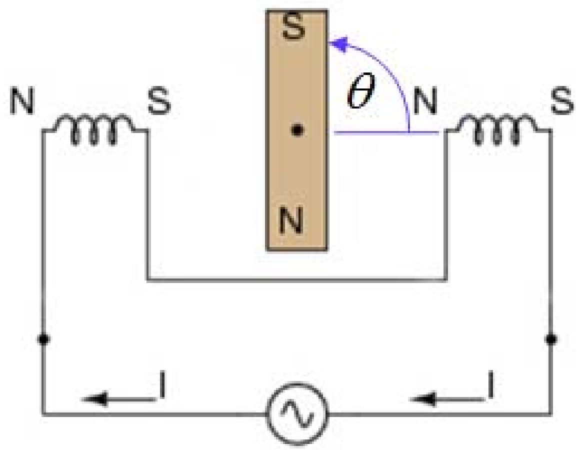
Self Inductance of a Solenoid with a Permanent-Magnet Core

Kirk T. McDonald
Joseph Henry Laboratories, Princeton University, Princeton, NJ 08544
(March 3, 2013; updated September 11, 2020)

1 Problem

Consider an oscillatory current $I(t)$ in a solenoid coil that contains a permanent magnet which can rotate about an axis perpendicular to its symmetry axis as well as to the axis of the solenoid, as shown in the figure below. Show that the $\mathcal{EMF}$ across the coil, due to the permanent magnet, has the character of a capacitance rather than an inductance.



The magnetic field due to the solenoid can be approximated at uniform over the (rotating) permanent magnet.[1]

2 Solution

If the magnetic field $\mathbf{B}_I \approx \mu_0 N I\,\hat{\mathbf{z}}/l$ due to the current I in the solenoid is uniform over the permanent magnet of volume V_M and magnetic moment $\mathbf{m} = \mathbf{M}V_M$ where V_M, then the force $\mathbf{F} + -\boldsymbol{\nabla}(\mathbf{m}\cdot\mathbf{B}_I)$ is zero. However, if the magnetic moment $\mathbf{m}V_M$ is not parallel to the magnetic field $\mathbf{B}_I$ of the coil, the permanent magnet experiences a torque,

$$\boldsymbol{\tau} = \mathbf{m}\times\mathbf{B}_I = \mathbf{M}V_M \times \mathbf{B}_I, \tag{1}$$

and will rotate as a consequence if the motion of the magnet is unconstrained.

Let θ be the variable angle between $\mathbf{M}$ and $\mathbf{B}_I$ and $I_{\rm mech}$ be the moment of inertia of the magnet about the its (fixed) axis of rotation.[2]

[1]The permanent magnetism can be thought of as due to permanent "supercurrents" that do not change (in the frame of the permanent magnet) as the current I in the rest of the circuit varies.

[2]Angle $\theta = 90^\circ$ in the figure above.

The rotational equation of motion of the magnet is,

$$I_{\rm mech} \frac{d^2\theta}{dt^2} = \tau = -\frac{\mu_0 M N V_M I(t)}{l} \sin\theta. \tag{2}$$

If the initial angle is θ_0 and we write $\theta = \theta_0 + \vartheta$, then the equation of motion (2) can be written as,

$$I_{\rm mech} \frac{d^2\vartheta}{dt^2} = -\frac{\mu_0 M N V_M I(t)}{l} (\sin\theta_0 \cos\vartheta + \cos\theta_0 \sin\vartheta). \tag{3}$$

In an AC circuit where the coil is in series with a resistor R and the current is $I(t) = I_0 \cos\omega t$, the equation of motion of the permanent magnet becomes,

$$\frac{d^2\vartheta}{dt^2} = -\frac{\mu_0 M N V_M I_0}{l I_{\rm mech}} \cos\omega t (\sin\theta_0 \cos\vartheta + \cos\theta_0 \sin\vartheta). \tag{4}$$

This has the oscillatory solution,

$$\cos\theta_0 = 0, \qquad \sin\theta_0 = \pm 1, \qquad \vartheta = \vartheta_0 \cos\omega t, \qquad \vartheta_0 = \sin\theta_0 \frac{\mu_0 M N V_M I_0}{\omega^2 l I_{\rm mech}}, \tag{5}$$

for small ϑ_0. The voltage source $V(t)$ does mechanical work on the rotating magnet at rate,

$$\begin{aligned} P_{\rm mech} = \frac{d}{dt} I_{\rm mech} \dot{\vartheta}^2 = \tau\dot{\vartheta} \;&\approx\; -\frac{\mu_0 M N V_M I_0 \cos\omega t}{l} \sin\theta_0 \left(-\omega \sin\theta_0 \frac{\mu_0 M N V_M I_0}{\omega^2 l I_{\rm mech}} \sin\omega t \right) \\ &= \frac{\mu_0^2 M^2 N^2 V_M^2}{\omega l^2 I_{\rm mech}} I_0^2 \cos\omega t \sin\omega t, \end{aligned} \tag{6}$$

so the total (instantaneous) power provided by the source is,

$$\begin{aligned} P = VI \;&=\; I^2 R + \frac{dU_{\rm mag}}{dt} + P_{\rm mech} \approx I^2 R + L I \dot{I} + \frac{\mu_0^2 M^2 N^2 V_M^2}{\omega l^2 I_{\rm mech}} I_0^2 \cos\omega t \sin\omega t \\ &= I \left[IR + \left(L - \frac{\mu_0^2 M^2 N^2 V_M^2}{\omega^2 l^2 I_{\rm mech}} \right) \dot{I} \right], \end{aligned} \tag{7}$$

