}$ has the form,

$$\mathbf{B} = \begin{cases} \frac{2\mathbf{m}}{b^3} & (r < b), \\ \frac{3(\mathbf{m}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}}-\mathbf{m}}{r^3} & (r > b), \end{cases} \tag{80}$$

for which the normal component of $\mathbf{B}$ is continuous across $r = b$. The system is in vacuum.

We consider the case that $a > b$, for which the Poynting form of the electromagnetic-field momentum $\mathbf{P}^{(\mathrm{P})}$ can be computed as,[54]

$$\begin{aligned}
\mathbf{P}^{(\mathrm{P})} &= \int \frac{\mathbf{E}\times\mathbf{B}}{4\pi c}\, d\mathrm{Vol} \\
&= \int_{r<b} \frac{-\mathbf{p}\times 2\mathbf{m}}{4\pi a^3 b^3 c}\, d\mathrm{Vol} + \int_{b<r<a} \frac{-\mathbf{p}\times[3(\mathbf{m}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}}-\mathbf{m}]}{4\pi a^3 r^3 c}\, d\mathrm{Vol} \\
&\quad + \int_{r>a} \frac{[3(\mathbf{p}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}}-\mathbf{p}]\times[3(\mathbf{m}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}}-\mathbf{m}]}{4\pi r^6 c}\, d\mathrm{Vol} \\
&= -\frac{2\mathbf{p}\times\mathbf{m}}{3a^3 c} - \frac{\mathbf{p}\times\mathbf{m}}{a^3 c}\ln\frac{a}{b} + \frac{\mathbf{p}\times\mathbf{m}}{a^3 c}\ln\frac{a}{b} - \frac{2\mathbf{p}\times\mathbf{m}}{3a^3 c} + \frac{\mathbf{p}\times\mathbf{m}}{3a^3 c} \\
&= \frac{\mathbf{m}\times\mathbf{p}}{a^3 c}\,.
\end{aligned} \tag{81}$$

For the special case that the magnetic moment $\mathbf{m}$ is along the y-axis, the field momentum (81) is,

$$\mathbf{P}^{(\mathrm{P})} = \frac{mp}{a^3 c}\,\hat{\mathbf{x}}. \tag{82}$$

The vector potential in the Coulomb gauge that corresponds to eq. (20) is,

$$\mathbf{A}^{(\mathrm{C})}(r<b) = \frac{mr\sin\theta'}{b^3}\,\hat{\boldsymbol{\phi}}', \qquad \mathbf{A}^{(\mathrm{C})}(r>b) = \frac{m\sin\theta'}{r^2}\,\hat{\boldsymbol{\phi}}', \tag{83}$$

[53] Romer's example is considered in greater detail in sec. 2.2 of [50], and in Appendix B of [51]

[54] If $a < b$ the result is $\mathbf{P}^{(\mathrm{P})} = \mathbf{m}\times\mathbf{p}/b^3 c$.

and for that case that $\hat{\mathbf{z}}' = \hat{\mathbf{y}}$ (and $\hat{\mathbf{x}}' = \hat{\mathbf{x}}$ and $\hat{\mathbf{y}}' = -\hat{\mathbf{x}}$), $\cos\theta = -\sin\theta' \sin\phi'$ in the spherical coordinate system based on the z'-axis.

The Maxwell form of the electromagnetic field momentum when $\hat{\mathbf{z}}' = \hat{\mathbf{y}}$ is,

$$\begin{aligned}\mathbf{P}^{(\mathrm{M})} &= \int \frac{\varrho \mathbf{A}^{(\mathrm{C})}}{c}\, d\mathrm{Vol} = \int_0^\pi a^2 \sin\theta'\, d\theta' \int_0^{2\pi} d\phi'\, \frac{-3p\sin\theta' \sin\phi'}{4\pi a^3} \frac{m\sin\theta'}{a^2 c}(-\sin\phi'\,\hat{\mathbf{x}}' + \cos\phi'\,\hat{\mathbf{y}}') \\ &= \frac{3mp}{4a^3 c}\int_0^\pi \sin^3\theta'\, d\theta'\,\hat{\mathbf{x}}' = \frac{mp}{a^3 c}\,\hat{\mathbf{x}}, \end{aligned} \tag{84}$$

in agreement with the result (82) for the Poynting form.

