Maxwell's Objection to Lorenz' Retarded Potentials

Kirk T. McDonald
Joseph Henry Laboratories, Princeton University, Princeton, NJ 08544
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1 Problem

Maxwell seems to have considered the great paper of L. Lorenz on retarded potentials [1] (published simultaneously in 1867 with a paper written in 1858 by B. Riemann on the same theme [2]) as insufficiently supportive of his vision of a dynamical theory of the electromagnetic field [3],[1] whereas the present attitude is that Riemann and Lorenz made important contributions to the Maxwellian view.[2]

In 1868, Maxwell made an objection, p. 653 of [6], that if a pair of equal and opposite charges move collinearly, then the retarded potential experienced by the charge in front has smaller magnitude than that experienced by the charge in the rear because the former retarded distance is larger than the latter; hence, there must be a net electrical force on the system, which accelerates it without limit, providing an infinite source of free energy.[3,4,5]

Is Maxwell's objection valid?[6]

2 Solution

Maxwell's objection was based on a misunderstanding of a (now) famous subtle issue about the use of the retarded potentials for moving charges. This issue is avoided by use of the

[1]In eq. (68) of [3], Maxwell nearly discovered the Lorenz gauge, which reads $kJ + 4\pi\mu\, d\Psi/dt = 0$ in the notation there. Instead, he argued after eq. (79) that $J\ (= \boldsymbol{\nabla}\cdot\mathbf{A})$ is either zero or constant for wave propagation. He was not bothered by the implication of eq. (79) that in this case the scalar potential φ "propagates" instantaneously, perhaps because of the great success that his assumptions about the potentials lead to propagation of the electric and magnetic fields at lightspeed $\sqrt{k/4\pi\mu}$.

As recounted in [4], Maxwell briefly thought in 1864 that transverse electromagnetic waves required $J = \boldsymbol{\nabla}\cdot\mathbf{A} = 0$, but he quickly realized that this condition is merely a choice.

[2]As will be reviewed in the Appendix, Lorenz identified light with oscillating currents, rather than with oscillating electric and magnetic fields. This identification was inspired by Lorenz' earlier work [5] on scalar sound waves in elastic solids, where he had used retarded "potentials" (*i.e.*, scalar fields). Hence, Lorenz' view of light was as a mechanical phenomenon, and not a dynamical aspect of the electromagnetic field as was Maxwell's vision. Lorenz must have been aware that his equations for (transversely) oscillating currents, whose wave velocity equals the speed of light, implied that the electromagnetic fields also undergo transverse oscillations that propagate at the speed of light, but he did not mention this.

[3]Maxwell's comment is reminiscent of a famous thought experiment of Galileo on why the acceleration of gravity must be independent of mass [7].

[4]Maxwell did not actually argue that the concept of retarded potentials was wrong, but he seems to have distrusted it, as he appears never to have referred to it again. Maxwell mentions Lorenz' paper [1] in sec. 805 of [8], but does not mention Lorenz' potentials there. The potentials used by Maxwell were, I believe, always in the Coulomb gauge, as in sec. 617 of [8]. Maxwell appears not to have commented/objected that Lorenz' potentials were different from the Coulomb-gauge potentials.

[5]This topic is discussed briefly on p. 671 of [9].

[6]Maxwell's views were likely reinforced by comments of Clausius [10] that Riemann's main application of his concept of a retarded scalar potential to electromagnetism was ill considered. See Appendix B.

relativistic transformation of the electromagnetic potentials, (see, for example, [11]), which is implicit in Maxwell's electrodynamics, but was not recognized as such until the efforts of Lorentz [12] and Einstein [13] in 1904-5. The electric field of a point charge along the direction of its velocity is the same in the rest frame of the charge and in a frame where the charge has velocity $\mathbf{v}$, contrary to Maxwell's overly hasty conclusion. That is, the total electrical force is zero on a pair of opposite charges that move with the same, constant velocity (no matter what the angle between the velocity vector and the line of centers of the charges).[7]

