

Methods of Calculating Forces on Rigid, Linear Magnetic Media

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Abstract

Despite 180 years of theory on magnetism, it appears that the practice of calculating forces on magnetic media is ambiguous, as illustrated by a recent article by Casperson [1]. Potentially troubling issues include: Which field \mathbf{B} or \mathbf{H} should be used? *This issue was raised by Einstein [2]-[5] who gave a now-nonstandard answer, which has no doubt prolonged this debate.* Should the total field be used, or only the external field? And if the latter, what is meant by “external”? Can/should effects of magnetization currents and/or effective magnetic poles be included? What is the force on only a portion of a circuit? We review several well-known approaches to magnetic force calculations on elements of rigid circuits associated with linear magnetic media, and find it helpful to use an explicit example to compare and contrast the various methods. Our discussion reinforces that of the most authoritative texts [6, 7], but corrects in an important way some previous attempts at a systematic review [8]-[11].

1 Introduction

The phenomenon of magnetism was first manifested via interactions of bulk magnetic materials (magnets). Following the discovery by Oersted [12] that an electric current exerted a force on a magnetic needle, Biot and Savart [13, 14] identified a corresponding force law. Ampère [16] soon noted that pairs of electric currents exert forces on one another, and made the conjecture that all magnetic phenomena are actually due to currents, some of which may be bound in “molecules” of magnetic materials. This view contrasted with the elegant work of Poisson [17] in which forces on magnetic materials (without conduction currents) could be explained in terms of a density of magnetic poles bound in the media in such a way that single poles can never be isolated.

Pedagogic discussions of forces on magnetic media tend to treat the case of current-carrying conductors separately from that of materials (typically conductors) that have nonzero bulk magnetization. The present article addresses the general case of linear magnetic media (where $\mathbf{B} = \mu\mathbf{H}$) in which conduction currents are flowing, but is restricted to the case of forces on elements of a rigid circuit.¹ The literature on this topic appears to be both sparse and erratic,² and we have the (possibly unrealistic) hope of clarifying the record.

Because the details prove to be intricate, the best impression that a reader may form is simply that one must approach magnetic-force problems with caution.

¹For discussion of forces when “permanent” magnetism is involved, see [18, 19, 20, 21].

²For a few examples, see [22, 23, 24].

Some more specific conclusions from this effort are:

1. The consequence of Newton's 1st law that an object with steady motion velocity does not exert a net force on itself can be used in the analysis of static forces on some rigid circuits.^{3,4} See sec. III. We do not pursue here the more complex case of deformable magnetic media.
2. In calculating the force on a circuit (or on a portion of a rigid circuit) using the Biot-Savart law one can use only that magnetic field which existed prior to the introduction of the circuit, since the fields produced directly (or indirectly via induced magnetization) cannot cause a net force on the circuit (or portion of a rigid circuit). See sec. III. That is, the Biot-Savart force density has the form $\mathbf{J}_{\text{cond}}/c \times \mathbf{B}_i = \mathbf{J}_{\text{total}}/c \times \mathbf{H}_i$ and not $\mathbf{J}_{\text{cond}}/c \times \mathbf{B}_{\text{total}}$ nor $\mathbf{J}_{\text{total}}/c \times \mathbf{B}_i$ nor $\mathbf{J}_{\text{total}}/c \times \mathbf{B}_{\text{total}}$ nor $\mathbf{J}_{\text{cond}}/c \times \mathbf{H}_i$ nor $\mathbf{J}_{\text{cond}}/c \times \mathbf{H}_{\text{total}}$ nor $\mathbf{J}_{\text{total}}/c \times \mathbf{H}_{\text{total}}$. See also sec. 4 of [28].
3. Methods also exist in which the total magnetic field can be used in the force calculation, even if the force on only a portion of a rigid circuit is desired. The most straightforward of these uses the Maxwell stress tensor (sec. VIII), which the author recommends be given more prominence in pedagogic treatments of electrostatic and magnetostatic forces.⁵
4. Considerable care must be given in the treatment of systems that include an interface across which the magnetic permeability changes discontinuously. See secs. IV-IX.
5. If the Helmholtz bulk force density (calculated from the total magnetic fields) is used to find the force by integration over a volume that contains such an interface, the volume integral must be supplemented by a surface integral over the (interior) interface. See sec. IX.
6. In a calculation of the force that uses the concept of magnetization currents, the magnetic field \mathbf{H} should be used rather than the field \mathbf{B} , and if the force is desired on only a circuit element, the field must be \mathbf{H}_i as existed prior to the introduction of the circuit (and not merely the introduction of the circuit element). See sec. VI.
7. In a calculation of the force that uses the concept of effective magnetic poles, the magnetic field \mathbf{B} should be used rather than the field \mathbf{H} ,⁶ and the prior field \mathbf{B}_i should

³Examples in which a current generates an "image current" whose field acts back on the first current do not obey this principle. For example, a current parallel to a perfectly conducting plane, or the problem in sec. 35 of [7] (discussed further in [26]).

