“Hidden” Momentum in a Link of a Moving Chain?

Kirk T. McDonald
Joseph Henry Laboratories, Princeton University, Princeton, NJ 08544
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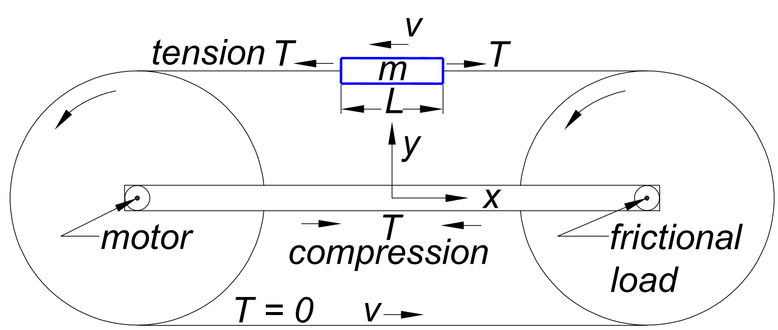
1 Problem

Discuss the momentum of stress in a link of a chain that moves at constant speed $v \ll c$, where c is the speed of light in vacuum,as sketched below. The link is subject to tension T, and has m when at rest under that tension.



Does the link contain hidden momentum, $\mathbf{P}_{\rm hidden}$, defined (when all velocities are small compared to the speed of light) for a subsystem by,

$$\mathbf{P}_{\rm hidden} \equiv \mathbf{P} - M\mathbf{v}_{\rm cm} - \oint_{\rm boundary} (\mathbf{x} - \mathbf{x}_{\rm cm})\,(\mathbf{p} - \rho\mathbf{v}_b) \cdot d\mathbf{Area}, \tag{1}$$

where $\mathbf{P}$ is the total momentum of the subsystem, $M = U/c^2$ is its total “mass”, U is its total energy, $\mathbf{x}_{\rm cm}$ is its center of mass/energy, $\mathbf{v}_{\rm cm} = d\mathbf{x}_{\rm cm}/dt$, $\mathbf{p}$ is its momentum density, $\rho = u/c^2$ is its “mass” density, u is its energy density, and $\mathbf{v}_b$ is the velocity (field) of its boundary?[1]

2 Solution

2.1 A Single Link

The mechanical properties of the moving link are best obtained via a Lorentz transformation from the its rest frame, in which quantities will be labeled with the superscript *.

We take the x axis to be in the direction of motion of the link in the lab frame.

The link has area A perpendicular to the x-axis, volume $V = AL$, and mass density $\rho^\star = m_0/V^\star$, where m_0 includes the mass/energy of the elastic strain due to tension T, and A and $L^\star$ hold for the link under tension and at rest. The length L of the link in the lab frame is Lorentz contracted compared to length $L^\star$,

$$L = \frac{L^\star}{\gamma}\sqrt{1 - \frac{v^2}{c^2}} \approx L^\star\left(1 - \frac{v^2}{2c^2}\right), \qquad \text{where} \qquad \gamma = \frac{1}{\sqrt{1 - v^2/c^2}}, \tag{2}$$

[1]The definition (11) was inspired by discussions with Daniel Vanzella [1]. See also [2].

for $v \ll c$. Similarly, the lab-frame volume is Lorentz contracted,

$$V = \frac{V^\star}{\gamma} \approx V^\star \left(1 - \frac{v^2}{2c^2}\right), \tag{3}$$

The stress-energy-momentum tensor in the rest frame of the link is,

$$\mathsf{T}^{\star\mu\nu} = \left(\begin{array}{c|ccc} \rho^\star c^2 & 0 & 0 & 0 \\ \hline 0 & -T/A & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right), \tag{4}$$