Returning to use of the retarded potentials, consider a localized charge density ρ that is in motion with velocity $\mathbf{v}$ where v is less than the speed of light c in the surrounding medium. Then the associated current density can be written as $\mathbf{J} = \rho\mathbf{v}$, and the retarded potentials are (in Gaussian units),

$$V(\mathbf{r}, t) = \int \frac{\rho(\mathbf{r}', t' = t - R/c)}{R}\, d\mathrm{Vol}', \qquad \mathbf{A}(\mathbf{r}, t) = \int \frac{\mathbf{J}(\mathbf{r}', t' = t - R/c)}{cR}\, d\mathrm{Vol}' = V\frac{\mathbf{v}}{c}, \quad (1)$$

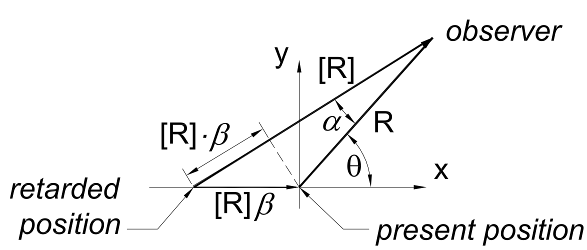
where R is the magnitude of the vector $\mathbf{R} = \mathbf{r} - \mathbf{r}'$. It is tempting to suppose that this can be simplified to read,

$$V(\mathbf{r}, t) \stackrel{?}{=} \frac{1}{[R]} \int \rho(\mathbf{r}')\, d\mathrm{Vol}' = \frac{q}{[R]}, \qquad \mathbf{A}(\mathbf{r}, t) = V\frac{\mathbf{v}}{c} = V\boldsymbol{\beta}, \quad (2)$$

where $q = \int \rho\, d\mathrm{Vol}$ is the total charge of the small object, and the retarded distance vector $[\mathbf{R}]$ is related to the present distance $\mathbf{R}$ as in the figure below,

$$[\mathbf{R}] = \mathbf{R} + [R]\boldsymbol{\beta}, \quad (3)$$

where $\boldsymbol{\beta} = \mathbf{v}/c = v\,\hat{\mathbf{x}}/c$.



Thus,

$$[R]^2 = R^2 + [R]^2\beta^2 + 2R[R]\beta\cos\theta, \quad (4)$$

and so the retarded distance $[R]$ is related to the present distance R by,

$$[R] = \gamma^2 R\left(\beta\cos\theta + \sqrt{1 - \beta^2\sin^2\theta}\right), \quad (5)$$

[7]Maxwell avoided discussion of two charges with noncollinear velocities, perhaps because of the ambiguity (first noted by Ampère in the 1820's) in extrapolating from the force law for pairs of current loops to that for pairs of moving charges. Extrapolation from the Biot-Savart force law leads to what is now called the Lorentz force law, which has the consequence that the total force is nonzero on a pair of charges which move with noncollinear velocities. Maxwell had only a partial understanding that this behavior is compatible with his theory in that the electromagnetic fields of carry momentum $\mathbf{P}_{\mathrm{EM}}$ such that $\mathbf{F}_{\mathrm{total}} = d\mathbf{P}_{\mathrm{mechanical}}/dt = -d\mathbf{P}_{\mathrm{EM}}/dt$, and the total momentum of the system is constant. See, for example, [14, 15].

where θ is the angle between vectors $\mathbf{v}$ and $\mathbf{R}$, and $\gamma = 1/\sqrt{1-\beta^2}$.

If eq. (2) were valid, then the electric field,

$$\mathbf{E} = -\boldsymbol{\nabla} V - \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t}\,, \tag{6}$$

of the moving charge would indeed be fore/aft asymmetric as implied by Maxwell.