⁴Accelerated charges can be subject to the so-called radiation-reaction force, which is a self force (first noted by Lorentz [27], and so should be considered as part of the "Lorentz force law." Not all accelerated charges are subject to the radiation reaction force, since interference of the fields of the various charges may cancel the total radiation, as for steady current loops. Also, a uniformly accelerated charge (which is a kind of steady motion) famously experiences no self/radiation-reaction force.

⁵In the literature of forces on permanent magnets the Maxwell stress tensor is frequently used, as well as the so-called method of virtual work which is not reviewed here. See, for example, [29].

⁶In 1908 Einstein and Laub [3, 5] argued that the force law should be,

$$\mathbf{F}_{\text{Einstein-Laub}} = \frac{1}{c} \int [\mathbf{J}_{\text{cond}} \times \mathbf{H} + 4\pi(\mathbf{M} \cdot \nabla)\mathbf{H}] d\text{Vol}. \quad (1)$$

be used if the force on only a circuit element is desired. See sec. VII.

While conclusions 5 and 6 may be known to those expert with magnetic media, the author has not found them in the literature. Because several of the conclusions amend conventional pedagogic wisdom, it has been critical to verify these conclusions with an example in which both $\mathbf{I} \times \mathbf{B}$ forces and magnetization forces are present, and which contains an interface with different permeabilities on either side.

If magnetic charges (monopoles) existed, the case of a magnetic-current-carrying “wire” embedded in a dielectric medium would be the dual of the present example. One could then ask whether the force density on a magnetic current density \mathbf{J}_m in the wire is $-\mathbf{J}_m \times \mathbf{D}$ (the canonical expression) or $-\mathbf{J}_m \times \mathbf{E}$? See, for example, [25, 33].

Other works on the theme of computation of forces in so-called ponderable media include [34, 35, 36].

2 An Example

What is the force per unit length on a wire of radius a and (relative) permeability μ' when it carries uniform conduction current density,

$$\mathbf{J}_{\text{cond}} = \frac{I}{\pi a^2} \hat{\mathbf{z}} \quad (3)$$

and is placed along the z axis in a magnetic field whose form is,

$$\mathbf{B}_i = B_0 \hat{\mathbf{x}} + B_1 \left[\frac{x}{a} \hat{\mathbf{x}} - \frac{y}{a} \hat{\mathbf{y}} \right] = \mu \mathbf{H}_i, \quad (4)$$

before the wire is placed in that field? The medium surrounding the wire is a nonconducting liquid with relative permeability $\mu \neq 1$. The wire is assumed to be part of a rigid circuit that is completely by a loop at “infinity”.

Note that in asking for the force on the wire along the z axis, we seek the force on only a portion of the complete circuit.

The form of the initial magnetic field has been chosen so that there will be both a $\mathbf{J} \times \mathbf{B}$ force associated with the uniform field $\mathbf{B}_0 = B_0 \hat{\mathbf{x}}$, as well as a force due to the interaction of the induced magnetization with the nonuniform field $\mathbf{B}_1 = B_1[(x/a) \hat{\mathbf{x}} - (y/a) \hat{\mathbf{y}}]$.

We have confirmed by the four methods of calculation discussed in secs. VI-IX that the force per unit length on the wire is, in Gaussian units,

$$\mathbf{F} = -\frac{\mu - \mu' a B_0 B_1}{\mu + \mu'} \frac{1}{2\mu} \hat{\mathbf{x}} + \frac{I B_0}{c} \hat{\mathbf{y}}, \quad (5)$$

It appears that this law is obtained from the “Lorentz” force law,

$$\mathbf{F}_{\text{“Lorentz”}} = \frac{1}{c} \int \mathbf{J}_{\text{total}} \times \mathbf{B}_{\text{total}} d\text{Vol}, \quad (2)$$

using Maxwell’s equations and vector identities to recast \mathbf{B} in another form. However, the forms (1)-(2) give incorrect results because the total field \mathbf{B} includes contributions from the current density $\mathbf{J}_{\text{total}}$, which leads to nonzero self forces. See, for example, sec. 4 of [28]. This error has propagated into a recent misunderstanding [30, 31].

More details are given in [32].

where the magnetic fields \mathbf{H} and,

$$\mathbf{B} = \mathbf{H} + 4\pi\mathbf{M}, \quad (6)$$

are related in a linear, isotropic medium of relative permeability μ by,

$$\mathbf{B} = \mu\mathbf{H}, \quad (7)$$

and hence the magnetization density \mathbf{M} obeys,

$$\mathbf{M} = \frac{\mu - 1}{4\pi}\mathbf{H}. \quad (8)$$

The magnetic field \mathbf{H} for this example is given in the Appendix, and details of the calculations can be found in [28].

If the initial magnetic field is uniform ($B_1 = 0$), then there is no net force on the magnetization of the wire or surrounding medium, and the force per unit length on the wire has the very simple form,

$$\mathbf{F} = \frac{IB_0}{c} \hat{\mathbf{y}} \quad (\text{uniform external field}). \quad (9)$$

The expression (9) has been verified in a recent experiment Casperson [1]. See also a related discussion by Lowes [37].