where the magnetic energy $U_{\rm mag}$ is the sum of the magnetic field energy and the interaction energy between the permanent magnet and the field of the solenoid,

$$\begin{aligned} U_{\rm mag} = U_{\rm field} + U_{\rm int} = \int \frac{(\mathbf{B}_I + \mathbf{B}_M)^2}{2\mu_0} d{\rm Vol} - \mathbf{m} \cdot \mathbf{B}_I \\ \approx \int \frac{B_I^2}{2\mu_0} d{\rm Vol} + \mathbf{B}_I \cdot \mathbf{M} V_M + \frac{\mu_0 M^2 V_M}{2} - \mathbf{m} \cdot \mathbf{B}_I = \frac{LI^2}{2} + + \frac{\mu_0 M^2 V_M}{2}, \end{aligned} \tag{8}$$

in the approximation that the permanent magnetic field is $\mathbf{B}_M = \mu_0 \mathbf{M}$ inside the permanent magnet and zero outside.[3] Here, L is the self inductance of the solenoid, such that the magnetic energy (in the absence of the permanent magnet) is $U_I = LI^2/2$.

[3] Difficulties in evaluating force and energy in systems with permanent magnets are discussed, for example, in [3, 4, 5].

If we now switch to complex notation, writing $V = V_0 \, e^{i\omega t}$, $I = I_0 \, e^{i\omega t}$ with I_0 a complex constant, then $\dot{I} = i\omega I$ and the impedance Z of the system is,

$$Z = \frac{V}{I} = \frac{P}{I^2} = R + i\omega L + \frac{\mu_0^2 M^2 N^2 V_M^2}{i\omega l^2 I_{\rm mech}} \equiv R + i\omega L + \frac{1}{i\omega C} \,. \tag{9}$$

We can say that the effect of the oscillating permanent magnet on the system is not so much to change the self inductance L of the coil,[4,5] but to give it an effective capacitance,

$$C = \frac{l^2 I_{\rm mech}}{\mu_0^2 M^2 N^2 V_M^2} \,. \tag{10}$$

The system behaves like a series R-L-C circuit, with resonant angular frequency,

$$\omega_0 = \frac{1}{\sqrt{LC}} \,, \tag{11}$$

at which frequency the magnitude of the current is maximal, with $I_0 = V_0/R$.

Electromechanical resonances have been observed in the apparatus described in [6] (private communication, David J. Jefferies). The capacitance induced by a rotating magnetic field underlies the sensor described in [7].

The magnet can also make full rotations, driven by the AC power source, in which case the system is a kind of single-phase motor, as first demonstrated by Bally [8] in 1879.[6]

References

[1] K.T. McDonald, *Electricity and Magnetism, Lecture 8*,
http://kirkmcd.princeton.edu/examples/ph501/ph501lecture8.pdf

[2] K.T. McDonald, *Electricity and Magnetism, Lecture 7*,
http://kirkmcd.princeton.edu/examples/ph501/ph501lecture7.pdf

[3] H.C. Lovatt and P.A. Watterson, *Energy Stored in Permanent Magnets*, IEEE Trans. Mag. **35**, 505 (1999), http://kirkmcd.princeton.edu/examples/EM/lovatt_ieeetm_35_505_99.pdf

[4] As the permanent magnet rotates, the flux of its magnetic field through the coil varies, and an $\mathcal{EMF}$ is induced in the circuit (as pointed out by Pei-Hsun Jiang). For $\theta = \theta_0 + \vartheta$ with small ϑ, this flux is proportional to ϑ, so the $\mathcal{EMF}$ is proportional to $\dot{\vartheta}$, which is proportional to I/ω, recalling eqs. (5)-(6). While this $\mathcal{EMF}$ (reactance) is associated with magnetism, its dependence on current and frequency is that of a capacitive, rather than an inductive, reactance. It is certainly not associated with the self inductance of the circuit, since the permanent magnet is not part of the nominal electric circuit. And, since the magnet is permanent, rather than an electromagnetic, it is not to be associated with a mutual inductance in the usual sense.

[5] If the permanent magnet were fixed with respect to the solenoid coil, then the mechanical energy of the permanent magnet would be zero, and in the approximation of eqs. (7)-(8) the circuit would behave as if the permanent magnet were not present.

[6] Variants of "flywheels" have long been used as energy storage devices in mechanical systems, and also in electromechanical systems such as motor-generator sets, although such flywheels are generally not magnets.

[4] P. Campbell, *Comments on Energy Stored in Permanent Magnets*, IEEE Trans. Mag. **36**, 401 (2000), http://kirkmcd.princeton.edu/examples/EM/campbell_ieeetm_36_401_00.pdf

[5] S. Sanz *et al.*, *Evaluation of Magnetic Forces in Permanent Magnets*, IEEE Trans. Appl. Semi. **20**, 846 (2010), http://kirkmcd.princeton.edu/examples/EM/sanz_ieeetap_20_846_10.pdf

[6] M.N. Wybourne *et al.*, *Frequency-crossing phonon spectrometer techniques*, Rev. Sci. Instr. **50**, 1634 (1979), http://kirkmcd.princeton.edu/examples/detectors/wybourne_rsi_50_1634_79.pdf

[7] F.-M. Hsu *et al.*, *A Novel Magnetic-Induced Capacitive-Sensing Rotation Sensor*, IEEE Sensors, 1 (2012), http://kirkmcd.princeton.edu/examples/EM/hsu_ieeesensors_1_12.pdf

[8] W. Bally, *A Mode of producing Arago's Rotation*, Proc. Phys. Soc. London **3**, 115 (1879), http://kirkmcd.princeton.edu/examples/EM/bally_ppsl_3_115_79.pdf