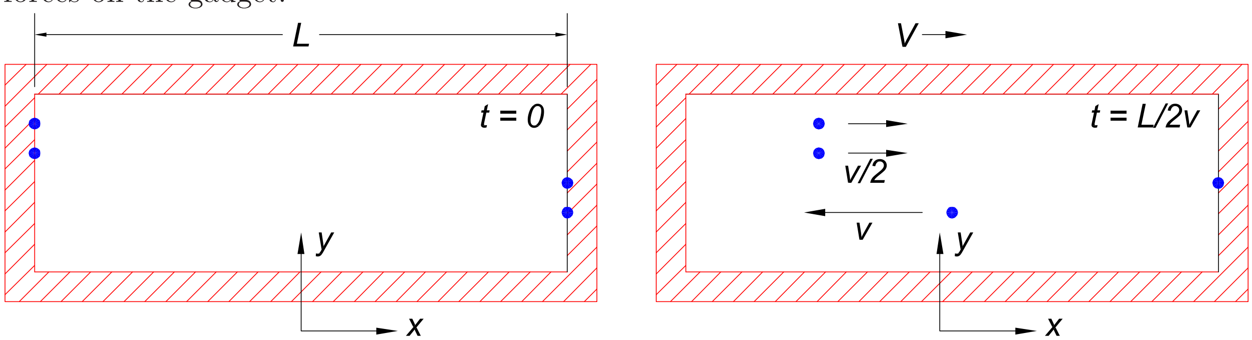
Little's Paradox

Kirk T. McDonald
Joseph Henry Laboratories, Princeton University, Princeton, NJ 08544
(April 21, 2017; updated September 21, 2020)

1 Problem

Scott Little recently proposed a gadget,[1] a version of which is sketched below, in which a frame (of rest mass M and rest length L) supports mechanisms that can launch (perhaps via precompressed springs) and catch four balls of rest mass m each, with two balls initially on the left, and two on the right, from opposite sides of the frame. There are no external forces on the gadget.



Initially, the frame is at rest, and no balls are in motion. At time $t = 0$ one ball is launched to the left with velocity v, and two balls are launched to the right with velocity $v/2$. At time $t = L/v$ the left-moving ball is caught, and the remaining ball is launched to the left with velocity v. For $0 < t < 2L/v$, one ball is in motion to the left and two balls are in motion to the right. At time $t = 2L/v$, three balls are caught, such that the system ends up at rest two balls on the left, and also two balls on the right.

When balls are in motion relative to the frame, the total momentum $\mathbf{P}$ of the balls is to the left, with magnitude,

$$P = m\gamma_v v - 2m\gamma_{v/2}\frac{v}{2} = m(\gamma_v - \gamma_{v/2})v \approx \frac{3v^2}{8c^2}mv, \quad \text{where} \quad \gamma_v = \frac{1}{\sqrt{1 - v^2/c^2}} \approx 1 + \frac{v^2}{2c^2}\,, \tag{1}$$

and c is the speed of light in vacuum, and the approximation holds for $v \ll c$. For momentum to be conserved, the frame (plus the one ball not in motion) must have momentum $P = (M + m)\gamma_V V$ to the right, where V is the velocity of the frame to the right,[2]

$$V = \frac{m(\gamma_v - \gamma_{v/2})v}{(M + m)\gamma_V} \approx \frac{3mv^2}{8(M + m)c^2}v, \tag{2}$$

where we approximate γ_V by 1, since $V \ll v \ll c$.

[1]Little gave a slightly different version of the paradox in a note of May, 31, 2017 [1], which is discussed in the Appendix below.

[2]We neglect that when the balls are first launched, and given kinetic energy, the energy of the frame is correspondingly reduced, which slightly reduces its rest mass to $M' = M[1 - \mathcal{O}(v^2/c^2)]$ while the frame is in motion. After all balls are caught by the frame, its rest mass returns to M.

Hence, at time $t = 2L/v$, the center of mass of this gadget is at rest at distance $d = Vt = 3mv^2L/4(M+m)c^2$ to the right of its initial position.

Can this be so?

2 Solution

The frame does move, but the there is a net transfer of the mass/energy of the frame from right to left, contrary to the misleading claim in the statement of the problem. As such, the center of mass of the system remains fixed while the frame moves to the right to compensate for the net flow of mass/energy the left.

For simplicity, we suppose that the frame has left and right segments of rest mass $M/2$, separated by two rigid, zero-mass segments of length L. That is, the left and right sides of the system have initial rest mass $M/2 + 2m$ each, and the total mass of the system is $M + 4m$.

For $t < 0$, the frame and the balls are at rest, and the center of mass of the system is at rest at $x = 0$, the initial horizontal coordinate of the center of the frame.

