Maxwell and Special Relativity

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It is now commonly considered that Maxwell's equations [28] in vacuum implicitly contain the special theory of relativity.[1]

For example, these equations imply that the speed c of light in vacuum is related by,[2]

$$c = \frac{1}{\sqrt{\epsilon_0 \mu_0}}\,, \tag{1}$$

where the constants ϵ_0 and μ_0 can be determined in any (inertial) frame via electrostatic and magnetostatic experiments (nominally in vacuum).[3,4,5] Even in æther theories, the velocity of the laboratory with respect to the hypothetical æther should not affect the results of these static experiments,[6] so the speed of light should be the same in any (inertial) frame. Then, the theory of special relativity, as developed in [69], follows from this remarkable fact.

Maxwell does not appear to have crisply drawn the above conclusion, that the speed of light is independent of the velocity of the observer, but he did make arguments in Arts. 599-600 and 770 of [55] that correspond to the low-velocity approximation to special relativity, as pointed out in sec. 5 of [86].[7]

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[1]Maxwell's electrodynamics was the acknowledged inspiration to Einstein in his 1905 paper [69].

[2]Equation (1) is a transcription into SI units of the discussion in Art. 80 of and Art. 758 of [35].

[3]This was first noted by Weber and Kohlsrausch (1856) [19, 20], as recounted by Maxwell on p. 21 of [26], sec. 96 of [28], p. 644 of [31], and Arts. 786-787 of [55].

[4]It is now sometimes said that electricity plus special relativity implies magnetism, but a more historical view is that (static) electricity plus magnetism implies special relativity. This theme is emphasized in, for example, [95].

[5]As reviewed in [99], examples of a "static" current-carrying wire involve effects of order v^2/c^2 where v is the speed of the moving charges of the current. A consistent view of this in the rest frame of the moving charges requires special relativity. These arguments could have been made as early as 1820, but it took 85 years for them to be fully developed.

[6]This *ansatz* is a weak form of Einstein's Principle of Relativity.

[7]These two arguments also correspond to use of the two types of Galilean electrodynamics [83], as noted in [87, 103, 105]. The notion of Galilean electrodynamics, consistent with Galilean relativity, *i.e.*, the coordinate transformation $x' = x - vt$, $y' = y$, $z' = z$, $t' = t$, seems to have been developed only in 1973 [83]. *The term Galilean relativity was first used in 1911 [72].* In Galilean electrodynamics there are no electromagnetic waves, but only quasistatic phenomena, so this notion is hardly compatible with Maxwellian electrodynamics as a whole. In contrast, electromagnetic waves can exist in the low-velocity approximation to special relativity, and, of course, propagate in vacuum with speed c.

In Galilean electrodynamics the symbol c does not represent the speed of light (as light does exist in this theory), but only the function $1/\sqrt{\epsilon_0\mu_0}$ of the (static) permittivity and permeability of the vacuum.

In fact, there are two variants of Galilean electrodynamics:

1. Electric Galilean relativity (for weak magnetic fields) in which the transformations between two inertial frames with relative velocity $\mathbf{v}$ are (sec. 2.2 of [83]), given here in Gaussian units, as will be used in the rest of this note,

$$\rho'_e = \rho_e, \qquad \mathbf{J}'_e = \mathbf{J}_e - \rho_e \mathbf{v}, \qquad (c\,|\rho_e| \gg |J_e|), \qquad V'_e = V_e, \qquad \mathbf{A}'_e = \, = \frac{\mathbf{v}}{c} V_e, \tag{2}$$

$$\mathbf{E}'_e = \mathbf{E}_e, \qquad \mathbf{B}'_e = \mathbf{B}_e - \frac{\mathbf{v}}{c} \times \mathbf{E}_e \qquad \mathbf{f}_e = \rho_e \mathbf{E}_e \qquad \text{(electric)}, \tag{3}$$

1 Articles 598-599 of Maxwell's *Treatise*

In his *Treatise* [55], Maxwell argued that an *element of a circuit* (Art. 598), or a *particle* (Art. 599) which moves with velocity $\mathbf{v}$ in electric and magnetic fields $\mathbf{E}$ and $\mathbf{B} = \mu\mathbf{H}$ experiences a (vector) *electromotive force* $\mathfrak{E}$ (Art. 598), given by eq. (B) of Art. 598 and eq. (10) of Art. 599,[8]

$$\mathfrak{E} = \mathfrak{V} \times \mathfrak{B} - \dot{\mathfrak{A}} - \boldsymbol{\nabla}\Psi, \tag{8}$$

where $\mathfrak{V}$ is the velocity $\mathbf{v}$, $\mathfrak{B}$ is the magnetic field $\mathbf{B}$, $\mathfrak{A}$ is the vector potential and Ψ *represents, according to a certain definition, the electric (*scalar*) potential.* If we interpret *electromotive force* to mean the force per charge q of the *particle*,[9] *i.e.*, $\mathfrak{E} = \mathbf{F}/q$, then we could write eq. (8) as,

$$\mathbf{F} = q\left(\mathbf{E} + \frac{\mathbf{v}}{c} \times \mathbf{B}\right), \tag{9}$$

noting that the electric field $\mathbf{E}$ is given (in emu) by $-\partial\mathbf{A}/\partial t - \boldsymbol{\nabla}\Psi$,[10,11,12] and that $\mathbf{v} \times \mathbf{B}$ in emu becomes $\mathbf{v}/c \times \mathbf{B}$ in Gaussian units.