Strictly speaking, eq. (9) describes the force on the conduction electrons, and not that on the lattice of positive ions through which the electrons flow. The force (9) results in a slight rearrangement of the distribution of the conduction electrons and positive lattice ions so that a transverse electric field is generated that acts on the lattice to provide the force experienced by an observer who holds the wire at rest. See [38] for further discussion.

Another example that illustrates the themes of this note is given in [26].

3 The Biot-Savart Force Law

The result (9) is to be expected from a simple argument, consistent with Newton's first law that an object does not exert a net force on itself. Namely, the Biot-Savart force law (for media of unit permeability) on a current-carrying circuit a due to a current-carrying circuit b is⁷

$$\mathbf{F}_a = \frac{I_a I_b}{c^2} \oint_a d\mathbf{l}_a \times \oint_b \frac{d\mathbf{l}_b \times \hat{\mathbf{r}}_{ab}}{r_{ab}^2} = \frac{I_a}{c} \oint_a d\mathbf{l}_a \times \mathbf{B}_b(a), \quad (11)$$

⁷Strictly, Biot and Savart [13, 14] only considered the force on an isolated magnetic pole p due to a steady current I in a circuit, $\mathbf{F} = p \oint \mathbf{I} d\mathbf{l} \times \hat{\mathbf{r}}/cr^2$. What is now called the Biot-Savart force law seems to have been first deduced by Grassmann [39], although not, of course, in vector notation. The vector relation (11) appears without attribution as eq. (11) of Art. 603 of Maxwell's *Treatise* [49], while Einstein may have been the first to call this the Biot-Savart law, in sec. 2 of [3].

In contrast, Ampère [16] proposed the force law,

$$\mathbf{F}_a = -\frac{I_a I_b}{c^2} \oint_a \oint_b \frac{(d\mathbf{l}_a \cdot d\mathbf{l}_b) \mathbf{r}_{ab}}{r_{ab}^2}. \quad (10)$$

The forms (10) and (11) are equivalent for steady currents but predict different forces between isolated current elements (moving charges), as discussed further on the next page.

where

$$\mathbf{B}_b(a) = \frac{I_b}{c} \oint_b \frac{d\mathbf{l}_b \times \hat{\mathbf{r}}_{ab}}{r_{ab}^2}. \quad (12)$$

In the example of sec. II, the initial field \mathbf{B}_i plays the role of the field \mathbf{B}_b not due the current in wire a , which quickly leads to the result (9). The reader may object that the example does not involve media of unit permeability, so the Biot-Savart force law may have to be modified in such a way as to lead to a different result than (9). This issue will be pursued in the following section.

The statement that an object cannot exert a net force on itself tacitly presumes that the object has mechanical integrity as a whole, and will generate internal elastic forces to counteract possible electromagnetic forces of one part of the object on another. If a magnetic circuit is mechanically flexible, we will not be content with an analysis of the force on the circuit as a whole. In addition, we desire a calculation of magnetic force on an element of the circuit, imagining it to be mechanically (but not electrically) disconnected from the rest of the circuit. In this case, the magnetic field to be used in differential form to the Biot-Savart force law is the field \mathbf{B}_{ext} due to all sources outside the element itself,

$$d\mathbf{F} = \frac{I}{c} d\mathbf{l} \times \mathbf{B}_{\text{ext}}. \quad (13)$$

The differential expression (13) can have meaning independent of the integral form (11) only if the circuit deforms, in which case the problem is not one of statics. It is well known that the differential force (13) does not satisfy Newton's 3rd law in cases of isolated current elements, because in such dynamical systems the electromagnetic field momentum is varying [40]. Newton's 3rd law is respected via the appropriate electromagnetic version of his 2nd law,

$$\sum \left(\mathbf{F} - \frac{d\mathbf{P}_{\text{mech}}}{dt} - \frac{d\mathbf{P}_{\text{EM}}}{dt} \right) = 0, \quad (14)$$

where \mathbf{P}_{mech} and \mathbf{P}_{EM} are the mechanical and electromagnetic momenta, respectively. For a discussion of deformable circuits, see [41]. The topic of forces on magnetic liquids, including striction effects, is treated in [42].

In the remainder of this article we assume that the object on which we desire to calculate the magnetic force is a rigid body at rest.

4 Microscopic and Macroscopic Magnetic Fields

In the macroscopic approach to calculation of forces on magnetic media one considers in general a current density \mathbf{J} (electric charge crossing a directed unit area per unit time) and a net magnetization density \mathbf{M} (net magnetic dipole moment per unit volume) as well as the magnetic fields \mathbf{B} and $\mathbf{H} = \mathbf{B} - 4\pi\mathbf{M}$. In a microscopic view, one considers collections of individual charges and/or magnetic moments, but averaged quantities like the magnetization density \mathbf{M} are not yet defined. Hence, the fields \mathbf{B} and \mathbf{H} are identical in the microworld, and it is a matter of convention which symbol is used for the microscopic magnetic field.