Appendix E: Fixed Charge plus Solenoid with Conducting Wires[55]

E.1 Linear Momentum

If a solenoid has its electric current flowing in/on a perfect conductor, then (for steady currents) the entire conductor is at the same electric potential V, while the current density $\mathbf{J}$ in the (idealized) solenoid is axially symmetry. Hence, the Furry form of the field momentum is zero,

$$\mathbf{P}^{(\mathrm{F})} = \int \frac{V\mathbf{J}}{c^2}\, d\mathrm{Vol} = 0, \tag{85}$$

for any length of the solenoid, finite or infinite.

For a finite solenoid, we have that $\mathbf{P}^{(\mathrm{F})} = \mathbf{P}^{(\mathrm{P})} = \mathbf{P}^{(\mathrm{M})}$, so all three forms of the field momentum are zero.

For an infinite solenoid, its perfectly conducting "coil" isolated the electric field in the two regions, $r < R$ and $r > R$, from one another. If the fixed charge is at $a > R$, where $\mathbf{B} = 0$, then for $r < R$ the electric field is zero, and $\mathbf{E} \times \mathbf{B} = 0$ everywhere. However if the fixed charge is at $(a, 0, 0)$, with $0 < a < R$, then for $r < R$ the electric field is the same as for the solenoid considered in Appendix C, so the (Poynting) field momentum in nonzero, as given by eq. (74).

E.1 Angular Momentum

For a finite solenoid, the Maxwell and Poynting forms of the field angular momentum are equal, as noted in Appendix A (and, in general, different from the Furry form, which we disregard).

For an infinite solenoid based on a perfect conductor, the electric field inside and outside the solenoid are decoupled; the field on the side where the fixed charge is located is simply the Coulomb field of that charge, while the field on the other side of the solenoid is zero.

In the limit that the thickness of the perfect conductor is zero, and the conductor is a cylindrical shell at radius R, the surface charges $Q(\theta)$ on the inner and outer surfaces of this shell are equal and opposite, such that $\int_{\mathrm{shell}} \varrho \mathbf{A}\, d\mathrm{Vol}/c = 0$, and the Maxwell angular

[55] This Appendix written May 2019, based on comments by V. Hnizdo. This case was also discussed in sec. II of [24].

momentum $\mathbf{L}^{(\mathrm{M})}$ is just that due to the fixed charge, *i.e.*, the same as eq. (10) for the case of a solenoid based on counterrotating rings of opposite charges as considered in the rest of this note.

And, in this limit, the Poynting angular momentum $\mathbf{L}^{(\mathrm{P})}$ is that found in eqs. (12)-(13), again different from the Maxwell angular momentum.

Acknowledgment

Thanks to D.J. Griffiths and V. Hnizdo for e-discussions of this topic. Griffiths has kindly provided two commentaries on this topic [53, 54].

References

[1] L.J. Tassie and M. Peshkin, *Symmetry Theory of the Aharonov-Bohm Effect: Quantum Mechanics in a Multiply Connected Region*, Ann. Phys. **16**, 177 (1961), http://kirkmcd.princeton.edu/examples/QM/tassie_ap_16_177_61.pdf

[2] M. Peshkin, I. Talmi and L.J. Tassie, *The Quantum Mechanical Effects of Magnetic Fields Confined to Inaccessible Regions*, Ann. Phys. **12**, 426 (1961), http://kirkmcd.princeton.edu/examples/QM/peshkin_ap_12_426_61.pdf

[3] H.J. Lipkin and M. Peshkin, *Angular Momentum Paradoxes with Solenoids and Monopoles*, Phys. Lett. **118B**, 385 (1982), http://kirkmcd.princeton.edu/examples/EM/lipkin_pl_118b_385_82.pdf

[4] K.T. McDonald, *Four Expressions for Electromagnetic Field Momentum* (April 10, 2006), http://kirkmcd.princeton.edu/examples/pem_forms.pdf

[5] L. Lorenz, *Ueber die Identität der Schwingungen des Lichts mit den elektrischen Strömen*, Ann. d. Phys. **207**, 243 (1867), http://kirkmcd.princeton.edu/examples/EM/lorenz_ap_207_243_67.pdf
On the Identity of the Vibration of Light with Electrical Currents, Phil. Mag. **34**, 287 (1867), http://kirkmcd.princeton.edu/examples/EM/lorenz_pm_34_287_67.pdf

