A Magnetic Linear Accelerator

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1 Problem

A delightful science toy, sold by Scitoys.com [1] is shown in Fig. 1.

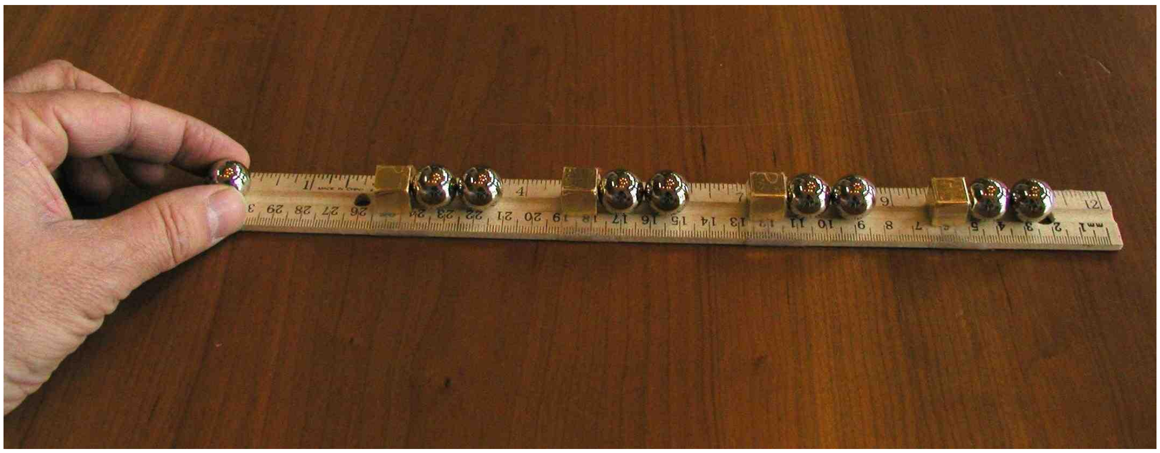

Figure 1: The magnetic linear accelerator, from Scitoys.com [1]

An iron ball of mass m, radius a and (relative) permeability $\mu \gg 1$ is released along the z axis under the attraction of the first permanent magnet, whose length is $2a$ and whose magnetic field can be approximated by that of a point dipole $\mathbf{p} = p\,\hat{\mathbf{z}}$. Initially, two more iron balls are in contact with the magnet, on the opposite side from the ball to be released. When the first ball collides with the magnet, it comes to rest and its momentum is transferred to the third ball with negligible loss of energy.

This process repeats for n cycles, where $n = 4$ in Fig. 1.

Estimate the velocity of the final ball after n such cycles, supposing that the strength of the induced magnetic field outside an iron ball is small compared to that of the permanent magnet, and that the field inside a ball is well approximated by the value at its center. The permanent magnets are far enough apart that the field of one does not affect the balls next to another.

2 Solution

The final kinetic energy of the last ball is equal to the change in the magnetic field energy stored in the system. That is, the magnetic forces in this example do work![1]

[1] Of course, as the magnetic field changes, a (weak) electric field is induced.

Each cell of the system changes from a configuration of one ball far from the magnet plus two balls in sequence on one side of the magnet to a configuration of one ball far from the magnet plus one ball one each side of the magnet.

We neglect the magnetic energy of the ball far from the magnet, and calculate the magnetic energy U of the two balls in contact with the magnet or one another. This energy has three terms,

$$U = U_{01} + U_{02} + U_{12}, \tag{1}$$

where U_{0j} is the magnetic energy of ball j in the field of the magnet, and U_{12} is the energy of interaction between the two balls.

In going from the initial to the final configuration, the magnetic energy of a cell changes by $U_i - U_f$. Hence, the velocity v attained by the final ball in a system of n cells (with all balls initially at rest) is related by,

$$\frac{1}{2} m v_f^2 = n(U_i - U_f). \tag{2}$$

The final velocity is,

$$v_f = \sqrt{\frac{2n(U_i - U_f)}{m}}. \tag{3}$$

To estimate the energies of an iron ball in the presence of the magnetic field of the permanent magnet plus the field of the other iron ball, we approximate those fields by their value at the center of the ball in question. Then, we recall that a permeable sphere takes on a uniform magnetization when placed in a uniform external field, and that the field outside the ball due to this magnetization is that of a dipole.