However, because the charge is moving the retarded integrals of eq. (1) actually imply that,

$$V(\mathbf{r},t) = \frac{q}{[R] - [\mathbf{R}\cdot\boldsymbol{\beta}]}\,, \qquad \mathbf{A}(\mathbf{r},t) = V\frac{[\mathbf{v}]}{c} = V[\boldsymbol{\beta}], \tag{7}$$

where $[\mathbf{v}]$ is the velocity at the retarded time, as first deduced by Liénard [16] (1898) and Wiechert [17] (1900). Their argument is subtle (see, for example, sec. 10.3 of [18]; the first English exposition of this may have been on pp. 254-255 of [19]).

When the velocity $\mathbf{v}$ is constant in time, we have (referring to the figure above),

$$[R] - [\mathbf{R}\cdot\boldsymbol{\beta}] = R\cos\alpha = R\sqrt{1-\beta^2\sin^2\theta}, \tag{8}$$

noting that,

$$\frac{\sin\alpha}{[R]\beta} = \frac{\sin(\pi-\theta)}{[R]}\,. \tag{9}$$

Thus,

$$V(\mathbf{r},t) = \frac{q}{R\sqrt{1-\beta^2\sin^2\theta}} = \frac{q}{\sqrt{x^2+(1-\beta^2)y^2}}\,, \qquad \mathbf{A}(\mathbf{r},t) = V\boldsymbol{\beta} = V\beta\,\hat{\mathbf{x}}, \tag{10}$$

at the moment when the moving charge is at the origin with velocity $\mathbf{v} = v\,\hat{\mathbf{x}}$, and the observer is at $(x, y, 0)$. Since the charge rather than the observer is moving, $dx/dt = -v$, and the components of the electric field are,

$$\begin{aligned}
E_x &= -\frac{\partial V}{\partial x} - \frac{1}{c}\frac{\partial A_x}{\partial t} = -\frac{\partial V}{\partial x} - \frac{1}{c}\frac{\partial A_x}{\partial x}\frac{dx}{dt} = -(1-\beta^2)\frac{\partial V}{\partial x} = \frac{qx}{\gamma^2 R^3(1-\beta^2\sin^2\theta)^{3/2}}\,, \\
E_y &= \frac{qy}{\gamma^2 R^3(1-\beta^2\sin^2\theta)^{3/2}}\,, \\
E_z &= 0,
\end{aligned} \tag{11}$$

so that the electric field is along the present vector $\mathbf{R}$ and is fore/aft symmetric,

$$\mathbf{E} = \frac{q\mathbf{R}}{\gamma^2 R^3(1-\beta^2\sin^2\theta)^{3/2}}\,. \tag{12}$$

The magnetic field at the observer has only a z-component,

$$B_z = -\frac{\partial A_x}{\partial y} = -\beta\frac{\partial V}{\partial y} = -\beta E_y = -\left|\boldsymbol{\beta}\times\mathbf{E}\right|, \qquad \text{and so} \qquad \mathbf{B} = \boldsymbol{\beta}\times\mathbf{E}. \tag{13}$$

The results (12)-(13) were first derived for $v \ll c$ by J.J. Thomson [20, 21] (1881), two years after Maxwell's death, and by Heaviside [22] (1889) for any $v < c$, which showed that the electric field at a given distance from the charge is fore/aft symmetric, although not isotropic.[8] Neither of them used the retarded potentials in their calculations.

An early application of the retarded potentials in electrodynamics was made in 1883 by FitzGerald [25, 26], who apparently reinvented them in a derivation of the radiation from an oscillating current loop (magnetic dipole).

A Appendix: Lorenz' Paper of 1867

This Appendix comments on Lorenz' paper [1], transcribing key results into Gaussian units.[9]

Lorenz used the symbol a to represent the speed of light in "air" (*i.e.*, vacuum), and also used the symbol c for $\sqrt{2}a$ following Weber [30, 31, 32]. Here, we define c to be the speed of light in vacuum, and C to be Weber's velocity, such that in our notation $C = \sqrt{2}c$.

Lorenz built on (and used the same notation as) two important papers of Kirchhoff (1857) [33, 34] that associated electrical effects in wires with the speed of light.[10,11] Lorenz did not pursue his theory of light after 1867, as recounted in [29].