The Lorentz transformation L_x from the rest frame to the lab frame in which the link has velocity $\mathbf{v} = -v\,\hat{\mathbf{x}}$ can be expressed in tensor form as,

$$\mathsf{L}_x^{\mu\nu} = \left(\begin{array}{c|ccc} \gamma & -\gamma v/c & 0 & 0 \\ \hline -\gamma v/c & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right), \tag{5}$$

where $\gamma = 1\ \sqrt{1 - v^2/c^2} \approx 1 + v^2/2c^2$ when $v \ll c$. Hence, the energy-momentum-stress tensor in the lab frame is given by,

$$\mathsf{T}^{\mu\nu} = (\mathsf{L}_x \mathsf{T}^\star \mathsf{L}_x)^{\mu\nu} = \left(\begin{array}{c|ccc} \gamma^2(\rho^\star c^2 - v^2 T/Ac^2) & -\gamma^2 v(\rho^\star c^2 - T/A)/c & 0 & 0 \\ \hline -\gamma^2 v(\rho^\star c^2 - T/A)/c & \gamma^2(\rho^\star v^2 - T/A) & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right). \tag{6}$$

The mass density $\rho = u/c^2 = \mathsf{T}^{00}/c^2$ in the lab frame is,

$$\rho = \frac{\gamma^2}{c^2}\left(\rho^\star c^2 - \frac{v^2 T}{Ac^2}\right) \approx \rho^\star \left(1 + \frac{v^2}{c^2}\right), \tag{7}$$

to order $1/c^2$. Hence, the mass M is,

$$M = \rho V = \frac{\rho V^\star}{\gamma} \approx m_0 \left(1 + \frac{v^2}{2c^2}\right), \tag{8}$$

the momentum density $\mathbf{p}$ is given by $p_i = \mathsf{T}^{0i}/c$,

$$\mathbf{p} = -\frac{\gamma^2 v}{c^2}\left(\rho^\star c^2 - \frac{T}{A}\right)\hat{\mathbf{x}} \approx \left(\rho - \frac{T}{c^2 A}\right)\mathbf{v}, \tag{9}$$

and the total momentum $\mathbf{P}$ of the link in the lab frame is,

$$\mathbf{P} = \mathbf{p}V \approx \left[m_0 \left(1 + \frac{v^2}{2c^2}\right) - \frac{TL}{c^2}\right] \mathbf{v} = \left(M - \frac{TL}{c^2}\right) \mathbf{v}\,. \tag{10}$$

The velocity of the center of mass/energy of the link, and of the boundary of the link, is,

$$\mathbf{v}_{\rm cm} = \mathbf{v}_b = \mathbf{v} = v\,\hat{\mathbf{x}}. \tag{11}$$

We consider the link at the instant when $\mathbf{x}_{\rm cm} = 0$, so the two ends of the link have x-coordinates $\pm L/2$. Then, according to definition (1), the link has no hidden momentum in the lab frame,

$$\begin{aligned} \mathbf{P}_{\rm hidden} &= \mathbf{P} - M\mathbf{v}_{\rm cm} - \oint_{\rm boundary} (\mathbf{x} - \mathbf{x}_{\rm cm})\,(\mathbf{p} - \rho\mathbf{v}_b) \cdot d\mathbf{Area} \\ &= \mathbf{P} - M\mathbf{v}_{\rm cm} - \mathbf{p}V + \rho V \mathbf{v}_b = 0, \end{aligned} \tag{12}$$

since $\mathbf{v}_{\rm cm} = \mathbf{v}_b$.

Although the total momentum $\mathbf{P}$ of the link in the lab frame has two "relativistic" terms of order $1/c^2$, as given in eq. (10), there is no hidden momentum in the link, according to definition (1).

2.1.1 Alternative Analysis

Daniel Vanzella noted that if the center-of-mass/energy velocity in definition (1) is the microscopic velocity, then the "hidden" momentum (of a subsystem) can also be written as,

$$\mathbf{P}_{\rm hidden} = -\int \frac{f^0}{c}(\mathbf{x} - \mathbf{x}_{\rm cm})\,d{\rm Vol}, \tag{13}$$

where,

$$f^\mu = \frac{\partial T^{\mu\nu}}{\partial x^\nu}\,. \tag{14}$$

is the 4-force density exerted on the subsystem by all other subsystems, and $T^{\mu\nu}$ is the stress-energy-momentum tensor of the subsystem.