2.1 Permeable Sphere in a Uniform External Field

There are no free currents in the region of interest, so the magnetic fields $\mathbf{B} = \mu \mathbf{H}$ can be deduced from a scalar potential Φ that satisfies Laplace's equation, $\nabla^2 \Phi = 0$, in regions of uniform permeability.

Outside the sphere of radius a, we have $\mu = 1$. For large r the field is uniform, say $B_0 = H_0$ in the $-z$ direction, so that the corresponding potential is $\Phi_0 = B_0 z = B_0 r P_1(\cos\theta)$. The potential of the induced magnetism is driven by this term, and so will depend only on $P_1(\cos\theta)$, will vanish at infinity, will be finite at the origin, and will be continuous at $r = a$. Hence, the total potential can be written as,

$$\Phi(r < a) = B_0 r P_1 + A \frac{r}{a} P_1, \tag{4}$$

$$\Phi(r < a) = B_0 r P_1 + A \frac{a^2}{r^2} P_1. \tag{5}$$

The radial component of the field $\mathbf{B} = -\mu \nabla \phi$ is continuous at the surface $r = a$ of the sphere, so we learn that,

$$-\mu \left(B_0 + \frac{A}{a} \right) = B_0 - 2\frac{A}{a}, \qquad \Rightarrow \qquad A = -\frac{\mu - 1}{\mu + 2} a B_0. \tag{6}$$

The potential is therefore,

$$\Phi(r<a) = \frac{3}{\mu+2}B_0 r P_1 = \frac{3}{\mu+2}B_0 z \tag{7}$$

$$\Phi(r<a) = B_0 r P_1 - \frac{\mu-1}{\mu+2}\frac{a^3}{r^2}B_0 P_1. \tag{8}$$

Inside the sphere the field is uniform, with,

$$\mathbf{H}_{\rm in} = \frac{3}{\mu+2}\mathbf{B}_0, \tag{9}$$

The magnetization (density) is therefore uniform also, with value,

$$\mathbf{M} = \frac{\mu-1}{4\pi}\mathbf{H}_{\rm in} = \frac{3}{4\pi}\frac{\mu-1}{\mu+2}\mathbf{B}_0. \tag{10}$$

The total dipole moment $\mathbf{p}_{\rm ind}$ of this induced magnetization is

$$\mathbf{p}_{\rm ind} = \frac{4\pi a^3}{3}\mathbf{M} = \frac{\mu-1}{\mu+2}a^3\mathbf{B}_0. \tag{11}$$

Outside the sphere, the potential due to the induced magnetism [the 2nd term of eq. (8)] is simply that due to the point dipole given in eq. (11). Hence, the induced exterior field is

$$\mathbf{B}_{\rm ind}(r>a) = \frac{\mu-1}{\mu+2}\frac{a^3}{r^3}[3(\mathbf{B}_0\cdot\hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{B}_0]. \tag{12}$$

2.2 The Induced Dipole Strengths

The induced magnetic moment $\mathbf{p}_{\rm ind}$ in an iron ball is due to the field of the permanent magnet, here approximated as that of a point dipole $\mathbf{p}$, plus the exterior field of the other iron ball, which we also approximate as due to a point dipole, $\mathbf{p}'_{\rm ind}$. All of these dipoles are along the $+z$ axis in the present problem, and so their fields are also along the $+z$ axis for any observation point on that axis.

For spheres of high permeability, eq. (11) tells us that the induced dipole moment is simply a^3 times the strength of the magnetic field at the center of the sphere due to external sources. Thus, we write,

$$p_1 = a^3 B_1, \qquad \text{and} \qquad p_2 = a^3 B_2, \tag{13}$$

where B_1 is the magnetic field at the center of ball 1 due to the permanent magnet plus ball 2, *etc.*

2.2.1 The Initial Configuration

In each cell of the linear accelerator, ball 1 is initially at $z = 2a$ and ball 2 is at $z = 4a$, taking the permanent magnet to be centered on the origin. (We ignore the 3rd ball for the time being, as it is initially far from the permanent magnet.)

