Maximum Power from DC Current and Voltage Sources

Kirk T. McDonald
Joseph Henry Laboratories, Princeton University, Princeton, NJ 08544
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1 Problem

In circuit analysis one often considers idealized current and voltage sources that can deliver arbitrarily large power into a resistive load at current I_0 and voltage V_0, respectively. More realistic sources have limitations to the power they can deliver. Discuss the maximum power that can be delivered by

1. A voltage source V_0 with an internal series resistance R_0.
2. A current source I_0 with a maximum output voltage V_0, at which voltage some of the current is shunted past the load (as if the current source has a diode in parallel that conducts only for $V > V_0$).[1]

2 Solution

2.1 Voltage Source with Internal Series Resistance

In case of a load of resistance R, the total resistance presented to the voltage source is $R+R_0$, so the current which flows is,

$$I = \frac{V_0}{R + R_0}\,. \tag{1}$$

The total power delivered by the voltage source into the load resistances is,

$$P = V_0 I = \frac{V_0^2}{R + R_0}\,. \tag{2}$$

The maximum power delivered by the source is for the case that $R = 0$,

$$P_{\rm max} = \frac{V_0^2}{R_0}\,. \tag{3}$$

However, none of this power is delivered to an "external" load, so this case is of little practical interest.

The power delivered to the load is,

$$P_{\rm load} = I^2 R = \frac{V_0^2 R}{(R + R_0)^2}\,, \tag{4}$$

[1]This is a simplified model of a silicon-solar-cell power source, which is not well represented by a Norton equivalent circuit with ideal current source and parallel resistance.

which is maximal for $R = R_0$,

$$P_{\text{load,max}} = \frac{V_0^2}{4R_0} = \frac{P_{\max}}{4} \qquad (R = R_0). \tag{5}$$

In this case, the power consumed by the internal resistance R_0 is $P_{\text{internal}} = V_0^2/4R_0$ also, such that the total power delivered by the source is,

$$P_{\text{total}} = \frac{V_0^2}{2R_0} = 2P_{\text{load,max}} = \frac{P_{\max}}{2} \qquad (R = R_0). \tag{6}$$

The results (5)-(6) are called the maximum power transfer theorem. Setting the load resistance equal to the internal resistance is often called impedance matching (even in the DC case as here).

2.2 Current Source with Maximum Voltage

If a current source of strength I_0 has a maximum output voltage V_0, it is useful to define a characteristic impedance R_0 (although this does not correspond to a physical resistance),

$$R_0 \equiv \frac{V_0}{I_0}. \tag{7}$$

If the load resistance R is less than R_0, then the voltage drop across the load is,

$$V_{\text{load}} = I_0 R < V_0 \qquad (R < R_0), \tag{8}$$

and the power delivered by the current source to the load is,

$$P_{\text{load}} = V_{\text{load}} I_0 = I_0^2 R < V_0 I_0 \qquad (R < R_0). \tag{9}$$

On the other hand, if $R > R_0$, current $I_0 - I$ is shunted past the load (and ideally delivers no power), such that the voltage drop across the load is,

$$V_{\text{load}} = IR = V_0 \qquad (R > R_0). \tag{10}$$

The power delivered to the load (= total power delivered) is

$$P_{\text{load}} = I^2 R = V_0 I = V_0 I_0 \frac{R}{R_0} < V_0 I_0 \qquad (R > R_0). \tag{11}$$

The two cases (9) and (11) indicate that the maximum power that can be delivered to the load by the current source occurs when $R = R_0$ (which can be regarded as a kind of impedance matching),

$$P_{\max} = V_0 I_0 = I_0^2 R_0 = \frac{V_0^2}{R_0} \qquad (R = R_0). \tag{12}$$

For $R > R_0 = V_0/I_0$ the current source of strength I_0 considered here behaves like an ideal voltage source of strength V_0. However, for $R < R_0$ where $V_{\text{load}} = I_0 R < V_0$, the current source behaves differently than a voltage source.

A Appendix: Maximum Power with a DC-DC Converter

It is sometimes convenient to alter the effective output current and voltage of DC power sources by a DC-DC converter, which is a kind of transformer (with internal time-dependent components) that ideally relates its input and output parameters according to,

$$V_{\rm out} = nV_{\rm in}, \qquad I_{\rm out} = \frac{I_{\rm in}}{n}, \qquad P_{\rm out} = V_{\rm out}I_{\rm out} = V_{\rm in}I_{\rm in} = P_{\rm in}, \tag{13}$$

where n is some positive real number. Here, we verify that use of an ideal DC-DC converter does not change the maximum power that can be delivered by the power sources considered above, although the value of the load resistor for maximum power changes from R_0 to n^2R_0.