The symbol \mathbf{H} tends to be associated with the magnetic force as studied by the early workers Ampère, Biot and Savart, and Poisson [17], which led authors such as Lorentz [43]

and Landau [7] to use this symbol for the microscopic magnetic field. The symbol \mathbf{B} was introduced by Thomson around 1850 [44] in the form (7), which suggests that it is to be derived from the more fundamental (or anyway more familiar at the time) field \mathbf{H} . However, as apparently first noted by Lorentz [43], the macroscopic average of the microscopic magnetic field is \mathbf{B} and not \mathbf{H} .

To maximize the continuity between the microscopic and macroscopic views, the author prefers that the microscopic magnetic field be labeled \mathbf{B} . Then, the force on an electric charge e with velocity \mathbf{v} in microscopic electromagnetic fields \mathbf{E} and \mathbf{B} is,

$$\mathbf{F} = e \left(\mathbf{E} + \frac{\mathbf{v}}{c} \times \mathbf{B} \right), \quad (15)$$

the Lorentz force law, which also represents the average force on the charge in the presence of macroscopically averaged fields \mathbf{E} and \mathbf{B} . It is always understood that the fields \mathbf{E} and \mathbf{B} in the Lorentz force law do not include the fields of the moving charge itself.

High-energy physicists such as the present author, consider the Lorentz force law (15) to be experimentally confirmed (and continually reconfirmed) for over 50 years [45] in the case of high-energy particles moving inside media such as iron where $\mathbf{B} = \mu\mathbf{H} \gg \mathbf{H}$. That is, when applying the Lorentz force law a charge inside a macroscopic medium, the appropriate macroscopic average of the microscopic magnetic field is indeed the macroscopic field \mathbf{B} . This insight is affirmed, for example, in sec. 8.2 of [46], and sec. 22.1.1 of [47],

We therefore expect that other methods of calculating forces on currents in macroscopic media will be consistent with the Lorentz force law using the macroscopic field \mathbf{B} . In particular, we expect that the Biot-Savart force law for a current \mathbf{I} in a macroscopic magnetic field \mathbf{B}_i not due to that current itself is,

$$\mathbf{F} = \frac{1}{c} \oint \mathbf{I} \, dl \times \mathbf{B}_i = \frac{I}{c} \oint d\mathbf{l} \times \mathbf{B}_i, \quad (16)$$

no matter what permeabilities exist. Thus, we reaffirm that eq. (9) is the force on the wire in our example, if the initial field is uniform.

5 The Biot-Savart Force Law in a Permeable Medium

Despite the simplicity of the result (16), care is needed when using the Biot-Savart force law in permeable media. We review this issue by starting with the case that all wires and their (linear, liquid) surrounding media have the same permeability $\mu \neq 1$. Then, there is neither a surface current nor an effective pole density at the interface between the wire and the liquid. However, there remains a volume current density,

$$\mathbf{J}_M = c\nabla \times \mathbf{M} = \frac{\mu - 1}{4\pi/c} \nabla \times \mathbf{H} = (\mu - 1)\mathbf{J}_{\text{cond}}, \quad (17)$$

using Ampère's law,

$$\nabla \times \mathbf{H} = \frac{4\pi}{c} \mathbf{J}_{\text{cond}}, \quad (18)$$

that relates the magnetic field \mathbf{H} to the conduction current density \mathbf{J}_{cond} . Thus, the total current density is,

$$\mathbf{J}_{\text{total}} = \mu \mathbf{J}_{\text{cond}}. \quad (19)$$

The fact that the total current density (19) does not equal the conduction current density in a permeable medium contradicts the view of Lorentz [43], as reaffirmed in sec. 30 of [7]. This appears to be one of the very few oversights in these distinguished works.

The force (16) on the wire that carries conduction current density \mathbf{J}_{cond} can be written as,

$$\mathbf{F} = \frac{1}{c} \int \mathbf{J}_{\text{cond}} \times \mathbf{B}_i \, d\text{Vol}. \quad (20)$$

If instead we wish to use the total current density (19) we must write,

$$\mathbf{F} = \frac{1}{c} \int \frac{\mathbf{J}_{\text{total}}}{\mu} \times \mathbf{B}_i \, d\text{Vol} = \frac{1}{c} \int \mathbf{J}_{\text{total}} \times \mathbf{H}_i \, d\text{Vol}. \quad (21)$$

Another aspect of the analysis of Biot and Savart is the calculation of the magnetic field from the current density. The microscopic version of Ampère's law,

$$\nabla \times \mathbf{B} = \frac{4\pi}{c} \mathbf{J}_{\text{total}}, \quad (22)$$

corresponds to the prescription that,

$$\mathbf{B} = \frac{1}{c} \int \frac{\mathbf{J}_{\text{total}} \times \hat{\mathbf{r}}}{r^2} \, d\text{Vol} = \frac{\mu}{c} \int \frac{\mathbf{J}_{\text{cond}} \times \hat{\mathbf{r}}}{r^2} \, d\text{Vol} = \mu \mathbf{H}, \quad (23)$$

supposing the permeability is uniform and may be taken outside the integral. Hence, the macroscopic version of Ampère's law, eq. (18), corresponds to the prescription that,

$$\mathbf{H} = \frac{1}{c} \int \frac{\mathbf{J}_{\text{cond}} \times \hat{\mathbf{r}}}{r^2} \, d\text{Vol} = \nabla \times \frac{1}{c} \int \frac{\mathbf{J}_{\text{cond}}}{r} \, d\text{Vol}, \quad (24)$$

independent of the permeability. This result, combined with Ampère's law (18), is consistent with Helmholtz' theorem [46] provided $\nabla \cdot \mathbf{H} = 0$, as holds within a medium of uniform permeability.