[6] K.T. McDonald, *Orbital and Spin Angular Momentum of Electromagnetic Fields* (Mar. 12, 2009), http://kirkmcd.princeton.edu/examples/spin.pdf

[7] Faraday anticipated that electric currents are associated with an "electrotonic" state, that Maxwell later called electromagnetic momentum. See Arts. 60, 1661, 1729 and 1733 of M. Faraday, *Experimental Researches in Electricity* (London, 1839), http://kirkmcd.princeton.edu/examples/EM/faraday_exp_res_v1.pdf
and Art. 3269 of *On the Physical Character of the Lines of Magnetic Force*, Phil. Mag. **3**, 401 (1852), http://kirkmcd.princeton.edu/examples/EM/faraday29a_pm_3_401_52_3243-3299.pdf

[8] Secs. 22-24 and 57 of J.C. Maxwell, *A Dynamical Theory of the Electromagnetic Field*, Phil. Trans. Roy. Soc. London **155**, 459 (1865),
http://kirkmcd.princeton.edu/examples/EM/maxwell_ptrsl_155_459_65.pdf

[9] J.H. Poynting, *On the Transfer of Energy in the Electromagnetic Field*, Phil. Trans. Roy. Soc. London **175**, 343 (1884),
http://kirkmcd.princeton.edu/examples/EM/Poynting_ptrsl_175_343_84.pdf

[10] J.J. Thomson, *On the Illustration of the Properties of the Electric Field by Means of Tubes of Electrostatic Induction*, Phil. Mag. **31**, 149 (1891),
http://kirkmcd.princeton.edu/examples/EM/thomson_pm_31_149_91.pdf
Recent Researches in Electricity and Magnetism (Clarendon Press, 1893),
http://kirkmcd.princeton.edu/examples/EM/thomson_recent_researches_in_electricity.pdf

[11] J.J. Thomson, *On Momentum in the Electric Field*, Phil. Mag. **8**, 331 (1904),
http://kirkmcd.princeton.edu/examples/EM/thomson_pm_8_331_04.pdf

[12] K.T. McDonald, *J.J. Thomson and "Hidden" Momentum* (Apr. 30, 2014),
http://kirkmcd.princeton.edu/examples/thomson.pdf

[13] G.R. Trammel, *Aharonov-Bohm Paradox*, Phys. Rev. **134**, B1183 (1964),
http://kirkmcd.princeton.edu/examples/EM/trammel_pr_134_B1183_64.pdf

[14] M.G. Calkin, *Linear Momentum of Quasistatic Electromagnetic Fields*, Am. J. Phys. **34**, 921 (1966), http://kirkmcd.princeton.edu/examples/EM/calkin_ajp_34_921_66.pdf

[15] H. Poincaré, *La Théorie de Lorentz et la Principe de Réaction*, Arch. Neer. **5**, 252 (1900), http://kirkmcd.princeton.edu/examples/EM/poincare_an_5_252_00.pdf.
Translation: The Theory of Lorentz and the Principle of Reaction,
http://kirkmcd.princeton.edu/examples/EM/poincare_an_5_252_00_english.pdf

[16] W.H. Furry, *Examples of Momentum Distributions in the Electromagnetic Field and in Matter*, Am. J. Phys. **37**, 621 (1969),
http://kirkmcd.princeton.edu/examples/EM/furry_ajp_37_621_69.pdf

[17] S. Coleman and J.H. Van Vleck, *Origin of "Hidden Momentum" Forces on Magnets*, Phys. Rev. **171**, 1370 (1968), http://kirkmcd.princeton.edu/examples/EM/coleman_pr_171_1370_68.pdf

[18] J.D. Jackson, *Surface charges on circuit wires and resistors play three roles*, Am. J. Phys. **64**, 855 (1996), http://kirkmcd.princeton.edu/examples/EM/jackson_ajp_64_855_96.pdf

[19] P. Penfield, Jr and H.A. Haus, *Electrodynamics of Moving Media*, p. 215 (MIT Press, 1967), http://kirkmcd.princeton.edu/examples/EM/penfield_haus_chap7.pdf

[20] H.A. Haus and P. Penfield, Jr, *Force on a Current Loop*, Phys. Lett. **26A**, 412 (1968),
http://kirkmcd.princeton.edu/examples/EM/penfield_pl_26a_412_68.pdf