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where ρ and $\mathbf{J}$ are the electric charge and current densities, V and $\mathbf{A}$ are the electromagnetic scalar and vector potentials, $\mathbf{E} = -\boldsymbol{\nabla}V - \partial\mathbf{A}/\partial ct$ is the electric field, $\mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A}$ is the magnetic (induction) field,.
2. Magnetic Galilean relativity (for weak electric fields, sec. 2.3 of [83]) with transformations,

$$\rho'_m = \rho_m - \frac{\mathbf{v}}{c^2} \cdot \mathbf{J}_m, \quad \mathbf{J}'_m = \mathbf{J}_m, \quad (c\,|\rho_e| \ll |J_e|), \quad V'_m = V_m - \frac{\mathbf{v}}{c} \cdot \mathbf{A}_m, \quad \mathbf{A}'_m = \mathbf{A}_m, \tag{4}$$

$$\mathbf{E}'_m = \mathbf{E}_m + \frac{\mathbf{v}}{c} \times \mathbf{B}_m, \qquad \mathbf{B}'_m = \mathbf{B}_m \qquad \mathbf{f}_m = \rho_m\left(\mathbf{E}_m + \frac{\mathbf{v}}{c} \times \mathbf{B}_m\right) \quad \text{(magnetic)}. \tag{5}$$

For comparison, the low-velocity limit of special relativity has the transformations,

$$\rho'_s \approx \rho_s - \frac{\mathbf{v}}{c^2} \cdot \mathbf{J}_s \qquad \mathbf{J}'_s \approx \mathbf{J}_s - \rho_s\mathbf{v}, \qquad V'_s \approx V_s - \frac{\mathbf{v}}{c} \cdot \mathbf{A}_s, \qquad \mathbf{A}'_s \approx \mathbf{A}_s - \frac{\mathbf{v}}{c}V_s, \tag{6}$$

$$\mathbf{E}'_s \approx \mathbf{E}_s + \frac{\mathbf{v}}{c} \times \mathbf{B}_s, \qquad \mathbf{B}'_s \approx \mathbf{B}_s - \frac{\mathbf{v}}{c} \times \mathbf{E}_s \qquad \text{(special relativity}, v \ll c). \tag{7}$$

[8] This result also appeared in eq. (77), p. 343, of [25] (1861), and in eq. (D), sec. 65, p. 485, of [28] (1864). The term *electromagentic force* was changed to *electromagnetic intensity* by J.J. Thomson in Arts. 598-601 of the 3rd edition of Maxwell's Treatise [55].

The evolution of Maxwell's thoughts on the "Lorentz" force are traced in Appendix A below. See also [85, 88, 91].

[9] In contrast to, for example, Weber [11], Maxwell did not present in his *Treatise* a view of an electric charge as a "particle", but rather as a state of "displaced" æther. However, in his earliest derivation of our eq. (8), his eq. (77), p. 342 of [25], Maxwell was inspired by his model of *molecular vortices* in which moving *particles* ("idler wheels") corresponded to an electric current (see also sec. A.2.7 below).

For comments on Maxwell's various views on electric charge, see [82].

[10] This assumes that Maxwell's $\dot{\mathfrak{A}}$ corresponds to $\partial\mathbf{A}/\partial t$, and not to the convective derivative $D\mathbf{A}/Dt = \partial\mathbf{A}/\partial t + (\mathbf{v} \cdot \boldsymbol{\nabla})\mathbf{A}$.

[11] Maxwell never used the term *electric field* as we now do, and instead spoke of the (vector) *electromotive force* or *intensity* (see Art. 44 of [54]). The distinction is important only when discussing a moving medium, as in Arts. 598-599.

[12] The relation $\mathbf{E} = -\partial\mathbf{A}/\partial t - \boldsymbol{\nabla}\Psi$ for the electric field holds in any gauge. However, Maxwell always worked in the Coulomb gauge, where $\boldsymbol{\nabla} \cdot \mathbf{A} = 0$, as affirmed, for example, in Art. 619. Maxwell was aware that, in the Coulomb gauge, the electric scalar potential Ψ is the instantaneous Coulomb potential, obeying Poisson's equation at any fixed time, as mentioned at the end of Art. 783. The discussion in Art. 783 is gauge invariant until the final comment about $\nabla^2\Psi$ (in the Coulomb gauge). That is, Maxwell missed an opportunity to discuss the gauge advocated by Lorenz [30], to which he was averse [97].

Our equation (9) is now known as the Lorentz force,[13,14] and it seems seldom noted that Maxwell gave this form, perhaps because he presented eq. (10) of Art. 599 as applying to an element of a circuit rather than to a charged particle. In Arts. 602-603, Maxwell discussed *the Electromotive Force acting on a Conductor which carries an Electric Current through a Magnetic field*, and clarified in his eq. (11), Art. 603 that the force on current density $\mathfrak{J}$ ($\mathbf{J}$) is,

$$\mathfrak{F} = \mathfrak{J} \times \mathfrak{B} \qquad \left(\mathbf{F} = \frac{\mathbf{J}}{c} \times \mathbf{B}\right). \tag{10}$$

If Maxwell had considered that a small volume of the current density is equivalent to an electric charge q times its velocity $\mathbf{v}$, then his eq. (11), Art. 602 could also have been written as,

$$\frac{\mathbf{F}}{q} = \frac{\mathbf{v}}{c} \times \mathbf{B} \qquad (= \mathfrak{V} \times \mathfrak{B}), \tag{11}$$

which would have confirmed the interpretation we have given to our eq. (9) as the Lorentz force law. However, Maxwell ended his Chap. VIII, Part IV of his *Treatise* with Art. 603, leaving ambiguous some of the meaning of that chapter.

In his Arts. 598-599, Maxwell considered a lab-frame view of a moving circuit. However, we can also interpret Maxwell's $\mathfrak{E}$ as the electric field $\mathbf{E}'$ in the frame of the moving circuit, such that Maxwell's transformation of the electric field is,[15]

$$\mathbf{E}' = \mathbf{E} + \frac{\mathbf{v}}{c} \times \mathbf{B}. \tag{12}$$

The transformation (12) is compatible with both magnetic Galilean relativity, eq. (5), and the low-velocity limit of special relativity, eq. (7). These two versions of relativity differ as to the transformation of the magnetic field. In particular, if $\mathbf{B} = 0$ while $\mathbf{E}$ were due to a single electric charge at rest (in the unprimed frame), then magnetic Galilean relativity predicts that the moving charge/observer would consider the magnetic field $\mathbf{B}'$ to be zero, whereas it is nonzero according to special relativity.

These themes were considered by Maxwell in Arts. 600-601, under the heading: *On the Modification of the Equations of Electromotive Force when the Axes to which they are referred are moving in Space*, which we review in sec. 2 below.

[13] Lorentz actually advocated the form $\mathbf{F} = q\,(\mathbf{D} + \mathbf{v} \times \mathbf{H})$ in eq. (V), p. 21, of [59], although he seems mainly to have considered its use in vacuum. See also eq. (23), p. 14, of [71]. That is, Lorentz considered $\mathbf{D}$ and $\mathbf{H}$, rather than $\mathbf{E}$ and $\mathbf{B}$, to be the microscopic electromagnetic fields.