The form of the eq. (11) for the force on circuit a due to circuit b supposing the wires and the surrounding media all have permeability μ is therefore,

$$\mathbf{F}_a = \mu \frac{I_a I_b}{c^2} \oint_a d\mathbf{l}_a \times \oint_b \frac{d\mathbf{l}_b \times \hat{\mathbf{r}}_{ab}}{r_{ab}^2}, \quad (25)$$

where I_a and I_b are the conduction currents in the circuits.

6 The Biot-Savart Force Law Plus Bound Current Densities

We also see that eq. (25) holds even if the wires have permeabilities μ_a and μ_b that differ from the permeability μ of the surrounding medium, since the magnetic field due to wire b

at the position of wire a before wire a was introduced is given by $\mathbf{B}_b = \mu\mathbf{H}_b$, which depends on neither μ_a nor μ_b . However, in this case there will exist effective surface currents,

$$\mathbf{K}_M = c\Delta\mathbf{M} \times \hat{\mathbf{n}}, \quad (26)$$

at the interface between the wires and the surrounding medium, where $\Delta\mathbf{M}$ is the difference between the magnetization on the two sides of the interface, and $\hat{\mathbf{n}}$ is the unit normal to the interface. The force on these surface currents follows the form (21) that uses the magnetic field \mathbf{H} rather than \mathbf{B} . Hence, the total force on a permeable wire surrounded by a permeable medium can be written as,

$$\mathbf{F} = \frac{1}{c} \int \mathbf{J}_{\text{total}} \times \mathbf{H}_i \, d\text{Vol} + \frac{1}{c} \int \mathbf{K}_M \times \mathbf{H}_i \, dS. \quad (27)$$

This is the appropriate version of the Biot-Savart law if we wish to include magnetization forces via the so-called bound current densities. However, the Coulomb Committee in their eq. (1.3-4') [10], and Jefimenko in his eq. (14-9.13a,b) [48], recommends that the initial field \mathbf{B}_i be used rather than \mathbf{H}_i , which would imply a force μ times the above.

For the example of sec. II, the surface current density is obtained from eq. (8) and the total magnetic field \mathbf{H} , eq. (46), as,

$$\mathbf{K}_M = \frac{\mu - \mu'}{4\pi} \left[\frac{2I}{a} - \frac{2\mu c}{\mu + \mu'} (H_0 \sin \theta + H_1 \sin 2\theta) \right] \hat{\mathbf{z}}, \quad (28)$$

and the total current density is $\mathbf{J}_{\text{total}} = \mu'I \hat{\mathbf{z}}/\pi a^2$ from eqs. (3) and (19). Then, evaluation of eq. (27) leads to eq. (5).

The correct result for the force on a portion of a rigid circuit would not be obtained from eq. (27) if we used the initial magnetic field \mathbf{B}_i , or if we used the total magnetic fields \mathbf{B} or \mathbf{H} on the wire. Furthermore, we would not obtain a correct result if we used the field in a vacuum cavity of radius a at the position of the wire either before or after the wire was inserted. The proper initial field \mathbf{H}_i is the one before the wire was inserted into the liquid dielectric.

When $\mu \neq \mu' \neq 1$, we cannot rewrite the first term of eq. (27) as $\int (\mathbf{J}_{\text{cond}}/c) \times \mathbf{B}_i \, d\text{Vol}$, which would incorrectly suggest that we could ignore the volume magnetization current density \mathbf{J}_M but not the surface current density \mathbf{K}_M .

7 The Biot-Savart Force Law Plus Effective Magnetic Poles

The forces on the magnetization can also be considered as due to a density of effective magnetic poles, rather than being due to currents \mathbf{J}_M and \mathbf{K}_M . Some care is required to use this approach, since a true magnetic pole density ρ_M would imply $\nabla \cdot \mathbf{B} = 4\pi\rho_M$, and the bulk force density on these poles would be $\mathbf{F} = \rho_M\mathbf{H}$ [17].⁸ However, in reality $0 = \nabla \cdot \mathbf{B} = \nabla \cdot (\mathbf{H} + 4\pi\mathbf{M})$, so we write,

$$\nabla \cdot \mathbf{H} = -4\pi\nabla \cdot \mathbf{M} = 4\pi\rho_{M,\text{eff}}, \quad (29)$$

⁸See [25] for additional discussion of true and effective magnetic charges.

