A Simple Experiment That Shows Ampère's (and Weber's) Force Law to Be Invalid

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Closed loops of steady electric current have a nonzero magnetic dipole moment, and exert equal and opposite forces on one another when at rest,[1] as first demonstrated by Ampère [1]. He argued that the force between two currents elements, $I_1\,d\mathbf{r}_1$ and $I_2\,d\mathbf{r}_2$, is directed along their line of centers $\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$: $d^2\mathbf{F}_{12}^{(\mathrm{A})} = -d^2\mathbf{F}_{21}^{(\mathrm{A})} \propto I_1 I_2\,(\mathbf{r_1} - \mathbf{r_2})$.[2] This law predicts that two current elements do not exert a total torque on themselves,

$$
\begin{aligned}
d^2\boldsymbol{\tau}_{12}^{(\mathrm{A})} + d^2\boldsymbol{\tau}_{21}^{(\mathrm{A})} &= \mathbf{r}_1 \times d^2\mathbf{F}_{12}^{(\mathrm{A})} + \mathbf{r}_2^{(\mathrm{A})} \times d^2\mathbf{F}_{21} \\
\propto \mathbf{r}_1 \times dI_1\,dI_2\,(\mathbf{r_1} - \mathbf{r_2}) &+ \mathbf{r}_2 \times dI_2\,dI_1\,(\mathbf{r_2} - \mathbf{r_1}) = 0. \qquad (1)
\end{aligned}
$$

A simple experiment with two neodymium magnets[3] demonstrates significant rotation when they are dropped from rest, with, say, the North pole of one magnet pointing along their line of centers and the North pole of the other magnet perpendicular that line.[4,5] In particular, when the two magnets collide and stick together, they continue to rotate as a whole, with nonzero mechanical angular momentum.

This result shows that Ampère's force law (and any magnetic force law in which the force between two current elements is along their line of centers, such as that of Weber [6]) cannot hold in general, although it does hold for two circuits at rest with steady currents.[6]

However, it could be that the total initial torque is zero, in agreement with eq. (1), and only after the two magnets start to move does the total torque on the system become nonzero. The experiment described above does not resolve this issue.

[1]The case of closed loops, at rest, of electric current can be called magnetostatics.

[2]In full, Ampère's force law can be written as $d^2\mathbf{F}_{12}^{(\mathrm{A})} = (I_1 I_2/c^2)\,[3(\hat{\mathbf{r}}\cdot d\mathbf{r}_1)(\hat{\mathbf{r}}\cdot d\mathbf{r}_2) - 2\,d\mathbf{r}_1\cdot d\mathbf{r}_2]\,\hat{\mathbf{r}}/r^2$ in Gaussian units, where c is the speed of light, for the force on current element 1 due to element 2.

[3]Such as at https://www.amazon.com/LOVIMAG-Neodymium-Magnets-Whiteboard-Magnets-24pack/dp/B0C7468DWQ/

[4]This experiment is more readily done today than it would have been in 1820, and seems little discussed in the literature of electromagnetism.

[5]Ampère argued [2] (see also [3]) that all magnetism is due to electric currents, rather than magnetic "poles". The latter do not exist so far as we know, and permanent magnetism can be ascribed to "molecular currents" in the language of Ampère. The confirmation that permanent magnetism, due to the magnetic moments of electrons, is Ampèrian (rather than Gilbertian = due to pairs of opposite magnetic charges) came only after detailed studies of positronium (e^+e^- "atoms") in the 1940's [4, 5].

[6]If Maxwell had been aware of this experiment, perhaps he would have been less enthusiastic about Ampère's force law in Art. 527 of his *Treatise* [7]. In any case, he displayed the Lorentz force law in Art. 598 of his *Treatise*, albeit not very clearly. Then, in eq. (11) of Art. 603, Maxwell presented the $\mathbf{J}/c \times \mathbf{B}$ law for the force on a current density $\mathbf{J}$ in a magnetic field $\mathbf{B}$, which is often called the Biot-Savart force.

Further discussion of these issues is given in Appendices A.24.4.7 and A.24.4.9 of [8].

