Marinov's Magnetic Puzzler

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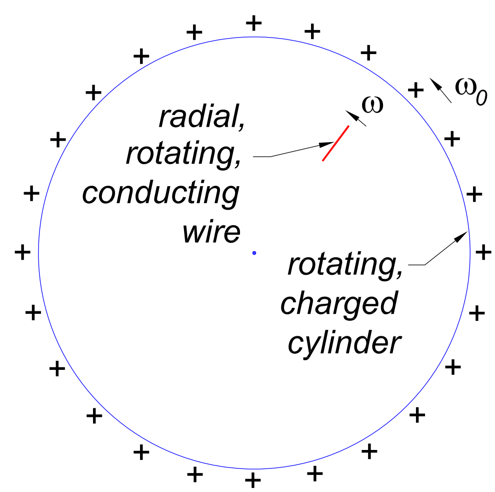
1 Problem

A uniform, axial magnetic field is generated inside a long cylinder of radius r_0 with uniform surface charge density λ when the cylinder is rotated at angular velocity ω_0. What is the voltage difference between the ends of a conducting wire that extends radially from $r = a$ to $r = b < r_0$ when this wire is rotated at angular velocity ω, always aligned along a radius, as sketched below?[1]



2 Solution

In this section, we work in the (inertial) lab frame.

While there exists a radial electric field (of magnitude $2\lambda/r$, in Gaussian units) outside the cylinder, the electric field is zero inside the cylinder.

The uniform, axial magnetic field inside the rotating cylinder of charge is (in Gaussian units),

$$B = \frac{4\pi}{c}\frac{\lambda\omega_0}{2\pi} = \frac{2\lambda\omega_0}{c}\,. \tag{1}$$

A conduction electron of charge $-e$ in the conducting wire at distance $a < r < b$ from the axis of the cylinder has azimuthal velocity $v = \omega r$, and experiences a radial magnetic force,

$$-\frac{evB}{c} = -\frac{2e\omega r\lambda\omega_0}{c^2}\,. \tag{2}$$

[1]This problem is a variant of the unipolar generator studied by Faraday, sec. 3091, *etc.*, of [1]. It was attributed to Stefan Marinov in [2].

If the wire is isolated, the conduction electrons move until a surface charge distribution generates a radial electric field inside the wire that opposes the magnetic force, such that the total force on an electron in the steady state is just the centripetal force,

$$F_r = -e\left(E_r + \frac{vB}{c}\right) = -m\omega^2 r, \tag{3}$$

where m is the mass of an electron. Hence, the steady-state, lab-frame, radial electric field inside an isolated conducting wire that rotates with angular velocity ω inside the uniform axial magnetic field (1) is,

$$E_r = -\frac{vB}{c} + \frac{m\omega^2 r}{e} \approx -\frac{vB}{c} = -\frac{2\lambda\omega\omega_0 r}{c^2}. \tag{4}$$

The voltage difference between the ends of the wire is,

$$\Delta V = V_b - V_a = -\int_a^b E_r\, dr = \frac{\lambda\omega\omega_0}{c^2}(b^2 - a^2). \tag{5}$$

This voltage difference is nonzero only if both angular velocities ω and ω_0 are nonzero in the lab frame.

In particular, the voltage difference is nonzero when $\omega = \omega_0$ and there is no motion of the wire relative to the cylinder, so long as both are moving in the lab frame.

The sign of the voltage difference depends on the signs of both ω and ω_0 (contrary to a statement in [2] that the sign depends on that of ω_0 but not on that of ω).

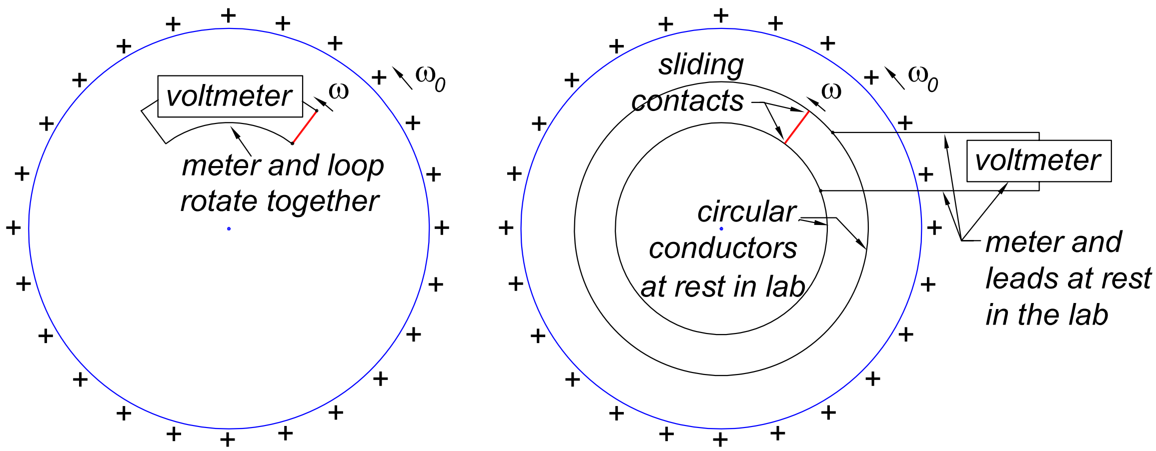
3 Comments

One might attempt to measure the voltage difference (5) with the apparatus sketched in the left figure on the next page, in which a voltmeter is attached to the radial wire, and the resulting loop + meter rotated together at angular velocity ω. Then, there would be no magnetic force on the conduction electrons in both circular segments of the loop, while in both radial segments the electrons experience a radially inward force. As such, no current would flow in the loop, and the voltmeter (which actually measures the current I in an internal resistor R and reports $V = IR$) would read zero.[2]

Instead, we could follow Faraday [1] (and Marinov [2]) and have the meter and its leads at rest in the lab frame, with the leads including two circular conductors (at rest in the lab) with which the moving, radial wire makes sliding contact, as shown in right figure on the next page. Now, the only magnetic force in the voltmeter + loop is in the moving, radial wire, such that this force acts as an $\mathcal{EMF}$ (electromotive force) of value (5), which is reported by the voltmeter.[3]

[2]This result is independent of the path of the voltmeter leads, so long as they rotate together with the conducting wire at angular velocity ω.