[14] It is generally considered that Heaviside first gave the Lorentz force law (9) for electric charges in [51], but the key insight is already visible for the electric case in [46] and for the magnetic case in [48].

[15] A more direct use of Faraday's law, without invoking potentials, to deduce the electric field in the frame of a moving circuit was made in sec. 9-3, p. 160, of [79], which argument appeared earlier in sec. 86, p. 398, of [67]. An extension of this argument to deduce the full Lorentz transformation of the electromagnetic fields $\mathbf{E}$ and $\mathbf{B}$ is given in Appendix C below.

1.1 Details

In Art. 598, Maxwell started from the integral form of Faraday's law, that the (scalar) *electromotive force* $\mathcal{E}$ in a circuit is related to the rate of change of the magnetic flux through it by his eqs. (1)-(2),

$$\mathcal{E} = -\frac{1}{c}\frac{d\Phi_m}{dt} = -\frac{1}{c}\frac{d}{dt}\int \mathbf{B}\cdot d\mathbf{S} = -\frac{1}{c}\frac{d}{dt}\oint \mathbf{A}\cdot d\mathbf{l} = -\frac{1}{c}\oint\left(\frac{\partial \mathbf{A}}{\partial t} + (\mathbf{v}\cdot\boldsymbol{\nabla})\mathbf{A}\right)\cdot d\mathbf{l}, \qquad (13)$$

where the last form, involving the convective derivative, holds for a circuit that moves with velocity $\mathbf{v}$ with respect to the lab frame.[16] In his discussion leading to eq. (3) of Art. 598, Maxwell argued for the equivalent of use of the vector-calculus identity,

$$\boldsymbol{\nabla}(\mathbf{v}\cdot\mathbf{A}) = (\mathbf{v}\cdot\boldsymbol{\nabla})\mathbf{A} + (\mathbf{A}\cdot\boldsymbol{\nabla})\mathbf{v} + \mathbf{v}\times(\boldsymbol{\nabla}\times\mathbf{A}) + \mathbf{A}\times(\boldsymbol{\nabla}\times\mathbf{v}), \qquad (14)$$

which implies for the present case,

$$(\mathbf{v}\cdot\boldsymbol{\nabla})\mathbf{A} = -\mathbf{v}\times(\boldsymbol{\nabla}\times\mathbf{A}) + \boldsymbol{\nabla}(\mathbf{v}\cdot\mathbf{A}) = -\mathbf{v}\times\mathbf{B} + \boldsymbol{\nabla}(\mathbf{v}\cdot\mathbf{A}), \qquad (15)$$

$$\mathcal{E} = \oint\left(\frac{\mathbf{v}}{c}\times\mathbf{B} - \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t}\right)\cdot d\mathbf{l} = \oint \mathfrak{E}\cdot d\mathbf{l}, \qquad (16)$$

since $\oint \boldsymbol{\nabla}(\mathbf{v}\cdot\mathbf{A})\cdot d\mathbf{l} = 0$. Our eq. (16) corresponds to Maxwell's eqs. (4)-(5), from which we infer that the vector *electromotive force* $\mathfrak{E}$ has the form,

$$\mathfrak{E} = \frac{\mathbf{v}}{c}\times\mathbf{B} - \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t} - \boldsymbol{\nabla}V, \qquad (17)$$

for some scalar field V (Maxwell's Ψ), that Maxwell identified with the electric scalar potential.

If it were clear that V (Ψ) is indeed the electric scalar potential, then Maxwell should be credited with having "discovered" the "Lorentz" force law. However, Helmholtz [eq. (5^d), p. 309 of [36] (1874)], Larmor [p. 12 of [44] (1884)], Watson [p. 273 of [50] (1888)), and J.J. Thomson [in his editorial note on p. 260 of [55] (1892)] argued that our eq. (15) leads to,

$$\mathcal{E} = \oint\left[\frac{\mathbf{v}}{c}\times\mathbf{B} - \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t} - \boldsymbol{\nabla}\left(\frac{\mathbf{v}}{c}\cdot\mathbf{A}\right)\right]\cdot d\mathbf{l}, \qquad (18)$$

so Maxwell's eq. (D) of Art. 598 and eq. (10) of Art. 599 should really be written as,

$$\mathfrak{E} = \frac{\mathbf{v}}{c}\times\mathbf{B} - \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t} - \boldsymbol{\nabla}\left(\Psi + \frac{\mathbf{v}}{c}\cdot\mathbf{A}\right), \qquad (19)$$

where Ψ is the electric scalar potential.[17] It went unnoticed by these authors that use of eq. (19) rather than (17) would destroy the elegance of Maxwell's argument in Arts. 600-601

[16]In Maxwell's notation, $E = \mathcal{E}$, $p = \Phi_m$, $(F, G, H) = \mathbf{A}$, $(F\,dx/ds + G\,dy/ds + H\,dz/ds)\,ds = \mathbf{A}\cdot d\mathbf{l}$, $(dx/dt, dy/dt, dz/dt) = \mathbf{v}$, and $(a, b, c) = \mathbf{B}$.

[17]A possible inference from eq. (19) is that the Lorentz force law should actually be,

$$\mathbf{F} = q\left[\mathbf{E} + \frac{\mathbf{v}}{c}\times\mathbf{B} - \boldsymbol{\nabla}\left(\frac{\mathbf{v}}{c}\cdot\mathbf{A}\right)\right] = q\left[\mathbf{E} + \left(\frac{\mathbf{v}}{c}\cdot\boldsymbol{\nabla}\right)\mathbf{A}\right] = -q\left(\boldsymbol{\nabla}V + \frac{1}{c}\frac{d\mathbf{A}}{dt}\right), \qquad (20)$$

Some debate persists on this issue, as discussed, for example, in [93] and references therein.

(discussed in sec. 2 below), as well as that Maxwell's earlier derivations of our eq. (17), on pp. 340-342 of [25] and in secs. 63-65 of [28],[18] used different methods which did not suggest the possible presence of a term $-\boldsymbol{\nabla}(\mathbf{v}\cdot\mathbf{A}/c)$ in our eq. (17). However, the practical effect of these doubts by illustrious physicists was that Maxwell has not been credited for having deduced the "Lorentz" force law, which became generally accepted only in the 1890's.