and we identify $\rho_{M,\text{eff}} = -\nabla \cdot \mathbf{M}$ as the volume density of effective magnetic poles. Inside linear magnetic media, such as those considered here, $\mathbf{B} = \mu\mathbf{H}$ and $\nabla \cdot \mathbf{B} = 0$ together imply that $\rho_{M,\text{eff}} = 0$. However, a surface density $\sigma_{M,\text{eff}}$ of effective poles can exist on an interface between two media, and we see that Gauss' law for the field \mathbf{H} implies that,

$$\sigma_{M,\text{eff}} = \frac{(\mathbf{H}_2 - \mathbf{H}_1) \cdot \hat{\mathbf{n}}}{4\pi}, \quad (30)$$

where unit normal $\hat{\mathbf{n}}$ points across the interface from medium 1 to medium 2. The effective surface pole density can also be written in terms of the magnetization $\mathbf{M} = (\mathbf{B} - \mathbf{H})/4\pi$ as,

$$\sigma_{M,\text{eff}} = (\mathbf{M}_1 - \mathbf{M}_2) \cdot \hat{\mathbf{n}}, \quad (31)$$

since $\nabla \cdot \mathbf{B} = 0$ insures that the normal component of \mathbf{B} is continuous at the interface.

The force on the surface density of effective magnetic poles is,

$$\mathbf{F} = \sigma_{M,\text{eff}}\mathbf{B}_i, \quad (32)$$

since the effective poles (which are representations of effects of Ampèrian currents) couple to the macroscopic average of the microscopic magnetic field \mathbf{B} .⁹ Equation (32) is in agreement with prob. 5.20 of [50]. However, the Coulomb Committee in their eq. (1.3-4) [10], and Jefimenko in his eq. (14-9.9a,b) [48], recommends that the initial field \mathbf{H}_i be used rather than \mathbf{B}_i when using the method of effective magnetic poles, which would imply a force $1/\mu$ times the above.

The total force on the medium in this view is the sum of the force on the conduction current plus the force on the effective surface poles, where to avoid calculating a spurious force of the rigid wire on itself we use the initial magnetic field \mathbf{B}_i ,

$$\mathbf{F} = \frac{1}{c} \int \mathbf{J}_{\text{cond}} \times \mathbf{B}_i \, d\text{Vol} + \int \sigma_{M,\text{eff}}\mathbf{B}_i \, dS. \quad (33)$$

In the example of sec. II, the density of effective magnetic poles on the surface $r = a$ is given by,

$$\begin{aligned} \sigma_{M,\text{eff}} &= \frac{H_r(r = a^+) - H_r(r = a^-)}{4\pi} \\ &= -\frac{1}{2\pi} \frac{\mu - \mu'}{\mu + \mu'} (H_0 \cos \theta + H_1 \cos 2\theta). \end{aligned} \quad (34)$$

Then, evaluation of eq. (33) again leads to the result (5). This reaffirms that one should use the initial field \mathbf{B}_i and not \mathbf{H}_i and not the total field \mathbf{B} in the Biot-Savart method (33) with effective magnetic poles when calculating the force on only a portion of a rigid circuit.

⁹Poisson [17] worked exclusively with the magnetic field \mathbf{H} , but realized that the effective force on a true (Gilbertian) magnetic pole p is not necessarily $\mathbf{F} = p\mathbf{H}$ if the pole is at rest inside a bulk medium, which results in an altered force on the pole depending on the assumed shape of the surrounding cavity. W. Thomson (Lord Kelvin) noted in 1871, sec. 517 of [44], that for a pole in a disk-shaped cavity with axis parallel to the magnetization \mathbf{M} of the medium, the force would be $\mathbf{F} = p(\mathbf{H} + 4\pi\mathbf{M})$, and therefore he introduced the magnetic field $\mathbf{B} = \mathbf{H} + 4\pi\mathbf{M}$ "according to the electromagnetic definition" (in Gaussian units). In sec. 400 of his *Treatise* [49], Maxwell follows Thomson in stating that the effective force on a true magnetic pole is usefully considered to be $\mathbf{F} = p\mathbf{B}$ (Gaussian units). This convention for the effective force on a true (Gilbertian) magnetic pole is the same as the "true" force on an effective (Ampèrian) magnetic pole, which latter is the topic of the sec. 7 of this note.

8 The Maxwell Stress Tensor

The methods of calculating the force on a circuit discussed thus far are require care in that only the fields \mathbf{B}_i and \mathbf{H}_i prior to the addition of the circuit appear explicitly in the force calculation, although the total fields \mathbf{B} and \mathbf{H} are needed in the calculation of the magnetization. The erratic literature on this topic is ample evidence that confusion as how to implement these calculation is likely.

Hence, it may be preferable to use methods that involve only the total fields \mathbf{B} and \mathbf{H} . In the author's view, the most reliable general method for the calculation of electromagnetic forces is that based on the Maxwell stress tensor,¹⁰ which is a formal transcription of Faraday's "tubes of force." The j -component of the electromagnetic force \mathbf{F} on the interior of a closed volume in a linear medium with oriented surface element $d\mathbf{S}$ is given by,

$$F_j = \int \sum_k T_{jk} dS_k, \quad (35)$$

where

$$T_{jk} = \frac{1}{4\pi} E_j D_k + B_j H_k - \frac{\delta_{jk}}{8\pi} (\mathbf{E} \cdot \mathbf{D} + \mathbf{B} \cdot \mathbf{H}). \quad (36)$$

The form (36) ignores interesting stricitive effects in compressible media [52].