A.1 Voltage Source with DC-DC Converter

The power delivered by the DC-DC converter to the load resistor R is,

$$P_{\rm out} = \frac{V_{\rm out}^2}{R} = \frac{n^2V_{\rm in}^2}{R} = P_{\rm in} = V_{\rm in}I_{\rm in}, \tag{14}$$

such that

$$n^2V_{\rm in} = I_{\rm in}R. \tag{15}$$

The input current to the DC-DC converter, when feed by a voltage source V_0 with a series resistance R_0 is,

$$V_{\rm in} = V_0 - I_{\rm in}R_0, \tag{16}$$

so eq. (15) tells us that,

$$I_{\rm in} = \frac{n^2V_0}{R + n^2R_0}, \qquad V_{\rm in} = \frac{V_0R}{R + n^2R_0}. \tag{17}$$

The power into the load resistor R connected to the output of the DC-DC converter is then,

$$P_{\rm out} = \frac{n^2V_{\rm in}^2}{R} = \frac{n^2V_0^2R}{(R + n^2R_0)^2}, \qquad P_{\rm out,max} = \frac{V_0^2}{4R_0}, \qquad \text{when} \qquad R = n^2R_0. \tag{18}$$

Hence, the maximum power into the load and is the same as that found in eq. (5) with no DC-DC converter.

A.2 Solar-Cell Current Source with DC-DC Converter

The solar cell has output current I_0 with maximum voltage V_0. As before, we define the characteristic resistance $R_0 = V_0/I_0$ (although this is not a physical resistor).

If the load R on the DC-DC converter is such that $V_{\rm in} < V_0$, we suppose that the input voltage $V_{\rm in}$ is established by a physical resistor R_1 in parallel with the DC-DC convertor (on the input side). For maximum power into the load, this resistor should not consume much power, so we anticipate that R_1 is large.

The current through resistor R_1 is,

$$I_1 = \frac{V_{\text{in}}}{R_1}, \tag{19}$$

such that the input current to the DC-DC converter is,

$$I_{\text{in}} = I_0 - I_1 = I_0 - \frac{V_{\text{in}}}{R_1}, \qquad V_{\text{in}} = (I_0 - I_{\text{in}})R_1. \tag{20}$$

The power into the load resistor R connected to the output of the DC-DC convertor is again related by eqs. (14)-(15), so that,

$$n^2 V_{\text{in}} = n^2(I_0 - I_{\text{in}})R_1 = I_{\text{in}}, \qquad I_{\text{in}} = \frac{n^2 I_0 R_1}{R + n^2 R_1}, \qquad V_{\text{in}} = \frac{I_0 R R_1}{R + n^2 R_1}. \tag{21}$$

The power into the load resistor R connected to the output of the DC-DC converter is then,

$$P_{\text{out}} = \frac{n^2 V_{\text{in}}^2}{R} = \frac{n^2 I_0^2 R R_1^2}{(R + n^2 R_1)^2}. \tag{22}$$

In this expression, both R and R_1 are unknown. For a given value of R, the output power (22) is maximal for very large R_1, as anticipated above. Then,

$$P_{\text{out}} \to \frac{I_0^2 R}{n^2}, \qquad \text{and} \qquad V_{\text{in}} \to \frac{I_0 R}{n^2} \qquad (R_1 \to \infty). \tag{23}$$

The power P_{out} increases with the value of R, which also increases V_{in}, until $V_{\text{in}} = V_0$ when $R = n^2 V_0 / I_0 = n^2 R_0$, and,

$$P_{\text{out}} = I_0^2 R_0, \qquad (R = n^2 R_0). \tag{24}$$

If R is further increased, the input and output voltages of the DC-DC converter do not change, so the power into the load decreases. Hence, the maximum power into the load is given by eq. (24), and is the same as that found in eq. (12) with no DC-DC converter.

This result was obtained assuming that a very large resistance R_1 is in parallel with the solar cell, although the precise value of this resistor is unimportant. Equivalent circuits of solar cells are typically drawn with such a resistor present, which permits this circuit to be ascribed an open-circuit voltage (V_0), as desirable in many types of circuit analysis.