The view of this author is that Maxwell did deduce the "Lorentz" force law, although in a manner that was "not beyond a reasonable doubt".

2 Articles 600-601 of Maxwell's *Treatise*

In Art. 600, Maxwell considered a moving point with respect to two coordinate systems, the lab frame where $\mathbf{x} = (x, y, z)$, and a frame moving with uniform velocity $\mathbf{v}$ respect to the lab in which the coordinates of the point are $\mathbf{x}' = (x', y', z')$, with quantities in the two frames related by Galilean transformations. Noting that a force has the same value in both frames, Maxwell deduced that the "Lorentz" force law has the same form in both frames, provided the electric scalar potential V' in the moving frame is related to lab-frame quantities by,

$$V' = V - \frac{\mathbf{v}}{c}\cdot\mathbf{A}. \tag{21}$$

This is the form according to the low-velocity Lorentz transformation (7), and also to the transformations of magnetic Galilean electrodynamics (5), which latter is closer in spirit to Maxwell's arguments in Arts. 600-601.

2.1 Details

In Art. 600, Maxwell consider both translations and rotations of the moving frame, but we restrict our discussion here to the case of translation only, with velocity $\mathbf{v} = (u, v, w) = (\delta x/dt, \delta y/dt, \delta z/dt)$ with respect to the lab.[19] Maxwell labeled the velocity of the moving point with respect to the moving frame by $\mathbf{u}' = d\mathbf{x}'/dt'$, while he called labeled its velocity with respect to the lab frame by $\mathbf{u} = d\mathbf{x}/dt$. Then, Maxwell stated the velocity transformation to be, eq. (1) of Art. 600,[20]

$$\mathbf{u}' = \mathbf{u} - \mathbf{v}, \qquad i.e., \qquad \mathbf{u} = \mathbf{v} + \mathbf{u}' \qquad \left(\frac{dx}{dt} = \frac{\delta x}{dt} + \frac{dx'}{dt}\right), \tag{22}$$

which corresponds to the Galilean coordinate transformation,

$$\mathbf{x}' = \mathbf{x} - \mathbf{v}t. \qquad t' = t, \qquad \boldsymbol{\nabla}' = \boldsymbol{\nabla}, \qquad \frac{\partial}{\partial t'} = \frac{\partial}{\partial t} + \mathbf{v}\cdot\boldsymbol{\nabla}. \tag{23}$$

[18]These derivations of Maxwell are reviewed in Appendix A below.

[19]For discussion of electrodynamics in a rotating frame (in which one must consider "fictitious" charges and currents, see, for example, [96].

[20]Equation (2) of Art. 600 refers to rotations of a rigid body about the origin of the moving frame.

Maxwell next considered the transformation of the time derivative of the vector potential $\mathbf{A} = (F, G, H)$ in his eq. (3), Art. 600,

$$\frac{\partial \mathbf{A}'}{\partial t'} = \frac{\partial \mathbf{A}}{\partial t} + (\mathbf{v} \cdot \boldsymbol{\nabla})\mathbf{A} \qquad \left(\frac{dF'}{dt} = \frac{dF}{dx}\frac{\delta x}{\delta t} + \frac{dF}{dy}\frac{\delta y}{\delta t} + \frac{dF}{dz}\frac{\delta z}{\delta t} + \frac{dF}{dt} \right), \tag{24}$$

which tacitly assumed that $\mathbf{A}' = \mathbf{A}$, and hence that $\mathbf{B}' = \mathbf{B}$.[21] In eqs. (4)-(7) of Art. 600, Maxwell argued for the equivalent of use of the vector-calculus identity (14), which implies eq. (15), and hence that,

$$\frac{\partial \mathbf{A}'}{\partial t'} = \frac{\partial \mathbf{A}}{\partial t} - \mathbf{v} \times \mathbf{B} + \boldsymbol{\nabla}(\mathbf{v} \cdot \mathbf{A}). \tag{25}$$

Then, in eqs. (8)-(9) of Art. 600, Maxwell combined his eq. (B) of Art. 598 with our eqs. (22) and (25) to write the *electromotive force* $\mathfrak{E}$ as, in the notation of the present section,

$$\mathfrak{E} = \frac{\mathbf{u}}{c} \times \mathbf{B} - \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t} - \boldsymbol{\nabla} V = \frac{\mathbf{u}'}{c} \times \mathbf{B} - \frac{1}{c}\frac{\partial \mathbf{A}'}{\partial t'} - \boldsymbol{\nabla}\left(V - \frac{\mathbf{v}}{c} \cdot \mathbf{A} \right). \tag{26}$$

Finally, since a force has the same value in two frames related by a Galilean transformation, Maxwell inferred that the *electromotive force* $\mathfrak{E}'$ in the moving frame can be written as,

$$\begin{aligned} \mathfrak{E}' = \frac{\mathbf{u}'}{c} \times \mathbf{B} - \frac{1}{c}\frac{\partial \mathbf{A}'}{\partial t'} - \boldsymbol{\nabla}\left(\Psi - \frac{\mathbf{v}}{c} \cdot \mathbf{A} \right) &= \frac{\mathbf{u}'}{c} \times \mathbf{B}' - \frac{1}{c}\frac{\partial \mathbf{A}'}{\partial t'} - \boldsymbol{\nabla}'\left(\Psi - \frac{\mathbf{v}}{c} \cdot \mathbf{A} \right) \\ &= \frac{\mathbf{u}'}{c} \times \mathbf{B}' - \frac{1}{c}\frac{\partial \mathbf{A}'}{\partial t'} - \boldsymbol{\nabla}' V' = \frac{\mathbf{u}'}{c} \times \mathbf{B}' + \mathbf{E}', \end{aligned} \tag{27}$$

where the electric scalar potential V' in the moving frame is related to lab-frame quantities by,

$$V' = V - \frac{\mathbf{v}}{c} \cdot \mathbf{A} \qquad (= \Psi + \Psi'). \tag{28}$$

This is the form according to the low-velocity Lorentz transformation (7), and also to the transformations of magnetic Galilean electrodynamics (5), which latter is closer in spirit to Maxwell's arguments in Arts. 600-601.