The surface over which the stress tensor is integrated need not correspond to a physical surface, which leads to the question of how the electromagnetic force is transmitted to the physical matter inside that surface. The answer is, of course, via the electromagnetic fields that enter into the stress tensor. Prior to, and even somewhat after, Maxwell, more physical explanations were considered necessary, which led to the diversionary search for the æther that, among other activities, would transmit the forces from the imaginary surface to the matter within.

We have already noted in sec. II that even the $\mathbf{J} \times \mathbf{B}$ force concept is at least once removed from a force on the positive ion lattice of a conductor – which caused confusion to Maxwell [54].

For the example of sec. II, we calculate the force on unit length of the wire by integrating the Maxwell stress tensor over a cylindrical surface of radius $r > a$, so that any effects at the surface $r = a$ are included. The result is, as expected, given by eq. (5), independent of radius r .

Equation (5) for the case that $B_1 = 0$ was deduced in a similar manner in ref. [6].

If we integrate the stress tensor over a cylinder of radius $r < a$ the result is,

$$\mathbf{F} = \frac{2\mu'}{\mu + \mu'} \frac{IB_0}{c} \frac{r^2}{a^2} \hat{\mathbf{y}}. \quad (37)$$

Since the limit of this as $r \rightarrow a$ does not equal the result (5) for $r > a$, we infer that there are important effects at the interface $r = a$. The permeable liquid is presumably contained in a tank of some characteristic radial scale $b \gg a$, at whose surface additional magnetization

¹⁰Maxwell's discussion of the stress tensor in secs. 639-459 of his *Treatise* [49] was for permanent magnetization. The stress tensor (35) for linear magnetic media was apparently first deduced by Lorentz [43], starting from the (Lorentz) force density $\mathbf{f} = \rho\mathbf{E} + \mathbf{J}/c \times \mathbf{B}$.

forces will arise. We consider these forces as distinct from those at the interface $r = a$, and that only the latter are part of the forces on the wire.¹¹

9 The Helmholtz Bulk Force Density

An expression for a bulk force density \mathbf{f} in magnetic media can be obtained by transformation of the surface integral of the stress tensor into a volume integral. See, for example, secs. 15 and 35 of [7]. The result, again ignoring magnetostriction, is,

$$\mathbf{f} = \frac{1}{c} \mathbf{J}_{\text{cond}} \times \mathbf{B} - \frac{H^2}{8\pi} \nabla \mu, \quad (38)$$

which is due to Helmholtz [51]. As for the Maxwell stress tensor, the fields \mathbf{B} and \mathbf{H} in eq. (38) are the total fields from all sources.

However, eq. (38) is not sufficient for the case that the permeability takes a discontinuous step at an interface within the volume of interest. To see this, we recall the usual derivation of the bulk force density, beginning with Gauss' law to transform the surface integral of the Maxwell stress tensor into a volume integral of a force density \mathbf{f} ,

$$F_i = \int \sum_j T_{ij} dS_j = \int \sum_j \frac{\partial T_{ij}}{\partial x_j} d\text{Vol} = \int f_i d\text{Vol}. \quad (39)$$

Recalling eq. (36), the force density \mathbf{f} is given by,

$$\begin{aligned} \mathbf{f} &= \nabla \cdot \mathbf{T} = \frac{1}{4\pi} \left[(\mathbf{B} \cdot \nabla) \mathbf{H} + \mathbf{H} (\nabla \cdot \mathbf{B}) - \frac{1}{2} \nabla (\mathbf{B} \cdot \mathbf{H}) \right] \\ &= \frac{\mu}{4\pi} (\mathbf{H} \cdot \nabla) \mathbf{H} - \frac{1}{8\pi} H^2 \nabla \mu - \frac{\mu}{8\pi} \nabla H^2, \end{aligned} \quad (40)$$

since $\nabla \cdot \mathbf{B} = 0$ always, and we first suppose that $\mathbf{B} = \mu \mathbf{H}$ involves a continuously varying permeability. The usual argument then proceeds by noting that,

$$\begin{aligned} \nabla H^2 &= 2(\mathbf{H} \cdot \nabla) \mathbf{H} + 2\mathbf{H} \times (\nabla \times \mathbf{H}) \\ &= 2(\mathbf{H} \cdot \nabla) \mathbf{H} - \frac{8\pi}{c} \mathbf{J}_{\text{cond}} \times \mathbf{H}, \end{aligned} \quad (41)$$

using Ampère's law (18). Inserting eq. (41) in (40) we arrive at eq. (38). However, if the volume of interest includes an interior interface across which the permeability takes a discontinuous step, we should revert to the first form of eq. (40) when performing the volume integral across the interface. Defining $\hat{\mathbf{n}}$ to be the unit normal to the interface, and noting that $\mathbf{B} \cdot \mathbf{H} = \mu H_t^2 + B_n^2/\mu$ where the tangential and normal components, H_t and B_n , are continuous across such an interface, the resulting surface integral is,

$$\begin{aligned} &\frac{1}{4\pi} \int \left[B_n \Delta H_n - \frac{1}{2} \Delta (\mathbf{B} \cdot \mathbf{H}) \right] \hat{\mathbf{n}} dS, \\ &= \frac{1}{8\pi} \Delta \left(\frac{1}{\mu} \right) \int B_n^2 \hat{\mathbf{n}} dS - \frac{\Delta \mu}{8\pi} \int H_t^2 \hat{\mathbf{n}} dS \end{aligned} \quad (42)$$

¹¹See sec. 2.24 of [6] for comments on the use of the stress tensor near a surface across which the conductivity, permeability, and/or permittivity are discontinuous. Further discussion appears, for example, in [55, 56].