When a ball is launched to the left with velocity v from the right side of the system, the mass of the right side is reduced by the "relativistic" mass,

$$\gamma_v m = m + \frac{\mathrm{KE}}{c^2} \approx m + \frac{mv^2}{2c^2}\,, \tag{3}$$

which mass ends up on the left side of the system after the ball is caught. Similarly, when a ball is launched to the right with velocity $v/2$ from the left side of the system, the mass of the left side is reduced by,

$$\gamma_{v/2} m \approx m + \frac{mv^2}{8c^2}\,, \tag{4}$$

which mass ends up on the right side of the system after this ball is caught.

Thus, at time $t = 2L/v$, after all four balls have been launched and caught, and the system is again at rest, the masses of the left and right sides of the system are,

$$M_L = \frac{M}{2} + 2m + 2\gamma_v m - 2\gamma_{v/2} m \approx \frac{M}{2} + 2m + \frac{3mv^2}{4c^2}\,, \tag{5}$$

$$M_R = \frac{M}{2} + 2m - 2\gamma_v m + 2\gamma_{v/2} m \approx \frac{M}{2} + 2m - \frac{3mv^2}{4c^2}\,, \tag{6}$$

rather than $M/2 + 2m$ as they were initially. As such, the center of mass of the final system is at distance,

$$D = \frac{(M_L - M_R)L/2}{M_L + M_R} = \frac{3mv^2L}{4(M+4m)c^2} \tag{7}$$

to the left of the center of the system.

If as expected the center of mass of the system remains fixed at all times, the system should have moved to the right by distance D at time $t = 2L/v$. However, during time $0 < t < 2L/v$, the center of the system moved to the right by distance,

$$d = Vt = \frac{3mv^2 L}{4(M+m)c^2}\,, \tag{8}$$

according to eq. (2).

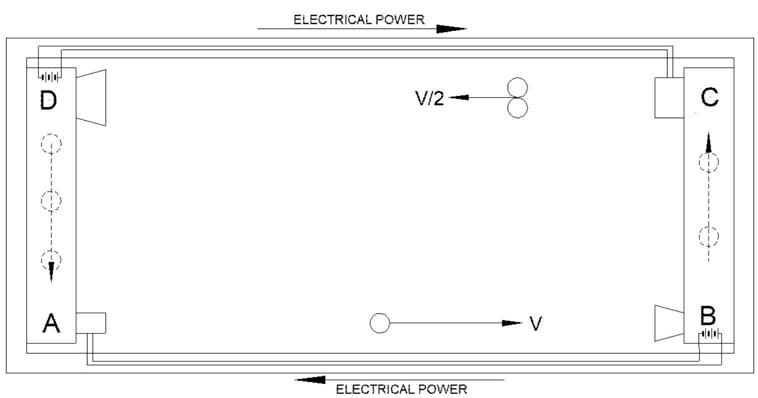
In the limit that $m \ll M$ the above analysis is consistent with the center of mass of the system remaining at rest, but if, say, the frame were massless, we find $D = d/4$.

The frame cannot be massless, in that after the three balls are initially launched, but before any are caught, the mass/energy of the frame must be reduced by the kinetic energy given to the balls. So, the mass of the frame must be greater than $3mv^2/4c^2 \ll m$, which is extremely small.

Appendix: Little's Paradox with Batteries

In [1], Scott Little first posed an "all mechanical" version of his paradox, using the figure below, but without the batteries and connections to the launchers A and C.

Note that the direction of motion of the various balls in the figure below is opposite to that in the figure on p. 1.



However, before completing analysis of this case, Little supposed that the batteries shown in the figure were present.[3] He still seemed to consider this system to be "all mechanical", although it now supports electrical currents, electric and magnetic fields, a nonzero Poynting vector $\mathbf{S}$, and a nonzero density of electromagnetic field momentum $\mathbf{p} = \mathbf{S}/c^2$.

In the version analyzed by Little [1], he supposed that the apparatus sketched above operates continuously, with three balls in motion at all times, as well as steady flows of energy from the two batteries to the two load resistors.

[3]Variants are possible. It could be that the launchers A and C contain load resistors that are independent of the "mechanics" of the launchers, which consist of precompressed springs that are used only once each to launch a ball; this variant retains the mechanics of the version without batteries. In contrast, Little may have imagined that the launchers contain motors that operate with 100% efficiency to convert electrical energy into compressing the springs of the launchers, which could thereby be reused.