Further, the force $\mathbf{F}'$ on a moving electric charge q in the moving frame is given by the "Lorentz" form,

$$\mathbf{F}' = q\left(\mathbf{E}' + \frac{\mathbf{u}'}{c} \times \mathbf{B}' \right), \tag{29}$$

which has the same form eq. (9) in the lab frame. As Maxwell stated at the beginning of Art. 601: *It appears from this that the electromotive force is expressed by a formula of the same type, whether the motions of the conductors be referred to fixed axes or to axes moving in space.*[22]

[21]While this assumption does not correspond to the low-velocity Lorentz transformation of the field between inertial frames, it does hold for the transformation from an inertial frame to a rotating frame. Faraday considered rotating magnets in [16], and in sec. 3090, p. 31, concluded that *No mere rotation of a bar magnet on its axis, produces any induction effect on circuits exterior to it.* That is, $\mathbf{B}' = \mathbf{B}$ relates the magnetic field in an inertial and a rotating frame. Possibly, this might have led Maxwell to infer a similar result for a moving inertial frame as well.

[22]Maxwell's equations in Art. 600 do not appear to be fully consistent with this "relativistic" statement, as

2.2 Articles 602-603. The "Biot-Savart" Force Law

In articles 602-603, Maxwell considered the force on a current element $I\,d\mathbf{l}$ in a circuit at rest in a magnetic field $\mathbf{B}$, and deduced the "Biot-Savart" form,[23]

$$d\mathbf{F} = \frac{I\,d\mathbf{l}}{c} \times \mathbf{B}, \qquad \mathbf{F} = \oint \frac{I\,d\mathbf{l}}{c} \times \mathbf{B}. \tag{30}$$

Some other comments on Arts. 602-603 were given around eqs. (10)-(11) above.

3 Articles 769-770 of Maxwell's *Treatise*

Faraday considered that a moving electric charge generates a magnetic field [6] (as quoted in [52]). Maxwell also argued for this in Arts. 769-770 of [55], where his argument can be transcribed as,

$$\mathbf{B} = \frac{\mathbf{v}}{c} \times \mathbf{E}, \tag{31}$$

for the magnetic field experienced by a fixed observer due to a moving charge. Maxwell noted that this is a very small effect, and claimed (1873) that it had never been observed.[24]

If $\mathbf{v}$ represents the velocity of a moving observer relative to a fixed electric charge, then eq. (31) implies that the magnetic field experienced by the moving observer would be,

$$\mathbf{B}' = -\frac{\mathbf{v}}{c} \times \mathbf{E}, \tag{32}$$

This corresponds to the low-velocity limit (7) of special relativity, and to form (3) of electric Galilean relativity).

It remains that while Maxwell used Galilean transformations as the basis for his considerations of fields and potentials in moving frames, he was rather deft in avoiding the contradictions between "Galilean electrodynamics" and his own vision. For a contrast, in which use of Galilean transformations for electrodynamics by J.J. Thomson [38] (1880) led to a result in disagreement with Nature (unrecognized at the time), see Appendix B below.[25]

he noted in Art. 601. That is, his eq. (9), Art. 600, is the equivalent to $\mathfrak{E}' = \mathbf{u}'/c \times \mathbf{B} - \partial \mathbf{A}'/\partial ct - \boldsymbol{\nabla}(\Psi + \Psi')$, where Ψ is the electric scalar potential in the lab frame, and $\Psi' = -\mathbf{v}/c \cdot \mathbf{A}$ (a lab-frame quantity) according to Maxwell's eq. (6), Art. 600. Maxwell did seem to realize that in addition to expressing the *electromotive force* $\mathfrak{E}'$ in the moving frame in terms of moving-frame quantities [our eq. (27)], as well as in terms of lab-frame quantities (our eq. (8), Maxwell's eq. (10), Art. 599), he had also deduced the relation of the electric scalar potential V' in the moving frame to the lab-frame quantities $\Psi + \Psi'$, as in our eq. (28).

[23] Maxwell had argued for this in his eqs. (12)-14), p. 172, of [24] (1861), which is the first statement of the "Biot-Savart" force law in terms of a magnetic field. Biot and Savart [2] discussed the force on a magnetic "pole" due to an electric circuit, and had no concept of the magnetic field.

[24] The magnetic field of a moving charge was detected in 1876 by Rowland [37, 52] (while working in Helmholtz' lab in Berlin). The form (31) was verified (in theory) more explicitly by J.J. Thomson in 1881 [39] for uniform speed $v \ll c$, and for any $v < c$ by Heaviside [51] and by Thomson [53] in 1889 (which latter two works gave the full special-relativistic form for $\mathbf{E}$ as well).

[25] For use by FitzGerald (1882) of Maxwell's "Galilean" arguments to deduce a result in agreement with special relativity as $v \to c$, see [42, 102], which was one of the first indications that the speed of light plays a role as a limiting velocity for particles.

Maxwell did not note the incompatibility (at order v^2/c^2) of his use of Galilean transformations in his Arts. 601-602 and 769-770 with his system of equations for the electromagnetic fields, but if he had, he might have mitigated this issue by deduction of self-consistent transformations for both $\mathbf{E}$ and $\mathbf{B}$ between the lab frame and a uniformly moving frame (*i.e.*, the full field transformations of special relativity), as in Appendix C below.

This note was stimulated by e-discussions with Dragan Redžić.

A Appendix: Maxwell's Derivations of the "Lorentz" Force Law[26]

This Appendix uses SI units, while the main text employs Gaussian units.

Maxwell published his developments of the theory of electrodynamics in four steps, *On Faraday's Lines of Force* [21] (1856), *On Physical Lines of Force* [24, 25, 26, 27] (1861-62), *A Dynamical Theory of the Electromagnetic Field* [28] (1864), and in his *Treatise on Electricity and Magnetism* [34, 35, 54, 55] (1873). In this Appendix we review his electrodynamic arguments related, in a broad sense, to the issue of the "Lorentz" force law.

As this Appendix has grown rather long, we first preview Maxwell's arguments most directly related to the force on a moving charge.