In the present example, $f^0 = \partial T^{0\nu}/\partial x^\nu$ vanishes inside the link, while having δ-function terms at the ends of the link. We consider that the integral in eq. (13) is taken over only the interior of the physical volume of the subsystem (*i.e.*, of the link), in which case we again find $\mathbf{P}_{\rm hidden} = 0$.[2]

[2] D. Vanzella appeared to argue that the boundary integral in eq. (12) can be neglected, and that the volume integral in eq. (13) should include the delta functions at the ends, in which case he found $\mathbf{P}_{\rm hidden} = \mathbf{P} - M\mathbf{v}_{\rm cm} = -TL\mathbf{v}/c^2$ by both forms. For a case where this procedure leads to different values of the "hidden" momentum using forms (12) and (13), see the footnote in [3].

2.1.2 Link with Variable Mass Density

Suppose the mass density $\rho(x)$ of the link varies with position. The internal stress T/A does not vary with position if the link is not accelerating. Then, $M = A\int\rho(x)\,dx$ at order $1/c^2$, such that ,

$$\mathbf{p}(x) = \left(\rho(x) - \frac{T}{Ac^2}\right)\mathbf{v}, \qquad \mathbf{P} = A\int\mathbf{p}(x)\,dx = M\mathbf{v} - \frac{TL\mathbf{v}}{c^2}\,, \tag{15}$$

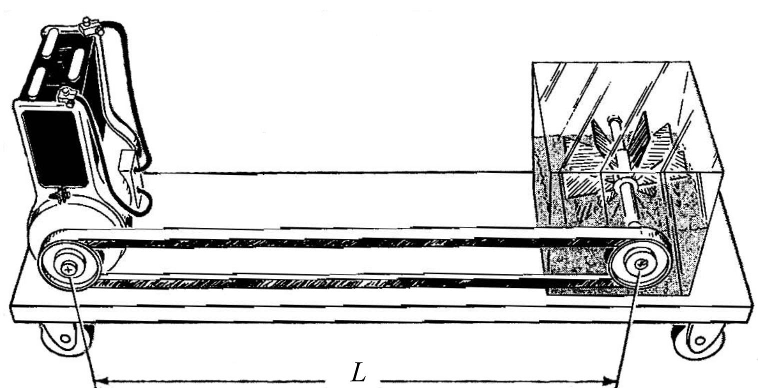
$$\mathbf{p} - \rho\mathbf{v}_b = -\frac{T\mathbf{v}}{Ac^2}\,, \qquad \oint_{\text{boundary}} (\mathbf{x} - \mathbf{x}_{\text{cm}})\,(\mathbf{p} - \rho\mathbf{v}_b)\cdot d\mathbf{Area} = -\frac{TL\mathbf{v}}{c^2}\,, \tag{16}$$

recalling that $\mathbf{v}_{\text{cm}} = \mathbf{v}_b = \mathbf{v}$, and hence,

$$\begin{aligned} \mathbf{P}_{\text{hidden}} &= \mathbf{P} - M\mathbf{v}_{\text{cm}} - \oint_{\text{boundary}} (\mathbf{x} - \mathbf{x}_{\text{cm}})\,(\mathbf{p} - \rho\mathbf{v}_b)\cdot d\mathbf{Area} \\ &= M\mathbf{v} - \frac{TL\mathbf{v}}{c^2} - M\mathbf{v} + \frac{TL\mathbf{v}}{c^2} = 0, \end{aligned} \tag{17}$$

as found previously for a link with uniform mass density.

2.2 A Belt/Chain That Transfers Energy



A common application of a belt/chain loop is to transfer energy from the drive pulley/sprocket (on the left in the figure on p. 1) to the driven pulley/sprocket (on the right). In this case the tension in the lower part of the belt/chain could be zero. Note that the direction of flow of energy is opposite to the direction of the velocity of the (upper) part of the chain loop that is under tension.[3]

[3]This problem was discussed by Hertz, Note 29, p. 276 of [4]: *Consider a steam engine which drives a dynamo by means of a strap running from the dynamo and back, and which in turn works an arc lamp by means of a wire reaching to the lamp and back again. In ordinary language we say—and no exception need be taken to such a mode of expression—that energy is transferred from the steam engine by means of the strap to the dynamo, and from this again to the lamp by the wire. But is there any clear meaning in asserting that the energy travels from point to point along the stretched strap in a direction opposite to that in which the strap itself move? And if not, can there be any more clear meaning in saying that the energy travels from point to point along the wires, or—as Poynting says—in the space between the wires? There are difficulties here which badly need clearly up.*

Consider now the entire system, which includes an energy source on the left, say a battery, and an energy sink on the right, say a pool of water driving by a paddle wheel.[4] We work in the frame of the system, the $'$ frame, which may or may not be at rest in the lab frame.