Then, for a ball launched with velocity $\mathbf{v}_{\rm ball}$ and kinetic energy $\mathrm{KE}_{\rm ball} = (\gamma_{\rm ball} - 1)mc^2$, the current in its associated battery is chosen such that it transfers energy $\mathrm{KE}_{\rm ball}$ to the launcher during the time $\Delta t_{\rm ball} L/v_{\rm ball}$ that the ball is in flight. Taking the Poynting vector $\mathbf{S}_{\rm ball}$ associated with the moving ball to flow opposite to the direction of $\mathbf{v}_{\rm ball}$ in a tube of effective area A and length L, we have that $\mathrm{KE}_{\rm ball} = S_{\rm ball} A \Delta t_{\rm ball} = S_{\rm ball} AL/v_{\rm ball}$, such that the field momentum density is,

$$\mathbf{p}_{\rm EM,ball} = \frac{\mathbf{S}_{\rm ball}}{c^2} = -\frac{\mathrm{KE}_{\rm ball}\, \mathbf{v}_{\rm ball}}{ALc^2}\,. \tag{9}$$

The total field momentum associated with a moving ball is eq. (9) times the volume AL,

$$\mathbf{P}_{\rm EM,ball} = \int \mathbf{p}_{\rm EM,ball}\, d\mathrm{Vol} = -\frac{\mathrm{KE}_{\rm ball}\, \mathbf{v}_{\rm ball}}{c^2} = -(\gamma_{\rm ball} - 1)m\mathbf{v}_{\rm ball} = -(\mathbf{P}_{\rm mech,ball} - m\mathbf{v}_{\rm ball}), \tag{10}$$

where $\mathbf{P}_{\rm mech,ball} = \gamma_{\rm ball} m \mathbf{v}_{\rm ball}$ is the mechanical momentum of a moving ball. Then, for each moving ball, $\mathbf{P}_{\rm mech,ball} + \mathbf{P}_{\rm EM,ball} = m\mathbf{v}_{\rm ball}$, so with two balls in motion with velocity $\mathbf{v}/2$ and one with velocity $-\mathbf{v}$, the total momentum of the balls and the electromagnetic fields is zero.[4] Furthermore, the frame remains at rest at all times, because the velocity of its recoil to the momentum of the launched balls is equal and opposite to the velocity of the frame associated with the mass transfer by the batteries.[5] That is, the total momentum of the system is zero at all times, as expected for an isolated system that is initially at rest.

This relatively brief argument omits mention that while the momentum of the frame itself is zero, its center of mass/energy is moving due to the mass transfer via the electromagnetic field, such that the quantity $\mathbf{P}_{\rm frame} - M_{\rm frame}\mathbf{v}_{\rm cm,frame} = -\mathbf{P}_{\rm EM}$ is nonzero. Such a quantity has been called "hidden" momentum. A general argument [2] indicates that in electromechanical examples like the present, if the field momentum $\mathbf{P}_{\rm EM}$ is nonzero, then a mechanical component of the system (here the moving balls) has an equal-and-opposite "momentum" of a "hidden" character.[6]

The version of Little's paradox without batteries, considered in sec. 2 above, does not contain any "hidden" momentum, in that $\mathbf{P}_{\rm frame} - M_{\rm frame}\mathbf{v}_{\rm cm,frame} = 0$ and $\mathbf{P}_{\rm ball} - M_{\rm ball}\mathbf{v}_{\rm cm,ball} = 0$ for each ball, where $M = U/c^2$ is the "relativistic" mass of an entity of total energy U.

References

[1] S. Little, *Can an all-mechanical system violate the center of energy theorem?* (May 31, 2017), http://kirkmcd.princeton.edu/examples/EM/little_170531.pdf

[4]Note that $\mathbf{P}_{\rm mech,balls} = \sum \mathbf{P}_{\rm mech,ball} = -\sum \mathbf{P}_{\rm EM,ball} = -\mathbf{P}_{\rm EM,balls}$, where the magnitude of $\mathbf{P}_{\rm mech,balls}$ is given by eq. (1).

[5]The recoil velocity due to the launch of a ball would be $\mathbf{v}_{\rm recoil} = -\mathbf{P}_{\rm ball}/M_{\rm frame}$. The displacement d of the frame needed to keep its center of mass/energy at rest while a battery transfers energy KE to a launcher at distance L from the battery would be related by $M_{\rm frame} d = (\mathrm{KE}/c^2)L$. This displacement occurs during time L/v, and is in the direction from the launcher to the battery, *i.e.*, the direction of the velocity $\mathbf{v}$ of the ball that was launched. Hence, the velocity of the center of mass energy of the frame associated with this transfer would be $\mathbf{v}_{\rm transfer} = \mathrm{KE}\,\mathbf{v}/M_{\rm frame}c^2 = -\mathbf{P}_{\rm EM}/M_{\rm frame}$. The sum of these (virtual) velocities is $-\sum(\mathbf{P}_{\rm ball} + \mathbf{P}_{\rm EM})/M_{\rm frame} = 0$.

[6]For discussion of "hidden" momentum in a system with a battery and resistor, see [3].

[2] K.T. McDonald, *On the Definition of "Hidden" Momentum* (July 9, 2012),
http://kirkmcd.princeton.edu/examples/hiddendef.pdf

[3] K.T. McDonald, *"Hidden" Momentum in a Coaxial Cable* (Mar. 28, 2002),
http://kirkmcd.princeton.edu/examples/hidden.pdf