A.1 Notation

As the first step in transcribing Lorenz' notation, we consider his eq. (2) which expresses charge conservation ("equation of continuity"). From this we infer that,

$$\epsilon \to \rho, \tag{14}$$

$$2(u, v, w) \to \mathbf{J}, \tag{15}$$

where ρ and $\mathbf{J}$ are the densities of electric charge and current.[12] Then, Lorenz' eq. (2) becomes,

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial e}{\partial z} = -\frac{1}{2}\frac{\partial \epsilon}{\partial t} \qquad \to \qquad \boldsymbol{\nabla} \cdot \mathbf{J} = -\frac{\partial \rho}{\partial t}\, . \tag{16}$$

Lorenz' eq. (1) reproduces Kirchhoff's eqs. (1)-(3) of [34], which expresses Ohm's law in the form,

$$(u, v, w) = -2k \left(\frac{\partial \Omega}{\partial(x, y, z)} + \frac{4}{C^2}\frac{\partial(U, V, W)}{\partial t} \right) \qquad \to \qquad \mathbf{J} = \sigma \mathbf{E} = -\sigma \left(\boldsymbol{\nabla} V + \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t} \right), \tag{17}$$

[8] Only if the speed of the charge exceeds that of light in the surrounding medium is the field pattern asymmetric, as first deduced by Heaviside [23], but now more familiar as the Čerenkov effect [24]. If two charges moved with parallel velocities, both greater than the speed of light in a dielectric medium, and the trailing particle was on the Čerenkov cone of the leader, then the trailing particle would be accelerated by the field of the leader. To avoid Maxwell's paradox in this case, we note that the effect of the acceleration is to move the trailing particle off the Čerenkov cone of the leader, which changes the cross term in the electromagnetic energy of the two particles such that total energy is conserved.

[9] A popular article [27] by Lorenz on his theory of light is now available in translation [28].

[10] Faraday (1845) [36] had provided the first experimental evidence for a relation between electromagnetism and light.

[11] The theories of Weber and Kirchhoff were based on action at a distance, to which Maxwell was not sympathetic.

[12] The factor of 2 in eq. (15) arose from the view of the Neumann, Weber and others that electric current consists of two counterpropagating flows.

with the identifications,[13]

$$
\begin{aligned}
4k &\to \sigma, && (18)\\
\Omega &\to V, && (19)\\
\frac{C}{\sqrt{2}}\ (= a) &\to c, && (20)\\
\frac{2\sqrt{2}(\alpha,\beta,\gamma)}{C}\ \left(= \frac{2(\alpha,\beta,\gamma)}{a}\right) &\to \mathbf{A}, && (21)
\end{aligned}
$$

for the electrical conductivity σ, the scalar potential V, the speed of light c and the vector potential $\mathbf{A}$.

Kirchhoff's eqs. (1)-(3) of [34], and their precursor on p. 199 of [33] may be the first appearance in the literature of the relation between the electric field $\mathbf{E}$ *and both potentials* V *and* $\mathbf{A}$*,*

$$
\mathbf{E} = -\boldsymbol{\nabla} V - \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t}\,. \qquad (22)
$$

Kirchhoff attributes this insight to Weber, who discussed the force on an "element of a conductor at rest" (i.e., an electric charge) due to a circuit with a time-dependent current in sec. 30 of [30] (p. 133 of the English translation). We recognize Weber as deducing that this force is proportional to the time derivative of the vector potential of the circuit, such that the force per unit charge (i.e., the electric field) is given by the second term of eq. (22).

Also noteworthy in Kirchhoff's paper [33] is the appearance just before eq. (2), p. 533, of the relation,

$$
\frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} + \frac{\partial W}{\partial z} = \frac{1}{2}\frac{\partial \Omega}{\partial t} \qquad \to \qquad \boldsymbol{\nabla}\cdot\mathbf{A} = \frac{1}{c}\frac{\partial V}{\partial t} \qquad \text{(Kirchhoff)}. \qquad (23)
$$

This seems to be the first statement of what is now called a gauge-condition [9] (defining the little-used Kirchhoff gauge, that differs from the Lorenz-gauge condition by a sign change).[14]

A.2 Wave Equations and Their Solutions

We turn now to the physics argument of Lorenz' paper.