[21] W. Shockley and R.P. James, *"Try Simplest Cases" Discovery of "Hidden Momentum" Forces on "Magnetic Currents,"* Phys. Rev. Lett. **18**, 876 (1967),
http://kirkmcd.princeton.edu/examples/EM/shockley_prl_18_876_67.pdf

[22] K.-H. Yang, *The physics of gauge transformations*, Am. J. Phys. **73**, 742 (2005), http://kirkmcd.princeton.edu/examples/EM/yang_ajp_73_742_05.pdf

[23] R.H. Romer, *Angular Momentum of Static Electromagnetic Fields*, Am. J. Phys. **34**, 772 (1966), http://kirkmcd.princeton.edu/examples/EM/romer_ajp_34_772_66.pdf

[24] F.S. Johnson, B.L. Cragin and R.L. Hodges, *Electromagnetic momentum density and the Poynting vector in static fields*, Am. J. Phys. **62**, 33 (1994), http://kirkmcd.princeton.edu/examples/EM/johnson_ajp_62_33_94.pdf

[25] V. Hnizdo, *Conservation of linear and angular momentum and the interaction of a moving charge with a magnetic dipole*, Am. J. Phys. **60**, 242 (1992), http://kirkmcd.princeton.edu/examples/EM/hnizdo_ajp_60_242_92.pdf

[26] R.H. Romer, *Electromagnetic Angular Momentum*, Am. J. Phys. **35**, 445 (1967), http://kirkmcd.princeton.edu/examples/EM/romer_ajp_35_445_67.pdf

[27] K.T. McDonald, *McKenna's Paradox: Charged Particle Exiting the Side of A Solenoid Magnet* (Apr. 12, 2006), http://kirkmcd.princeton.edu/examples/mckenna.pdf

[28] O. Lodge, *On an Electrostatic Field produced by varying Magnetic Induction*, Phil. Mag. **27**, 116, 469 (1889), http://kirkmcd.princeton.edu/examples/EM/lodge_pm_27_469_89.pdf

[29] A.Einstein and W.J. De Haas, *Experimental proof of the existence of Ampère's molecular currents*, Proc. KNAW **181**, 696 (1915), http://kirkmcd.princeton.edu/examples/EM/einstein_knawp_181_696_15.pdf

[30] S.C. Tiwari, *Physical reality of electromagnetic potentials and the classical limit of the Aharonov-Bohm effect*, Quant. Stud. Math. Found. **5**, 279 (2018), http://kirkmcd.princeton.edu/examples/EM/tiwari_qsmf_5_279_18.pdf

[31] M. Wakamatsu *et al.*, *The role of electron orbital angular momentum in the Aharonov-Bohm effect revisited*, Ann. Phys. **397**, 259 (2018), http://kirkmcd.princeton.edu/examples/QM/wakamatsu_ap_397_259_18.pdf

[32] J.A. Heras and R. Heras, *Can classical electrodynamics predict nonlocal effects?* Eur. Phys. J. Plus **136**, 847 (2021), http://kirkmcd.princeton.edu/examples/EM/heras_epjp_136_847_21.pdf

[33] K.T. McDonald, *Vector Potential of a Long Solenoid in the Poincaré Gauge* (Jan. 15, 2017), http://kirkmcd.princeton.edu/examples/poincare.pdf

[34] D.J. Griffiths, *Introduction to Electrodynamics*, 3rd ed. (Prentice Hall, 1999), http://kirkmcd.princeton.edu/examples/EM/griffiths_em3
http://kirkmcd.princeton.edu/examples/EM/griffiths_em3_prob8.14.pdf
In the 4th ed., see prob. 8.22.

[35] T.J. Boyer, *Classical Electromagnetic Interaction of a Charged Particle with a Constant-Current Solenoid*, Phys. Rev. D **8**, 1667 (1973), http://kirkmcd.princeton.edu/examples/EM/boyer_prd_8_1667_73.pdf

[36] H.B. Dwight, *Tables of Integrals and Other Mathematical Data*, 4$^{\rm th}$ ed. (Macmillan, 1961), http://kirkmcd.princeton.edu/examples/EM/dwight_57.pdf

[37] J. Belcher and K.T. McDonald, *Feynman Cylinder Paradox* (1983), http://kirkmcd.princeton.edu/examples/feynman_cylinder.pdf