The mass of the links is the same in both the upper and lower parts of the chain, recalling eq. (8). However, the momentum densities have different magnitudes because the tension is zero in the lower part of the chain. Recalling eq. (9), we now have that,

$$\mathbf{p}'_{\rm upper} = -\frac{\gamma^2 v}{c^2}\left(\rho^\star c^2 - \frac{T}{A}\right)\hat{\mathbf{x}} \approx \left[\rho^\star\left(1+\frac{v^2}{c^2}\right) - \frac{T}{c^2 A}\right]\mathbf{v}, \tag{18}$$

$$\mathbf{p}'_{\rm lower} = \frac{\gamma^2 v \rho^\star c^2}{c^2} \approx -\rho^\star\left(1+\frac{v^2}{c^2}\right)\mathbf{v}, \tag{19}$$

and the total momentum $\mathbf{P}^\star$ of the chain is,

$$\mathbf{P}' = (\mathbf{p}'_{\rm upper} + \mathbf{p}'_{\rm lower})V \approx -\frac{TL}{c^2}\mathbf{v}\,, \tag{20}$$

which is also the total momentum of the system in the $'$ frame.

If we regard the belt/chain as continuous, then the center of mass/energy of the belt is fixed in the frame of the system, so the moving belt makes no contribution to the velocity of the center-of-mass-energy.[5]

Associated with the momentum density in the belt is a flux $\mathbf{S}^\star$ of energy,

$$\mathbf{S}' = c^2(\mathbf{p}'_{\rm upper} + \mathbf{p}'_{\rm lower}) = -\frac{T}{A}\mathbf{v}. \tag{21}$$

Integrating this over the cross section of the belt, the power transmitted to the load is,

$$\frac{dU'}{dt'} = S'A = Tv. \tag{22}$$

The corresponding rate of increase of the load (ignoring radiation losses) is,

$$\frac{dM'}{dt'} = \frac{1}{c^2}\frac{dU'}{dt'} = \frac{Tv}{c^2}\,. \tag{23}$$

The mass transfer is over distance D' between the battery and the load, so the velocity of the center of mass/energy of the system (in the frame of the system) is given by,

$$M'\mathbf{v}'_{\rm cm} = \frac{dM'}{dt'}D' = -\frac{TD'}{c^2}\mathbf{v}. \tag{24}$$

The load end of distance D' is in line with the right horizontal end of the belt. If the battery is not in line with the left horizontal end of the belt, then flow of electrical energy from the battery to the motor would have an x-component, which would then require us

[4]This example is from p. 147 of [5].

[5]If we consider the upper, horizontal portion of the belt to be a subsystem, its center of mass/energy is fixed, and its boundary is not moving. Then, as in eq. (12), there is no "hidden" momentum in this subsystem, according to the definition (1).

to consider electromagnetic momentum. For simplicity, we take $D' = L' = L^\star/\gamma$ = length of the horizontal portion of the belt in the $'$ frame, so that we can regard this example as purely mechanical, and so,

$$M'\mathbf{v}'_{\rm cm} = \frac{dM'}{dt'} L' = -\frac{TL'}{c^2}\mathbf{v} = \mathbf{P}'. \tag{25}$$

According to definition (1) the "hidden" momentum of the entire system (which has no boundary) in the $'$ frame is,

$$\mathbf{P}'_{\rm hidden} = \mathbf{P}' - M'\mathbf{v}'_{\rm cm} = 0. \tag{26}$$

The frame of the system is not its center-of-mass/energy frame, because the velocity of the center of mass in this frame is,

$$\mathbf{v}'_{\rm cm} = -\frac{TL'}{Mc^2}\mathbf{v}. \tag{27}$$

Taking the lab frame to be that in which the center of mass of the system is at rest, the entire system has lab-frame velocity,

$$\mathbf{v}_{\rm system} = \frac{TL'}{Mc^2}\mathbf{v}. \tag{28}$$

2.2.1 Relativity of Steady Energy Flow

In the $'$ frame (the rest frame of the battery and load) we identified the flow of energy (21) from the battery to the load as taking place inside the upper horizontal portion of the belt.