Kirchhoff [33] found a second-order differential equation for the behavior of electricity in wires (but gave only two first-order differential equations in [34]). He showed that these were consistent with damped waves whose speed of propagation was $C/\sqrt{2} = c$.[15]

Lorentz had success in 1861 [5] in applying the method of retarded potentials to solving the second-order differential wave equation for an elastic solid, so he sought to modify Kirchhoff's equations (1)-(3), our eq. (17), into a wave equation for the current density $\mathbf{J}$,

[13]In the early part of his paper, Lorenz used Kirchhoff's notation U, V, W for components of the vector potential, and then switched to α, β, γ in his eq. (A).

[14]See, for example, sec. 2.3.5 of [35].

[15]Kirchhoff's analysis was not in the context of a field theory, so for him the waves existed only inside the wires. Kirchhoff made no connection between his waves and light.

such that retarded quantities would be the solution. He summarized the method of retarded "potentials" in eq. (5) of [1], that for a scalar field ψ (the "potential") with source function ϕ, which obeys the wave equation,

$$\nabla^2\psi - \frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2} = -4\pi\phi, \tag{24}$$

a solution is,

$$\psi(\mathbf{r},t) = \int \frac{\phi(\mathbf{r}', t' = t - R/c)}{R}\, d\mathrm{Vol}', \tag{25}$$

where $R = |\mathbf{r} - \mathbf{r}'|$. By assuming that the vector potential obeyed wave equations of the form (24), with source function $\mathbf{J}/c$, Lorenz found that Ohm's law (17) was indeed compatible with a second-order wave equation for the current density $\mathbf{J}$, provided that scalar and vector potentials were related by (shortly before eq. (8) of [1]),

$$\frac{\partial\Omega}{\partial t} = -2\left(\frac{\partial\alpha}{\partial x} + \frac{\partial\beta}{\partial y} + \frac{\partial\gamma}{\partial z}\right) \quad \to \quad \frac{1}{c}\frac{\partial V}{\partial t} = -\boldsymbol{\nabla}\cdot\mathbf{A} \qquad \text{(Lorenz)}. \tag{26}$$

This is now called the Lorenz-gauge condition, and was a key to his success, where Kirchhoff's gauge condition (23) was less felicitous.[16]

Lorenz then had two vector wave equations (given on pp. 293-294 of the English version of [1]),

$$\nabla^2\mathbf{A} - \frac{1}{c^2}\frac{\partial^2\mathbf{A}}{\partial t^2} = -\frac{4\pi}{c}\mathbf{J}, \qquad \nabla^2\mathbf{J} - \frac{1}{c^2}\frac{\partial^2\mathbf{J}}{\partial t^2} = 4\pi\sigma\left(\boldsymbol{\nabla}\rho + \frac{1}{c}\frac{\partial\mathbf{J}}{\partial t}\right). \tag{27}$$

Lorenz did not then recall Ohm's law, $\mathbf{J} = \sigma\mathbf{E}$, to write a third wave equation for the electric field $\mathbf{E}$ (which also follows from Maxwell's equations),[17]

$$\nabla^2\mathbf{E} - \frac{1}{c^2}\frac{\partial^2\mathbf{E}}{\partial t^2} = 4\pi\left(\boldsymbol{\nabla}\rho + \frac{1}{c}\frac{\partial\mathbf{J}}{\partial t}\right). \tag{28}$$

Had he done so, and identified light with the (transverse) vibrations of $\mathbf{E}$ rather than of $\mathbf{J}$ (which vibrations have the same character in view of Ohm's law), Lorenz' argument might have been better received by Maxwell and his contemporaries in England.[18]

Instead, Lorenz interpreted his results "mechanically", that all space was electrically conductive and that light was oscillations of electric charge in this medium, which, however, was not necessarily the elastic-solid æther often considered by others at the time. This model was sketched in the penultimate paragraph of [1], and bears some resemblance to a plasma below the plasma frequency. Lorenz ended his paper with the remarks:

[16]Lorenz' view was that the condition (26) follows from the assumption that the vector potential can be found via the method of retarded potentials. A Maxwellian view is that the wave equations for the potentials that follow from Maxwell's equations take the form that Lorenz supposed if the Lorenz-gauge condition is first assumed.