The total magnetic force can now be written,

$$\begin{aligned} \mathbf{F} = & \int \frac{1}{c} \mathbf{J}_{\text{cond}} \times \mathbf{B} \, d\text{Vol} - \frac{1}{8\pi} \int H^2 \nabla \mu \, d\text{Vol} \\ & + \frac{1}{8\pi} \Delta \left(\frac{1}{\mu} \right) \int B_n^2 \hat{\mathbf{n}} \, dS - \frac{\Delta \mu}{8\pi} \int H_t^2 \hat{\mathbf{n}} \, dS \end{aligned} \quad (43)$$

where the surface integral is over the interface which is, in general, interior to the volume of integration, and ΔA is the difference of quantity A on the two sides of the interface. To apply eq. (43) to example of sec. II, we note that the second integral is zero (considering the integral over the interior interface as distinct from the bulk volume integrals), and that $\hat{\mathbf{n}} = \hat{\mathbf{r}}$ on the interface at $r = a$. Therefore, we need only those terms of B_r^2 and H_θ^2 that vary as $\sin \theta$ or $\cos \theta$. Referring to the Appendix, we find,

$$\begin{aligned} \Delta \left(\frac{1}{\mu} \right) B_r^2 - \Delta \mu H_\theta^2 \\ = 8 \frac{\mu - \mu'}{\mu + \mu'} \left(\frac{IB_0}{ca} \sin \theta - \frac{B_0 H_1}{2} \cos \theta \right) + \dots \end{aligned} \quad (44)$$

Using this in eq. (43), we again obtain the result (5).

A Appendix: The Field \mathbf{H} for the Example

The magnetic field \mathbf{H} for the example of sec. II is, in both cylindrical and rectangular coordinates,

$$\mathbf{H} = \begin{cases} \left[\frac{2\mu}{\mu + \mu'} (H_0 \cos \theta + H_1 \frac{r}{a} \cos 2\theta) \hat{\mathbf{r}} \right. \\ \left. + \left[\frac{2Ir}{ca^2} - \frac{2\mu}{\mu + \mu'} (H_0 \sin \theta + H_1 \frac{r}{a} \sin 2\theta) \right] \hat{\boldsymbol{\theta}} \right] \quad (r < a), \\ \left[H_0 \left(1 - \frac{\mu - \mu'}{\mu + \mu'} \frac{a^2}{r^2} \right) \cos \theta \right. \\ \left. + H_1 \left(\frac{r}{a} - \frac{\mu - \mu'}{\mu + \mu'} \frac{a^3}{r^3} \right) \cos 2\theta \right] \hat{\mathbf{r}} \\ \left. + \left[\frac{2I}{cr} - H_0 \left(1 + \frac{\mu - \mu'}{\mu + \mu'} \frac{a^2}{r^2} \right) \sin \theta \right. \right. \\ \left. \left. - H_1 \left(\frac{r}{a} + \frac{\mu - \mu'}{\mu + \mu'} \frac{a^3}{r^3} \right) \sin 2\theta \right] \hat{\boldsymbol{\theta}} \quad (r > a), \end{cases} \quad (45)$$

$$= \begin{cases} \left[-\frac{2Ir}{ca^2} \sin \theta + \frac{2\mu}{\mu + \mu'} (H_0 + H_1 \frac{r}{a} \cos \theta) \right] \hat{\mathbf{x}} \\ \left. + \left(\frac{2Ir}{ca^2} \cos \theta - \frac{2\mu}{\mu + \mu'} H_1 \frac{r}{a} \sin \theta \right) \hat{\mathbf{y}} \quad (r < a), \right. \\ \left[-\frac{2I}{cr} \sin \theta + H_0 \left(1 - \frac{\mu - \mu'}{\mu + \mu'} \frac{a^2}{r^2} \cos 2\theta \right) \right. \\ \left. + H_1 \left(\frac{r}{a} \cos \theta - \frac{\mu - \mu'}{\mu + \mu'} \frac{a^3}{r^3} \cos 3\theta \right) \right] \hat{\mathbf{x}} \\ \left. + \left[\frac{2I}{cr} \cos \theta - \frac{\mu - \mu'}{\mu + \mu'} H_0 \frac{a^2}{r^2} \sin 2\theta \right. \right. \\ \left. \left. - H_1 \left(\frac{r}{a} \sin \theta + \frac{\mu - \mu'}{\mu + \mu'} \frac{a^3}{r^3} \sin 3\theta \right) \right] \hat{\mathbf{y}} \quad (r > a). \end{cases} \quad (46)$$

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