For further insight, we use the Biot-Savart force law[7] to compute the torque of one circuit on another, supposing both circuits are at rest and carry steady currents,

$$
\begin{aligned}
\boldsymbol{\tau}_{12}^{(\mathrm{BS})} &= \oint_1 \mathbf{r}_1 \times d\mathbf{F}_{12}^{(\mathrm{BS})}(\mathbf{r}_1) = \oint_1 \mathbf{r}_1 \times \frac{I_1 I_2}{c^2} \oint_2 \frac{d\mathbf{r}_1 \times (d\mathbf{r}_2 \times \mathbf{r})}{r^3} \\
&= \frac{I_1 I_2}{c^2} \oint_1 \oint_2 \frac{\mathbf{r}_1 \times [(d\mathbf{r}_1 \cdot \mathbf{r})\, d\mathbf{r}_2 - (d\mathbf{r}_1 \cdot d\mathbf{r}_2)\, \mathbf{r}]}{r^3} \\
&= \frac{I_1 I_2}{c^2} \oint_2 \left(\oint_1 \frac{(d\mathbf{r}_1 \cdot \mathbf{r})\, \mathbf{r}_1}{r^3} \right) \times d\mathbf{r}_2 - \frac{I_1 I_2}{c^2} \oint_1 \oint_2 \frac{(d\mathbf{r}_1 \cdot d\mathbf{r}_2)\, \mathbf{r}_1 \times \mathbf{r}_2}{r^3} \\
&= \frac{I_1 I_2}{c^2} \oint_2 \oint_1 \frac{d\mathbf{r}_1 \times d\mathbf{r}_2}{r} - \frac{I_1 I_2}{c^2} \oint_1 \oint_2 \frac{(d\mathbf{r}_1 \cdot d\mathbf{r}_2)\, \mathbf{r}_1 \times \mathbf{r}_2}{r^3} = -\boldsymbol{\tau}_{21}^{(\mathrm{BS})},
\end{aligned} \tag{2}
$$

with $\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$, and recalling[8] that $(d\mathbf{r}_1 \cdot \mathbf{r}/r^3) = -d(1/r)$, the integral in parentheses in the third line of eq. (2) can be integrated by parts (with respect to $\mathbf{r}_1$) to yield the form of the first integral in the last line of eq. (2).

Hence, the total torque, $\boldsymbol{\tau}_{12}^{(\mathrm{BS})} + \boldsymbol{\tau}_{21}^{(\mathrm{BS})}$, on two circuits (at rest with steady currents) is zero according to the Biot-Savart force law, in agreement with eq. (1) based on Ampère's force law.[9]

We infer that the observed total angular momentum of the system of two small magnets is due to a total nonzero total torque that is generated only after the system is released from rest, which could not happen if either Ampère's force law or the Biot-Savart force law still held for moving current elements.

However, the force law $\mathbf{J}/c \times \mathbf{B}$ does apply in general, being a version of the Lorentz force law. But, the form (often attributed to Biot and Savart) $\mathbf{B} = \int d\mathrm{Vol}\, (\mathbf{J}/c \times \mathbf{r})\, / r^3$ for the magnetic field due to current density $\mathbf{J}$ applies only to magnetostatics. Generalizations of the Biot-Savart magnetostatic field are reviewed, for example, in [11].

The initial angular momentum of the system is zero, so we expect that total, final angular momentum of the system to also be zero. This requires there to be a nonzero electromagnetic-field angular momentum equal and opposite to the mechanical angular momentum of the spinning magnets.

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[7] What is called the Biot-Savart force law, $d^2\mathbf{F}_{12}^{(\mathrm{BS})} = (I_1 I_2/c^2)\, [d\mathbf{r}_1) \times (d\mathbf{r}_2) \times \mathbf{r})] / r^3$, in, for example, Sec. 5.2 of [9], is actually due to Ampère, p. 29 of [1], p. 366 of the English translation. Biot and Savart only considered the force between a hypothetical magnetic monopole and an electric current loop, as reviewed, for example, in Sec. A.11 of [8]. Ampère's version of the Biot-Savart force law was almost universally ignored until it was independently (re)discovered by Grassmann in 1845 [10].

[8] See, for example, sec. 5.2 of [9], or footnote 58, p. 23 of [8].

[9] Already in 1825, Ampère [1] showed that his force law and that of Biot and Savart imply the same force of one circuit on another (for circuits at rest with steady currents). This is also true for the torque of one circuit on another, as can be seen from eq. (7) of [12], $d^2\left(\mathbf{F}_{12}^{(\mathrm{A})} - \mathbf{F}_{12}^{(\mathrm{BS})}\right) = -(I_1 I_2/c^2)\, d[\mathbf{r} \cdot d\mathbf{r}_1)\mathbf{r}/r^3]$ for the difference between the two force laws on element $I_1\, d\mathbf{r}_1$ due to element $I_2\, d\mathbf{r}_2$. This difference is a perfect differential, so the difference in the torque on circuit 1 due to circuit 2 according to the two forces laws is zero, $\boldsymbol{\tau}_{12}^{(\mathrm{A})} - \boldsymbol{\tau}_{12}^{(\mathrm{BS})} = \oint_1 \mathbf{r}_1 \times \oint_2 d^2\left(\mathbf{F}_{12}^{(\mathrm{A})} - \mathbf{F}_{12}^{(\mathrm{BS})}\right) = 0$, as the integral around closed circuit 2 of a perfect differential is zero.