Appendix A.1.7 notes Maxwell's first discussion of a *moving* (charge) *particle* on p. 64 of [21] (1856), in which he took the convective derivative of the vector potential to obtain our eq. (48), whose meaning was not very apparent. It it interesting to this author that Maxwell did not in 1856 or later convert our eq. (48) into our (50), which was subsequently claimed by Helmholtz, Larmor, Watson, and J.J. Thomson to have been what Maxwell should have done.

Appendix A.2.6 reviews Maxwell's use (1861) [25] of his theory of molecular vortices and an energy argument to deduce our eq. (69), which is the curl of the Lorentz force law. Maxwell then integrated this by adding the term $\boldsymbol{\nabla}\Psi$ to the argument of the curl operator, which yields the "Lorentz" force law, our eq. (70), if we accept Maxwell's interpretation of Ψ as the *electric tension* (electric scalar potential).

Appendix A.3.6 discusses Maxwell's consideration (1864), sec. 63 of [28], of a circuit at rest which led him to identify the *electromotive force* vector on a charge at rest as our eq. (95). Then, Appendix A.3.7 recounts Maxwell's extrapolation in sec. 64 of [28] to a moving charge (circuit element) by the addition of the single-charge version of the Biot-Savart force law, $\mathbf{F} = q\mathbf{v} \times \mathbf{B}$, to arrive at the "Lorentz" force law, our eq. (100), in the same form, our eq. (70), as he had previously found in [25].

The "Lorentz" force law, our eqs. (70) and (100) were also deduced by Maxwell (1873) in Arts. 598-599 of [35] via a slightly different argument, as discussed above in sec. 1 above. Maxwell included mention of this force law in Art. 619 of [35], the summary of his theory of the electromagnetic field, but in a somewhat unfortunate manner, as reviewed in Appendix A.4.2.

[26] This Appendix also appears as Appendix A.28 in [106].

A.1 In *On Faraday's Lines of Force* [21]

For a discussion of Maxwell's thoughts in 1855, which culminated in the publication [21], see [90].

A.1.1 *Theory of the Conduction of Current Electricity* and *On Electro-motive Forces*

On p. 46 of [21], Maxwell stated: *According to the received opinions we have here a current of fluid moving uniformly in conducting circuits, which oppose a resistance to the current which has to be overcome by the application of an electro-motive force at some part of the circuit.*

He continued on pp. 46-47: *When a uniform current exists in a closed circuit it is evident that some other forces must act on the fluid besides the pressures. For if the current were due to difference of pressures, then it would flow from the point of greatest pressure in both directions to the point of least pressure, whereas in reality it circulates in one direction constantly. We must therefore admit the existence of certain forces capable of keeping up a constant current in a closed circuit. Of these the most remarkable is that which is produced by chemical action. A cell of a voltaic battery, or rather the surface of separation of the fluid of the cell and the zinc, is the seat of an electro-motive force which can maintain a current in opposition to the resistance of the circuit. If we adopt the usual convention in speaking of electric currents, the positive current is from the fluid through the platinum, the conducting circuit, and the zinc, back to the fluid again.*

Here, Maxwell seemed to accept the *received opinions*[27] that electric current is a fluid; and actually two counterpropagating fluids.

A.1.2 Ohm's Law, Electromotive Force and the Electric Field

On p. 47 of [21], Maxwell wrote a version of Ohm's Law as $F = IK$ for an electrical circuit of resistance K that carries current I. He calls F the *electro-motive force*, which is consistent with a more contemporary notation,

$$\mathcal{E} = \oint \mathbf{E} \cdot d\mathbf{l} = IR \qquad (F = IK), \tag{33}$$

where $\mathcal{E} = F$ is the electromotive "force" (with dimensions of electric potential (voltage) rather than of force), $\mathbf{E}$ is the electrical field and $d\mathbf{l}$ is line element inside the wire of the circuit.

On p. 53, Maxwell introduced the (free/conduction) electric current-density vector $\mathbf{J} = (a_2, b_2, c_2)$, the electric scalar potential $\Psi = p_2$, and the electric field $\mathbf{E} = (\alpha_2, \beta_2, \gamma_2)$, writing in his eq. (A),

$$\mathbf{E} = \mathbf{E}_{\text{other}} - \boldsymbol{\nabla}\Psi \qquad \left(\alpha_2 = X_2 - \frac{dp_2}{dx}\,,\ etc.\right), \tag{34}$$

with $\mathbf{E}_{\text{other}} = (X_2, Y_2, Z_2)$ being a possible contribution to the electric field not associated with a scalar potential. To possible confusion, Maxwell called the vector $\mathbf{E}$ an *electro-motive force*, which term he also used for the scalar $\mathcal{E}$.

[27]Maxwell cited French translations of papers by Kirchhoff [13] and Quincke [18]. He seemed unaware that Kirchhoff had also published an English version of his paper [13].

Also on p. 53, Maxwell introduced the electrical resistivity $\varrho = k_2$ (reciprocal of the electrical conductivity $\sigma = 1/\varrho$, so the Ohm's Law can be written as Maxwell's eq. (B),

$$\mathbf{E} = \varrho\,\mathbf{J}_{\rm free} = \frac{\mathbf{J}_{\rm free}}{\sigma} \qquad (\alpha_2 = k_2\,a_2,\ \ etc.)\,, \tag{35}$$

On p. 54, Maxwell noted that for any closed curve eq. (34) implies,

$$\mathcal{E} = \oint \mathbf{E}\cdot d\mathbf{l} = \oint \mathbf{E}_{\rm other}, \tag{36}$$

He also introduced the concept of the flux (*conduction*) $\mathbf{E}\cdot d\mathbf{S}$ of a field $\mathbf{E}$ across a surface element $d\mathbf{S}$, and noted (Gauss' Law) that for a closed surface,

$$\int \mathbf{E}\cdot d\mathbf{S} = \int \boldsymbol{\nabla}\cdot\mathbf{E}\,d{\rm Vol}, \tag{37}$$

He indicated in his eq. (C) that he will often write the divergence of a vector field as $4\pi\rho$.

