Radiation by a Steady Current Loop Whose Center Oscillates in the Plane of the Loop

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1 Problem

Deduce the electromagnetic radiation pattern of a small (electrically neutral) loop of steady current I in the x-y plane, with magnetic moment $\mathbf{m} = IA\,\hat{\mathbf{z}}/c$ (in Gaussian units, A is the area of the loop and c is the speed of light in vacuum), supposing the center of the loop oscillates according to $x = x_0 \cos\omega t$, where the peak velocity $v = x_0\omega$ is much less than c.

Discuss also the case of an infinite solenoid, whose axis is parallel to the z-axis, with magnetic moment $\mathbf{m}$ per unit length, and a point on the axis of the solenoid oscillates as above.

This problem was inspired by a discussion of scattering off a cosmic string [1], and related consideration of the radiation of scalar particles by a transversely oscillating infinite solenoid (a prototype of a cosmic string) [2].

2 Solution

2.1 Current Loop

In the first approximation, the magnetic moment $\mathbf{m}$ of the oscillating current loop is constant in time, so there is no magnetic dipole radiation. But, the oscillating magnetic moment has a time-dependent quadrupole moment, whose radiation fields depend on the third time derivative, which varies as ω^3.[1]

Also, a moving magnetic moment, $\mathbf{m}$ when at rest, takes on a time-dependent electric-dipole moment $\mathbf{p}$ according to,[2]

$$\mathbf{p} \approx \frac{\mathbf{v}}{c} \times \mathbf{m}, \tag{1}$$

where the approximation holds for $v \ll c$. The radiation fields of a time-dependent electric dipole depend on its second time derivative, which also varies as ω^3 in the present example.

2.1.1 Electric-Dipole Radiation

The velocity of the center of the current loop (magnetic moment $\mathbf{m} = m\,\hat{\mathbf{z}}$) is $\mathbf{v} = -A\omega \sin\omega t\,\hat{\mathbf{x}}$, so its electric-dipole moment (1) is,

$$\mathbf{p} \approx \frac{A\omega m \sin\omega t\,\hat{\mathbf{y}}}{c}, \qquad \ddot{\mathbf{p}} \approx -\frac{A\omega^3 m \sin\omega t\,\hat{\mathbf{y}}}{c}. \tag{2}$$

[1]See, for example, §71 of [3].
[2]See, for example, [4].

For an observer at (x,y,z), where $\hat{\mathbf{n}} = (x,y,z)/r$, the radiation fields of this oscillating electric dipole are, with the time derivative evaluated at the retarded time $t - r/c = t - kr/\omega$,

$$\mathbf{E}_{E1} = \frac{(\ddot{\mathbf{p}} \times \hat{\mathbf{n}}) \times \hat{\mathbf{n}}}{c^2 r} = -\frac{A\omega^3 m \sin(kr - \omega)}{c^3 r^3}(xy, -x^2 - z^2, yz), \tag{3}$$

$$\mathbf{B}_{E1} = \frac{\ddot{\mathbf{p}} \times \hat{\mathbf{n}}}{c^2 r^2} = -\frac{A\omega^3 m \sin(kr - \omega t)}{c^3 r^2}(z, 0, x), \tag{4}$$

for which $B = E$. The time-average pattern for the electric-dipole part of the radiation can be computed as,

$$\frac{d\langle U_{E1}\rangle}{d\Omega} = \frac{c}{4\pi}\langle B^2\rangle r^2 = \frac{A^2\omega^6 m^2}{8\pi c^5}\frac{x^2+z^2}{r^2} = \frac{A^2\omega^6 m^2}{8\pi c^5}\left(1 - \frac{y^2}{r^2}\right) = \frac{A^2\omega^6 m^2}{8\pi c^5}\left(1 - \sin^2\theta \sin^2\phi\right), \tag{5}$$

where the last form is for spherical coordinates (r, θ, ϕ).

2.1.2 Magnetic-Quadrupole Radiation

The forms of radiation fields of an oscillating magnetic quadrupole can be obtained from those of an electric quadrupole, given in §71 of [3], via the duality transformation $q_e \to q_m$, $\mathbf{E}_e \to \mathbf{B}_m$, $\mathbf{B}_e \to -\mathbf{E}_m$,[3] which imply,

$$\mathbf{E}_{M2} = \frac{1}{6c^3 r}\hat{\mathbf{n}} \times \dddot{\mathbf{Q}} \qquad \mathbf{B}_{M2} = \frac{1}{6c^3 r}\hat{\mathbf{n}} \times (\hat{\mathbf{n}} \times \dddot{\mathbf{Q}}), \tag{6}$$

where the magnetic quadrupole vector $\mathbf{Q}$ is related to the magnetic quadrupole tensor Q by,

$$\mathsf{Q}_{ij} = \sum q_m(3x_i x_j - \delta_{ij} r^2), \qquad \mathbf{Q} = \mathsf{Q} \cdot \hat{\mathbf{n}}, \tag{7}$$

and the q_m are the effective magnetic charges of the current distribution. In the present example of a small current loop in the x-y plane, centered at $(x_m, 0, 0)$, with magnetic moment $\mathbf{m} = m\,\hat{\mathbf{z}}$, we can take the magnetic charges to be $\pm q_m$, located at $(x_m, 0, \pm d/2)$ for fixed d, where $m = q_m d$. Then, $r^2 = x_m^2 + d^2/4$ for both magnetic charges such that the only nonzero elements of the quadrupole tensor are $\mathsf{Q}_{xz} = \mathsf{Q}_{zx} = 6q_m x_m d = 6m x_m$. For an observer at (x,y,z), we have that $\hat{\mathbf{n}} = (x,y,z)/r$, and,

$$\mathbf{Q} = \frac{6m x_m}{r}(z, 0, x), \qquad \hat{\mathbf{n}} \times \mathbf{Q} = \frac{6m x_m}{r^2}(xy, z^2 - x^2, yz), \tag{8}$$

$$\hat{\mathbf{n}} \times (\hat{\mathbf{n}} \times \mathbf{Q}) = \frac{6m x_m}{r^3}(x^2 z + y^2 z - z^3, 0, xz^2 - xy^2 - x^3). \tag{9}$$