The field at the center of ball 1 due to the dipole fields of the permanent magnet and ball 2 is therefore,

$$B_1 = \frac{[3(\mathbf{p}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{p}]_z}{(2a)^3} + \frac{[3(\mathbf{p_2}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{2}]_z}{(2a)^3} = \frac{p + p_2}{4a^3}\,. \tag{14}$$

Similarly,

$$B_2 = \frac{2p}{(4a)^3} + \frac{2p_1}{(2a)^3} = \frac{p + 8p_1}{32a^3}\,. \tag{15}$$

Inserting these in eq. (13), we obtain the simultaneous equations,

$$4p_1 - p_2 = p, \qquad \text{and} \qquad -8p_1 + 32p_2 = p, \tag{16}$$

whose solutions are,

$$p_1 = \frac{11}{40}p, \qquad \text{and} \qquad p_2 = \frac{p}{10}\,. \tag{17}$$

If we neglect the induced moment of ball 2, we would have $p_1 = p/4$.

2.2.2 The Final Configuration

In the final configuration of each cell of the linear accelerator, there is one ball at $z = -2a$ and another at $z = 2a$. In this symmetrical configuration, the induced moments are equal, $p_1 = p_2$.

The field at the center of ball 1 due to the dipole fields of the permanent magnet and ball 2 is therefore,

$$B_1 = \frac{2p}{(2a)^3} + \frac{2p_2}{(4a)^3} = \frac{8p + p_1}{32a^3} = B_2. \tag{18}$$

Inserting this in eq. (13), we find,

$$p_1 = p_2 = \frac{8}{31}p. \tag{19}$$

If we neglect the effect of ball 2 on ball 1, we would have $p_1 = p_2 = p/4$.

2.3 The Magnetic Energies

The magnetic "potential" energy of a dipole moment $\mathbf{p}$ in an external magnetic field $\mathbf{B}$ is $U = -\mathbf{p}\cdot\mathbf{B}$, in the sense that the force on the moment $\mathbf{p}$ is $\mathbf{F} = -\boldsymbol{\nabla} U$.[2]

2.3.1 The Initial Configuration

Referring to sec. 2.2.1, we have,

$$U_i = U_{01} + U_{02} + U_{12} = -p_1\frac{2p}{(2a)^3} - p_2\frac{2p}{(4a)^3} - p_1\frac{2p_2}{(2a)^3} = -\frac{63}{800}\frac{p^2}{a^3} = -0.079\frac{p^2}{a^3}\,. \tag{20}$$

If we neglect the energies associated with the ball 2, we would have $U_i = U_{01} = -p^2/16a^3 = -0.0625p^2/a^3$.

[2]One can also speak of an "interaction" energy $U_{\rm int} = \mathbf{p}\cdot\mathbf{B}$, as reviewed in [2].

2.3.2 The Final Configuration

Referring to sec. 2.2.2, we have,

$$U_f = U_{01} + U_{02} + U_{12} = -p_1 \frac{2p}{(2a)^3} - p_2 \frac{2p}{(2a)^3} - p_1 \frac{2p_2}{(4a)^3} = -\frac{126}{961}\frac{p^2}{a^3} = -0.131 \frac{p^2}{a^3}\,. \qquad (21)$$

If we neglect the interaction energy between balls 1 and 2, we would have $U_f = -p^2/8a^3 = -0.125p^2/a^3$.

2.4 The Final Velocity

The change in magnetic energy in n cells is,

$$\Delta U = U_i - U_f = 0.052 n p^2/a^3\,. \qquad (22)$$

If we neglect the effect of ball 2 except for $U_{02,f}$ the energy change would be $\Delta U = p^2/16a^3 = 0.0625np^2/a^3$.

For the kit sold by scitoys.com, $n = 4$, $a = 0.6$ cm and (I estimate) the field at the surface of the permanent magnet is $B = 2p/a^3 = 1000$ gauss. Then, $m = 8$ g, using density $\rho = 8.9$ g/cm^3, $\Delta U = 11,200$ ergs, and so we predict that $v_f = \sqrt{2\Delta U/m} \approx 50$ cm/s (ignoring the effect of inelastic collisions).

References

[1] *The Gauss Rifle: A Magnetic Linear Accelerator*,
https://scitoys.com/gauss_rifle.html

[2] K.T. McDonald, *Is There a Mass Shift of a Permanent Magnetic Moment in an External, Static Magnetic Field?* (Dec. 14, 2022), http://kirkmcd.princeton.edu/examples/mu.pdf