Note that this result, together with eq. (1) for Ampère's force law, implies that the sum $\boldsymbol{\tau}_{12}^{(\mathrm{BS})} + \boldsymbol{\tau}_{21}^{(\mathrm{BS})}$ is zero, as found above by a different argument.

Once the magnets start to move, an electric field $\mathbf{E}$ is generated in addition to the initial magnetic field of the system. Then, the field angular momentum $\mathbf{L}_{\rm EM}$ can/must be nonzero,

$$\mathbf{L}_{\rm EM} = \int \mathbf{r} \times \mathbf{p}_{\rm EM}\, d{\rm Vol} = \int \mathbf{r} \times \left(\frac{c}{4\pi} \mathbf{E} \times \mathbf{B} \right) d{\rm Vol} = -\mathbf{L}_{\rm mech}. \tag{3}$$

Appendix: Total Torque on Two Magnetic Dipoles

It is also instructive to consider two small ("point") magnetic dipoles initially at rest, $\boldsymbol{\mu}_1 = \mu_1\, \hat{\mathbf{x}}$ initially located at the origin, $\mathbf{r}_1 = (x_1, y_1, z_1) = (0, 0, 0)$, and $\boldsymbol{\mu}_2 = \mu_2\, \hat{\mathbf{y}}$ initially at $\mathbf{r}_2 = (x_2, y_2, z_2) = (r, 0, 0)$. The analysis below is in the context of Maxwell's electrodynamics; see, for example, Sec. 5.7 of [9] or Sec. 6.1.2 of [13].

The initial magnetic field at $\boldsymbol{\mu}_1$, due to $\boldsymbol{\mu}_2$, is (in Gaussian units)

$$\mathbf{B}_2(\mathbf{r}_1) = \frac{3(\boldsymbol{\mu}_2 \cdot \hat{\mathbf{r}})\, \hat{\mathbf{r}} - \boldsymbol{\mu}_2}{r^3} = -\frac{\mu_2\, \hat{\mathbf{y}}}{r^3}\,. \tag{4}$$

with $\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$. Similarly, the initial magnetic field at $\boldsymbol{\mu}_2$, due to $\boldsymbol{\mu}_1$, is

$$\mathbf{B}_1(\mathbf{r}_2) = \frac{3(\boldsymbol{\mu}_1 \cdot \hat{\mathbf{r}})\, \hat{\mathbf{r}} - \boldsymbol{\mu}_1}{r^3} = \frac{2\mu_1\, \hat{\mathbf{x}}}{r^3}\,. \tag{5}$$

The initial force on $\boldsymbol{\mu}_1$ is[10]

$$\mathbf{F}_1 = \boldsymbol{\nabla}(\boldsymbol{\mu}_1 \cdot \mathbf{B}_2) = \mu_1 \boldsymbol{\nabla}_1 B_{2,x} = \mu_1\mu_2 \boldsymbol{\nabla}_1 \frac{3(y_1 - y_2)(x_1 - x_2)}{r_{12}^5} = -\frac{3\mu_1\mu_2}{r^4}\, \hat{\mathbf{y}}, \tag{6}$$

with $r_{12} = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}$, noting that $\partial r_{12}/\partial y_1 = \partial r_{12}/\partial z_1 = 0$ as the dipoles have $y_1 = z_1 = y_2 = z_2 = 0$. Similarly the initial force on $\boldsymbol{\mu}_2$ is[11]

$$\mathbf{F}_2 = \boldsymbol{\nabla}(\boldsymbol{\mu}_2 \cdot \mathbf{B}_1) = \mu_2 \boldsymbol{\nabla}_2 B_{1,y} = \mu_1\mu_2 \boldsymbol{\nabla}_2 \frac{3(x_1 - x_2)(y_1 - y_2)}{r_{12}^5} = \frac{3\mu_1\mu_2}{r^4}\, \hat{\mathbf{y}} = -\mathbf{F}_1. \tag{10}$$

[10]Use of the gradient operator $\boldsymbol{\nabla}$ requires consideration of the magnetic field in the vicinity of the dipoles, and not just at their location. Of course, the result of the gradient operation is evaluated for the actual location of the dipoles, at $\mathbf{r}_1 = (0, 0, 0)$ and $\mathbf{r}_2 = (r, 0, 0)$.