[17]Equation (28) does not follow from eq. (27) if the conductivity varies, which may have been the cause of Lorenz' hesitation. A Maxwellian view is that eq. (28) is the more fundamental, and that the wave equation (27) for $\mathbf{J}$ holds only if σ is constant.

[18]Lorenz did not explicitly display the retarded solution to the wave equation for $\mathbf{J}$ (or $\mathbf{E}$). Had he done so, he would have obtained what are often called Jefimenko's equations. See the Appendix to [37].

> This hypothesis as to the nature of light and of electrical currents will probably, as science progresses, either assume a new form, or be totally rejected. But the result of the present investigation, that vibrations of light are electrical currents, has been obtained without the assumption of a physical hypothesis, and will therefore be independent of one.

However, most people today are unaware that Lorenz advocated that the "vacuum" is electrically conductive and that light is the vibration of electrical charges therein; rather, he is remembered more for the first application of the method of retarded potentials to electrodynamics, and for the corresponding Lorenz-gauge condition.[19]

B Appendix: Riemann and Clausius

In Art. 862 of his *Treatise* [8], Maxwell mentioned the paper of Riemann [2], but not that of Lorenz [1],[20] and cited a paper [10] of Clausius, who argued that Riemann's retarded scalar potential *does not lead...to the known laws of electrodynamics* (Maxwell's words).

While Riemann introduced the concept of retarded time and distance in [2], he did not use different symbols for the present and retarded distance, which confusion was noted by Clausius in [10].

The second to last equation on p. 370 of the English version of [2], gives an expression for electric scalar potential in the Lorenz gauge due to charge ϵ at the location of charge ϵ', multiplied by ϵ' to become a (potential) energy. This expression is valid from a contemporary perspective.

However, Riemann wanted to justify that the velocity α of propagation of the retarded potential equaled that of light. For this, the rest of his paper considered a peculiar quantity, the time integral of his "potential", and at the end of the paper he declared that his analysis agreed with some unspecified experiment if indeed α was the speed of light.

Clausius objected to this argument as containing mathematical errors, while we might object to it as not physically meaningful.

It remains unfortunate that the earliest discussion of retarded potentials (by Riemann) included unsound arguments, which no doubt slowed the acceptance of this concept.[21]

C Appendix: Helmholtz

In 1870, Helmholtz published a paper [42] that discussed the electromagnetic potentials in a manner (not easy to follow) that is largely equivalent to what is now called the velocity gauge (see, for example, sec. VII of [43] and sec. 2.3.1 of [35], wherein the scalar potential

[19] As H.A. Lorentz was one of the "popularizers" of the method of retarded potentials [19, 38], the Lorenz condition is often called the Lorentz condition.

[20] Lorenz' paper was mentioned in an addendum to Art. 805 of Maxwell's *Treatise*, without reference to the retarded potentials.

[21] As Clausius noted in [10], Riemann withdrew his paper from publication in 1857, perhaps recognizing that it contained some doubtful analysis.

Clausius claimed that Riemann was inspired by comments of Gauss to Weber in 1845, and noted that these comments also led to papers by Neumann [39] and by Betti [40].

For additional commentary on these early considerations of the speed of propagation of electromagnetic effects, see secs.8.1-2 of the English version of [41].

propagates with an arbitrary velocity v, while the vector potential has terms that propagate at both v and c.[22] Maxwell cited this paper in Art. 854 of [8], without mention of the potentials.

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