However, there is no flow of energy in that portion of the belt there in its rest frame, the * frame, as considered in sec. 2.1.

In the * frame the lower portion of the belt has velocity $-2\mathbf{v}$ and has a flow of energy $-2\gamma^2\rho^\star c^2\mathbf{v}$ due to the bulk mass transport in that portion of the belt. This does not account for the flow of energy from the battery to the load.

Rather, we should consider the baseplate of the system, which is under compression, and has velocity $-\mathbf{v}$ in the * frame.

First, we note that in the $'$ frame the stress-energy-momentum tensor of the upper portion of the belt is, from eq. (6),

$$\mathsf{T}'^{\mu\nu}_{\rm upper\ belt} = \left(\begin{array}{c|ccc} \gamma^2(\rho^\star c^2 - v^2T/Ac^2) & -\gamma^2 v(\rho^\star c^2 - T/A)/c & 0 & 0 \\ \hline -\gamma^2 v(\rho^\star c^2 - T/A)/c & \gamma^2(\rho^\star v^2 - T/A) & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right). \tag{29}$$

The stress-energy-momentum tensor of the portion of the baseplate that is under compression (which has mass density ρ_B,[6] length $L' = L^\star/\gamma$ and cross sectional area A_B) is,

$$\mathsf{T}'^{\mu\nu}_{\rm baseplate} = \left(\begin{array}{c|ccc} \rho_B c^2 & 0 & 0 & 0 \\ \hline 0 & \gamma^2(\rho^\star v^2 - T/A)(A/A_B) & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right), \tag{30}$$

such that (in the $'$ frame) the compressive force in the baseplate has the same magnitude as the tension in the belt. Transforming this tensor to the $\star$ frame, we find,

$$\mathsf{T}^{\star\mu\nu}_{\rm baseplate} = \tag{31}$$
$$\left(\begin{array}{c|ccc} \gamma^2[\rho_B c^2 + \gamma^2 v^2(\rho^\star v^2 - T/A)(A/A_B)/c^2] & \gamma^2 v[\rho_B c^2 - \gamma^2(\rho^\star v^2 - T/A)(A/A_B)]/c & 0 & 0 \\ \hline \gamma^2 v[\rho_B c^2 - \gamma^2(\rho^\star v^2 - T/A)(A/A_B)]/c & \gamma^2[\rho_B v^2 + \gamma^2(\rho^\star v^2 - T/A)(A/A_B)] & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right).$$

The element $\mathsf{T}^{\star 01}_{\rm baseplate}$ includes the term $\gamma^4 Tv/A_B c$, which corresponds to the flux of energy from the battery to the load in the $\star$ frame.[7]

For related discussion of the relativity of steady energy flow, see [5, 6].

References

[1] D. Vanzella, Private communication (June 29, 2012).

[2] K.T. McDonald, *On the Definition of "Hidden" Momentum* (July 9, 2012),
http://kirkmcd.princeton.edu/examples/hiddendef.pdf

[3] K.T. McDonald, *"Hidden" Momentum in a River* (July 5, 2012),
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[4] H. Hertz, *Electric Waves* (Macmillan, 1900),
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[5] E.F. Taylor and J.A. Wheeler, *Spacetime Physics*, 1st ed. (WH Freeman, 1966),
http://kirkmcd.princeton.edu/examples/GR/taylor_66.pdf
http://kirkmcd.princeton.edu/examples/mechanics/taylor_spacetime_physics_p147.pdf

[6] K.T. McDonald, *Relativity of Steady Energy Flow* (Nov. 3, 2007),
http://kirkmcd.princeton.edu/examples/1dgas.pdf

[6] The mass density of the portion of the baseplate under compression is different from that in the regions where there is no compression.

[7] The other terms in $\mathsf{T}^{\star 01}_{\rm baseplate}$ describe the bulk mass transport in the moving baseplate.