[38] R.P. Feynman, R.B. Leighton and M. Sands, *The Feynman Lectures on Physics*, Vol. II (Addison-Wesley, 1964), http://www.feynmanlectures.caltech.edu/II_17.html#Ch17-S4

[39] V. Hnizdo, *Hidden momentum and the electromagnetic mass of a charge and current carrying body*, Am. J. Phys. **65**, 55 (1997), http://kirkmcd.princeton.edu/examples/EM/hnizdo_ajp_65_55_97.pdf

[40] J.D. Jackson, *Classical Electrodynamics*, 3$^{\rm rd}$ ed. (Wiley, 1999), http://kirkmcd.princeton.edu/examples/EM/jackson_ce3_99.pdf

[41] Y. Aharonov and D. Bohm, *Significance of Electromagnetic Potentials in Quantum Theory*, Phys. Rev. **115**, 485 (1959), http://kirkmcd.princeton.edu/examples/QM/aharonov_pr_115_485_59.pdf

[42] K.T. McDonald, *Classical Aspects of the Aharonov-Bohm Effect* (Nov. 29, 2013), http://kirkmcd.princeton.edu/examples/aharonov.pdf

[43] Y. Aharonov, P. Pearle and L. Vaidman, *Comment on "Proposed Aharonov-Casher effect: Another example of an Aharonov-Bohm effect arising from a classical lag,"* Phys. Rev. A **37**, 4052 (1988), http://kirkmcd.princeton.edu/examples/QM/aharonov_pra_37_4052_88.pdf

[44] K.T. McDonald, *On the Definition of "Hidden" Momentum* (July 9, 2012), http://kirkmcd.princeton.edu/examples/hiddendef.pdf

[45] K.T. McDonald, *Electrodynamics of Rotating Systems* (Aug. 6, 2008), http://kirkmcd.princeton.edu/examples/rotatingEM.pdf

[46] K.T. McDonald, *"Hidden" Momentum in a Coaxial Cable* (Mar. 28, 2002), http://kirkmcd.princeton.edu/examples/hidden.pdf

[47] K.T. McDonald, *Mansuripur's Paradox* (May 2, 2012), http://kirkmcd.princeton.edu/examples/mansuripur.pdf

[48] J.D. Jackson, *Relation between Interaction terms in Electromagnetic Momentum* $\int d^3\mathbf{x}\,\mathbf{E}\times\mathbf{B}/4\pi c$ *and Maxwell's* $e\mathbf{A}^{(\rm C)}(\mathbf{x},t)/c$*, and Interaction terms of the Field Lagrangian* $\mathcal{L}_{\rm em} = \int d^3\mathbf{x}\,[E^2 - B^2]/8\pi$ *and the Particle Interaction Lagrangian,* $\mathcal{L}_{\rm int} = e\phi - e\mathbf{v}\cdot\mathbf{A}^{(\rm C)}/c$ (May 8, 2006), http://kirkmcd.princeton.edu/examples/EM/jackson_050806.pdf

[49] C.G. Darwin, *The Dynamical Motions of Charged Particles*, Phil. Mag. **39**, 537 (1920), http://kirkmcd.princeton.edu/examples/EM/darwin_pm_39_537_20.pdf

[50] K.T. McDonald, *Abraham, Minkowski and "Hidden" Mechanical Momentum* (June 6, 2012), http://kirkmcd.princeton.edu/examples/abraham.pdf

[51] K.T. McDonald, *"Hidden" Momentum in a Current Loop* (June 30, 2012), http://kirkmcd.princeton.edu/examples/penfield.pdf

[52] D.J. Griffiths, *Comment on "Electromagnetic momentum conservation in media", by I. Brevik and S.Å. Ellingsen, Ann. Physics, 326 (2011) 754*, Ann. Phys. (N.Y.) **327**, 1290 (2012), http://kirkmcd.princeton.edu/examples/EM/griffiths_ap_327_1290_12.pdf

[53] D.J. Griffiths, *Momentum and Angular Momentum in Static Fields* (Jan. 27, 2015), http://kirkmcd.princeton.edu/examples/EM/griffiths_012715.pdf

[54] D.J. Griffiths, *Angular Momentum of a Point Charge and Infinite Solenoid* (Jan. 28, 2015), http://kirkmcd.princeton.edu/examples/EM/griffiths_012815.pdf