The electric field (6) of the magnetic-quadrupole radiation is, with $x_m = A\cos\omega t$,

$$\mathbf{E}_{M2} = -\frac{A\omega^3 m \sin(kr - \omega t)}{c^3 r^3}(xy, z^2 - x^2, yz). \tag{10}$$

[3]See, for example, Appendix D.4 of [5].

2.1.3 Total Radiation

The electric field (10) is in phase with the electric-dipole field (3), so we should consider the total radiation electric field,

$$\mathbf{E} = \mathbf{E}_{E1} + \mathbf{E}_{M2} = -\frac{2A\omega^3 m \sin(kr - \omega t)}{c^3 r^3}(xy, -x^2, yz). \tag{11}$$

The time-average pattern for the total radiation can be computed as, recalling eq. (66.6) of [3],

$$\frac{d\langle U\rangle}{d\Omega} = \frac{c}{4\pi}\left\langle E^2\right\rangle r^2 = \frac{A^2\omega^6 m^2}{2\pi c^5}\frac{x^2y^2 + x^4 + y^2z^2}{r^4}$$
$$= \frac{A^2\omega^6 m^2}{2\pi c^5}\sin^2\theta(\sin^2\theta\cos^2\phi + \cos^2\theta\sin^2\phi). \tag{12}$$

The radiated energy comes from the force that drives the motion of the current loop. In the classical view, there exists a radiation-reaction force on the current loop, equal and opposite to the drive force, that depends on the time derivative of electric charges in the loop. While the oscillating loop is electrically neutral, its "conduction" charges have different acceleration than its the "lattice" charges, so the radiation-reaction force is nonzero, as required.

We note that in cylindrical coordinates (ρ, ϕ, z), where $\cos\phi = x/\rho$ and $\sin\phi = y/\rho$, the electric field (11) can be written as, with $r^2 = x^2 + y^2 + z^2 = \rho^2 + z^2$,

$$E_\rho = E_x\cos\phi + E_y\sin\phi, \qquad E_\phi = -E_x\sin\phi + E_y\cos\phi, \tag{13}$$

$$\mathbf{E}(\rho, \phi, z) = -\frac{2A\omega^3 m \sin(kr - \omega t)}{c^3 r^3}(0, \rho^2\cos\phi, \rho z\sin\phi). \tag{14}$$

The radiation magnetic field in cylindrical coordinates is, with $\hat{\mathbf{n}} = (\rho\,\hat{\boldsymbol{\rho}} + z\,\hat{\mathbf{z}})/r$,

$$\mathbf{B}(\rho, \phi, z) = \hat{\mathbf{n}} \times \mathbf{E} = -\frac{2A\omega^3 m \sin(kr - \omega t)}{c^3 r^4}(\rho^2 z, \rho^2 z\sin\phi, \rho^3\cos\phi). \tag{15}$$

2.2 Infinite Solenoid

We now consider an infinite solenoid, along the z-axis. If this is based on currents in a conductor, that conductor "shields" the electromagnetic fields inside and outside the solenoid from one another. For simplicity, we instead suppose that the infinite solenoid is made of a nonconducting, "permanent" magnetic material, such that we can avoid considerations of "shielding".

We also suppose that the infinite solenoid has a very small cross-sectional area, and has magnetic moment $\mathbf{m} = m\,\hat{\mathbf{z}}$ per unit length. That is, our infinite solenoid is a kind of "cosmic string", particularly as imagined by Dirac [6].

The electric field at $(\rho, \phi, 0)$ can be computed (in cylindrical coordinates) from the field (14) as, with $r^2 = \rho^2 + z^2$,

$$\mathbf{E}(\rho, \phi, 0) = -\int_{-\infty}^{\infty} dz\,\frac{2A\omega^3 m \sin(kr - \omega t)}{c^3}\,\frac{(0, \rho^2\cos\phi, \rho z\sin\phi)}{r^3}$$
$$= -\frac{2A\omega^3 m\rho^2\cos\phi}{c^3}\int_{-\infty}^{\infty} dz\,\frac{(0, \sin(kr - \omega t), 0)}{r^3}, \tag{16}$$

which field is purely azimuthal. The integral in eq. (16) has dimensions of length^{-2} and presumably goes as $1/k\rho^3$ times some Bessel function of argument $k\rho - \omega t$, so the field falls off as $1/k\rho$. If so, the dependence of the field on angular frequency $\omega = kc$ would be reduced from ω^3 to ω^2.

By a similar argument for the magnetic field as the z-integral of eq. (15), it has only a z-component, equal in magnitude to the ϕ-component of the electric field (16).

The radiation pattern depends on the square of the radiation electric field, so its angular distribution is just $\cos^2\phi$.

While an infinite solenoid at rest has "zero" magnetic (and electric) field outside it, we see that any transverse motion of the solenoid leads to (small) **E** and **B** fields everywhere in its exterior.

References

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