[11]The force $\mathbf{F}$ on a permanent magnetic-dipole moment $\boldsymbol{\mu}$ in an external magnetic field $\mathbf{B}$ can be rewritten as

$$\mathbf{F} = \boldsymbol{\nabla}(\boldsymbol{\mu} \cdot \mathbf{B}) = \boldsymbol{\mu} \times (\boldsymbol{\nabla} \times \mathbf{B}) + \mathbf{B} \times (\boldsymbol{\nabla} \times \boldsymbol{\mu}) + (\boldsymbol{\mu} \cdot \boldsymbol{\nabla})\mathbf{B} + (\mathbf{B} \cdot \boldsymbol{\nabla})\boldsymbol{\mu} = (\boldsymbol{\mu} \cdot \boldsymbol{\nabla})\mathbf{B}, \tag{7}$$

noting that $\boldsymbol{\nabla} \times \mathbf{B} = 0$ in the region of the dipole $\boldsymbol{\mu}$ provided the source currents of the external field are elsewhere. For the present example,

$$\mathbf{F}_1(\mathbf{r}_1) = (\boldsymbol{\mu}_1 \cdot \boldsymbol{\nabla})\mathbf{B}_2(\mathbf{r}_1) = \mu_1 \frac{\partial}{\partial x_1} \left(\frac{3\mu_2 (y_1 - y_2)\mathbf{r}}{r_{12}^5} - \frac{\mu_2\, \hat{\mathbf{y}}}{r_{12}^3} \right) = -\frac{3\mu_1\mu_2\, \hat{\mathbf{y}}}{r^4}\,, \tag{8}$$

while

$$\mathbf{F}_2(\mathbf{r}_2) = (\boldsymbol{\mu}_2 \cdot \boldsymbol{\nabla})\mathbf{B}_1(\mathbf{r}_2) = \mu_2 \frac{\partial}{\partial y_2} \left(\frac{3\mu_1 (x_1 - x_2)\mathbf{r}}{r_{12}^5} - 2\frac{\mu_1\, \hat{\mathbf{x}}}{r_{12}^3} \right) = \frac{3\mu_1\mu_2\, \hat{\mathbf{y}}}{r^4} = -\mathbf{F}_1. \tag{9}$$

However, once the magnets start rotating, the forces between them become attractive, and they soon stick to one another, while continuing to rotate as a whole. Experiment with two neodymium magnets confirms these results.

The torque about an arbitrary reference point on a magnetic dipole $\boldsymbol{\mu}$ at position $\mathbf{r}_\mu$ is the sum of the torque $\mathbf{r}_\mu \times \mathbf{F}$, due to the magnetic force $\mathbf{F} = \boldsymbol{\nabla}(\boldsymbol{\mu} \cdot \mathbf{B})$ on the dipole, and the torque $\boldsymbol{\mu} \times \mathbf{B}$ about the center of the dipole.

The total initial torque $\boldsymbol{\tau}$ (about an arbitrary, fixed reference point) on the system of two magnetic dipoles is

$$\begin{aligned} \boldsymbol{\tau} &= \mathbf{r}_1 \times \mathbf{F}_1 + \boldsymbol{\mu}_1 \times \mathbf{B}_2 + \mathbf{r}_2 \times \mathbf{F}_2 + \boldsymbol{\mu}_2 \times \mathbf{B}_1 \\ &= \mathbf{r} \times \mathbf{F}_2 + \mu_1\,\hat{\mathbf{x}} \times -\frac{\mu_2\,\hat{\mathbf{y}}}{r^3} + \mu_2\,\hat{\mathbf{y}} \times \frac{2\mu_1\,\hat{\mathbf{x}}}{r^3} = r\,\hat{\mathbf{x}} \times \frac{3\mu_1\mu_2}{r^4}\,\hat{\mathbf{y}} - \frac{\mu_1\mu_2\,\hat{\mathbf{z}}}{r^3} - \frac{2\mu_1\mu_2\,\hat{\mathbf{z}}}{r^3} = 0, \end{aligned} \tag{11}$$

The total initial torque on the system (which is initially at rest) is zero, such that the Maxwellian force and torque laws for "point" magnetic dipoles are in agreement with Ampère's force law about this.

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