A.1.3 The Magnetic Fields H and $\mu\mathbf{H} = \mathbf{B}$

On p. 54 Maxwell introduced magnetic phenomena, via a formal parallel with electric phenomena, that implies there are two types of magnetic fields. He labeled the magnetic ("quantity") field $\mathbf{H}$ as $(\alpha_1, \beta_1, \gamma_1)$ and the magnetic (induction/"intensity") field $\mathbf{B}$ as (a_1, b_1, c_1).[28] He called the (relative) magnetic permeability μ the reciprocal of the *resistance to magnetic induction*, k_1, and noted (in words) that the parallel to our eq. (35), his eq. (B), is,

$$\mathbf{H} = \frac{\mathbf{B}}{\mu} \qquad (\alpha_1 = k_1\,a_1,\ \ etc.)\,, \tag{38}$$

and that in the relation $\boldsymbol{\nabla}\cdot\mathbf{B} = \mu\rho_m$, ρ_m is the density of *real magnetic matter*.

A.1.4 Ampère's Law

On p. 56, Maxwell stated Ampère's Law in the form,[29]

$$\boldsymbol{\nabla}\times\mathbf{H} = \mathbf{J}_{\rm free} \qquad \left(a_2 = \frac{d\beta_1}{dz} - \frac{d\gamma_1}{dy}\,,\ \ etc.\right), \tag{43}$$

[28] Maxwell did not use the symbol $\mathbf{B}$ for the magnetic (induction) field until 1873, in his *Treatise* [35], when he followed W. Thomson (1871), eq. (r), p. 401 of [45], who first defined $\mathbf{B} = \mu_0\mathbf{H} + \mathbf{M}$, where $\mathbf{M}$ is the density of magnetization.

[29] This is probably the first statement of Ampére's Law as a differential equation.

Ampère did not have a concept of fields, but discussed the force between two circuits, carrying currents I_1 and I_2, on pp. 21-24 of [3] (1822) and inferred that this could be written (here in vector notation) as,

$$\mathbf{F}_{\rm on\ 1} = \oint_1\oint_2 d^2\mathbf{F}_{\rm on\ 1}, \qquad d^2\mathbf{F}_{\rm on\ 1} = \frac{\mu}{4\pi}I_1I_2[3(\hat{\mathbf{r}}\cdot d\mathbf{l}_1)(\hat{\mathbf{r}}\cdot d\mathbf{l}_2) - 2\,d\mathbf{l}_1\cdot d\mathbf{l}_2]\frac{\hat{\mathbf{r}}}{r^2} = -d^2\mathbf{F}_{\rm on\ 2}, \tag{39}$$

where $\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$ is the distance from a current element $I_2\,d\mathbf{l}_2$ at $\mathbf{r}_2$ to element $I_1\,d\mathbf{l}_1$ at $\mathbf{r}_1$.

Ampère also noted that,

$$d\mathbf{l}_1 = \frac{\partial\mathbf{r}}{\partial l_1}\,dl_1, \qquad \mathbf{r}\cdot d\mathbf{l}_1 = \mathbf{r}\cdot\frac{\partial\mathbf{r}}{\partial l_1}\,dl_1 = \frac{1}{2}\frac{\partial r^2}{\partial l_1}\,dl_1 = r\frac{\partial r}{\partial l_1}\,dl_1, \qquad d\mathbf{l}_2 = -\frac{\partial\mathbf{r}}{\partial l_2}\,dl_2, \qquad \mathbf{r}\cdot d\mathbf{l}_2 = -r\frac{\partial r}{\partial l_2}\,dl_2, \tag{40}$$

and on p. 57 he noted that the divergence of eq. (43) is zero, so that his discussion is limited to *closed currents* that obey $\boldsymbol{\nabla} \cdot \mathbf{J} = 0$ (*i.e.*, to magnetostatics). Indeed, he added: *in fact we know little of the magnetic effects of any current that is not closed.*[30]

A.1.5 Helmholtz' Theorem

There followed an interlude on various theorems, some due to Green [4], and also Helmholtz' theorem [22] that "any" vector field $\mathbf{E}$ can be related to a scalar potential Ψ and a vector potential $\mathbf{A}$ as $\mathbf{E} = \boldsymbol{\nabla}\Psi + \boldsymbol{\nabla} \times \mathbf{A}$, where $\Psi = 0$ if $\boldsymbol{\nabla} \cdot \mathbf{E} = 0$ and $\mathbf{A} = 0$ if $\boldsymbol{\nabla} \times \mathbf{E} = 0$.[31]

A.1.6 Magnetic Field Energy and the Electric Field Induced by a Changing Current

After this, Maxwell considered the energy stored in the magnetic field. On p. 63, he first argued that if the magnetic field were due to a density ρ_m (ρ_1) of magnetic charges, the field could be deduced from a magnetic scalar potential Ψ_m ($= p_1$) and the energy stored in the field during the assembly of this configuration could be written,[32]

$$U_m = \frac{1}{2} \int \rho_m \Psi_m \, d\mathrm{Vol} \qquad \left(Q = \int\int\int \rho_1 p_1 \, dxdydz \right). \tag{44}$$

He then noted that this form can be transformed to,

$$U_m = \int \frac{\mathbf{B} \cdot \mathbf{H}}{2} \, d\mathrm{Vol} \qquad \left(Q = \frac{1}{4\pi} \int\int\int (a_1\alpha_1 + b_1\beta_1 + c_1\gamma_1) \, dxdydz \right). \tag{45}$$

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where l_1 and l_2 measure distance along the corresponding circuits in the directions of their currents. Then,

$$d\mathbf{l}_1 \cdot d\mathbf{l}_2 = -d\mathbf{l}_1 \cdot \frac{\partial \mathbf{r}}{\partial l_2} \, dl_2 = -\frac{\partial}{\partial l_2}(\mathbf{r} \cdot d\mathbf{l}_1) \, dl_2 = -\frac{\partial}{\partial l_2} \left(r \frac{\partial r}{\partial l_1} \right) dl_1 \, dl_2 = -\left(\frac{\partial r}{\partial l_1} \frac{\partial r}{\partial l_2} + r \frac{\partial^2 r}{\partial l_1 \partial l_2} \right) dl_1 \, dl_2, \tag{41}$$

and eq. (39) can also be written in forms closer to that used by Ampère,

$$d^2\mathbf{F}_{\mathrm{on}\ 1} = \frac{\mu}{4\pi} I_1 I_2 dl_1 dl_2 \left[2r \frac{\partial^2 r}{\partial l_1 \partial l_2} - \frac{\partial r}{\partial l_1} \frac{\partial r}{\partial l_2} \right] \frac{\hat{\mathbf{r}}}{r^2} = \frac{\mu}{2\pi} I_1 I_2 dl_1 dl_2 \frac{\partial^2 \sqrt{r}}{\partial l_1 \partial l_2} \frac{\hat{\mathbf{r}}}{\sqrt{r}} = -d^2\mathbf{F}_{\mathrm{on}\ 2}. \tag{42}$$