[3]If the angular velocity ω_0 were time dependent, the reading of the voltmeter would be rather counter-intuitive, as discussed in [3].



3.1 Relativity

This problem involves activity in both the lab frame and in the frame of the moving, radial wire, so the theory of relativity should be somehow relevant. However, it appears (as in [2]) that it is easy to misconstrue what the implications of relativity might be for this problem.

Besides a lab-frame observer, we consider an observer in the rotating frame of the (radial) wire.[4] The transformation between these observers is not a usual Lorentz transformation, as the frame of the wire is a rotating, rather than inertial, frame. For discussion by the author of electrodynamics in rotating frames, see [4].

The fields $\mathbf{E}'$ and $\mathbf{B}'$ according to a rotating observer are related to the lab-frame fields $\mathbf{E}$ and $\mathbf{B}$ as, eqs. (9)-(10) of [4],

$$\mathbf{E}' \approx \mathbf{E} + \frac{\mathbf{v}}{c} \times \mathbf{B}, \qquad \mathbf{B}' = \mathbf{B}, \tag{6}$$

where $\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r} = \omega\, \hat{\mathbf{z}} \times \mathbf{r}$ is the velocity of the observer (at distance $\mathbf{r}$ from the axis of rotation) with respect to the lab frame, and the approximation holds for $v \ll c$.

In the present example, in the absence of the conducting wire, but with a rotating observer, $\mathbf{E} = 0$, and $\mathbf{B} = (2\lambda\omega_0/c)\,\hat{\mathbf{z}}$ inside the charged cylinder, so in this region,

$$\mathbf{E}' = \frac{\omega}{c}(\hat{\mathbf{z}} \times \mathbf{r}) \times \frac{2\lambda\omega_0}{c}\,\hat{\mathbf{z}} = \frac{2\lambda\omega\omega_0 r}{c^2}\,\hat{\mathbf{r}}, \qquad \mathbf{B}' = \mathbf{B} = \frac{2\lambda\omega_0}{c}\,\hat{\mathbf{z}}, \tag{7}$$

even when $\omega = \omega_0$! The radial field $\mathbf{E}'$ has nonzero divergence, so the rotating observer tends to associated the "empty" space inside the charged cylinder with a nonzero (fictitious) electric charge density.[5]

[4] In sec. 14-4 of [5], which discusses a close variant of the present problem, it is stated: *But are you wondering: "What if I put myself in the frame of reference of the rotating cylinder? Then there is just a charged cylinder at rest, and I know that the electrostatic equations say there will be no electric fields inside, so there will be no force pushing charges to the center. So something must be wrong." But there is nothing wrong. There is no "relativity of rotation". A rotating system is not an inertial frame, and the laws of physics are different.*

[5] This disconcerting aspect of electrodynamics in a rotating frame makes analyses of electrical systems in such frames sparse in the literature. See, for example, [6].

When the rotating, conducting wire is present but isolated, the lab-frame electric field (4) transforms to zero electric field inside the wire, according to a rotating observer. Hence, if that rotating observer attempted to measure the voltage difference between the ends of the wire with apparatus at rest in the rotating frame, the result would be zero, as noted at the beginning of sec. 3.

In contrast, if a voltmeter is attached to the two ends of the moving, radial wire in the manner described at the bottom of p. 2 (right figure on p. 3), with the meter and leads at rest in the lab frame, the measured voltage difference would be given by eq. (5), as in this case the quantity ΔV represents a kind of "battery" (electromotive force) that can drive a (tiny) current $\Delta V/R$ in the voltmeter circuit, where R is the (large) electrical resistance of the voltmeter.

References

[1] M. Faraday, *On Lines of Magnetic Force*, Phil. Trans. Roy. Soc. London **142**, 25 (1852), http://kirkmcd.princeton.edu/examples/EM/faraday_ptrsl_142_25_52.pdf

[2] J. Maddox, *Stefan Marinov's seasonal puzzle*, Nature **346**, 103 (1990), http://kirkmcd.princeton.edu/examples/EM/maddox_nature_346_103_90.pdf

[3] K.T. McDonald, *Lewin's Circuit Paradox* (May 7, 2010), http://kirkmcd.princeton.edu/examples/lewin.pdf

[4] K.T. McDonald, *Electrodynamics of Rotating Systems* (Aug. 6, 2008), http://kirkmcd.princeton.edu/examples/rotatingEM.pdf

[5] R.P. Feynman, R.B. Leighton and M. Sands, *The Feynman Lectures on Physics*, Vol. II (Addison-Wesley, 1963), http://www.feynmanlectures.caltech.edu/II_14.html

[6] L.I. Schiff, *A Question in General Relativity*, Proc. Nat. Acad. Sci. **25**, 391 (1939), http://kirkmcd.princeton.edu/examples/EM/schiff_proc_nat_acad_sci_25_391_39.pdf