The integrand $d^2\mathbf{F}_{\mathrm{on}\ 1}$ of eq. (39) has the appeal that it changes sign if elements 1 and 2 are interchanged, and so suggests a force law for current elements that obeys Newton's third law. However, the integrand does not factorize into a product of terms in the two current elements, in contrast to Newton's gravitational force, and Coulomb's law for the static force between electric charges (and between static magnetic poles, whose existence Ampère doubted). As such, Ampère (correctly) hesitated to interpret the integrand as providing the force law between a pair of isolated current elements, *i.e.*, a pair of moving electric charges.

[30]This statement can be regarded as a precursor to Maxwell's later vision (first enunciated in eq. (112), p. 19, of [26]) that all currents are closed if one considers the "displacement-current" (density) $d\mathbf{D}/dt$ in addition to the conduction-current density $\mathbf{J}_{\mathrm{free}}$. But it also indicates that Maxwell chose not to consider the notion of moving charged particle as elements of an electrical current, as advocated by Weber (1846) [11] (see p. 88 of the English translation) as a way of understanding Ampère's expression for the force between two current loops.

For discussion of the "displacement current" of a uniformly moving charge, see [98].

[31]We also write that $\mathbf{E} = \mathbf{E}_{\mathrm{irr}} + \mathbf{E}_{\mathrm{rot}}$ where $\mathbf{E}_{\mathrm{irr}} = \boldsymbol{\nabla}\Psi$ obeys $\boldsymbol{\nabla} \times \mathbf{E}_{\mathrm{irr}} = 0$ and $\mathbf{E}_{\mathrm{rot}} = \boldsymbol{\nabla} \times \mathbf{A}$ obeys $\boldsymbol{\nabla} \cdot \mathbf{E}_{\mathrm{rot}} = 0$. Many people write $\mathbf{E}_{\mathrm{irr}} = \mathbf{E}_{\parallel}$ and $\mathbf{E}_{\mathrm{rot}} = \mathbf{E}_{\perp}$.

[32]It seems to this author that Maxwell omitted a factor of 1/2 in throughout his discussion on pp. 63-64.

He next argued that since this form does not include any trace of the origin of the magnetic field, it should also hold if the field is due to electrical currents, it can be transformed to,

$$U_m = \int \frac{\mathbf{J}\cdot\mathbf{A}}{2}\, d\mathrm{Vol} \qquad \left(Q = \frac{1}{4\pi}\int\!\!\int\!\!\int \left\{p_1\rho_1 - \frac{1}{4\pi}(\alpha_0\alpha_2 + \beta_0\beta_2 + \gamma_0\gamma_2)\right\} dxdydz\right). \tag{46}$$

where $\mathbf{B} = \boldsymbol{\nabla}\times\mathbf{A}$,[33,34,35]

The time rate of change of energy in the magnetic field is $-\mathbf{J}\cdot\mathbf{E}$,[36] so, by taking the time derivative of eq. (46), and noting that $\mathbf{A} = (\alpha_0, \beta_0, \gamma_0)$ scales linearly with $\mathbf{J}$, he infers (on p. 64) that the *electro-motive force due to the action of magnets and currents* is,[37]

$$\mathbf{E}_{\mathrm{induced}} = -\frac{\partial\mathbf{A}}{\partial t} \qquad \left(\alpha_2 = -\frac{1}{4\pi}\frac{d\alpha_0}{dt}\ ,\ \ etc.\right). \tag{47}$$

On p. 66, he stated this equation in words as: *Law VI. The electro-motive force on any element of a conductor is measured by the instantaneous rate of change of the electro-tonic intensity on that element, whether in magnitude or direction.* Here, Maxwell describes the vector potential $\mathbf{A}$ as the *electro-tonic intensity*, following Faraday, Art. 60 of [5].[38]

A.1.7 The Electro-Motive Force on a Moving Particle

At the bottom of p. 64, Maxwell made a statement that anticipated his later efforts towards the "Lorentz" force law: *If α_0 be expressed as a function of x, y, x, and t, and if x, y, z be the co-ordinates of a moving particle, then the electro-motive force measured in the direction of x is,*

$$\alpha_2 = -\frac{1}{4\pi}\left(\frac{d\alpha_0}{dx}\frac{dx}{dt} + \frac{d\alpha_0}{dy}\frac{dy}{dt} + \frac{d\alpha_0}{dz}\frac{dz}{dt} + \frac{d\alpha_0}{dt}\right), \qquad \mathfrak{E} = -\left(\frac{\partial\mathbf{A}}{\partial t} + (\mathbf{v}\cdot\boldsymbol{\nabla})\mathbf{A}\right), \tag{48}$$

[33]Maxwell seems to have made a sign error in his integration by parts of the integrand $\mathbf{H}\cdot\mathbf{B} = \mathbf{H}\cdot\boldsymbol{\nabla}\times\mathbf{A}$. Note that $\boldsymbol{\nabla}(\mathbf{A}\times\mathbf{H}) = \mathbf{H}\cdot\boldsymbol{\nabla}\times\mathbf{A} - \mathbf{A}\cdot\boldsymbol{\nabla}\times\mathbf{H}$.

[34]Maxwell did not seem to suppose in [21] that there is no *real magnetic matter*, so his $\mathbf{B}$ was also related to a scalar potential, and his version of eq. (46) has an additional term related to possible magnetic charges and the scalar potential.

[35]W. Thomson, p. 63 of [12] (1846), described the *electrical force* due to a unit charge at the origin *exerted at the point* (x, y, z) as $\mathbf{r}/4\pi\epsilon\, r^3$, without explicit statement that a charge exists at the point to experience the force. In the view of this author, that statement is the first mathematical appearance of the electric field in the literature, although neither vector notation nor the term "electric field" were used by Thomson.

Thomson immediately continued with the example of a *small magnet*, *i.e.*, a "point" magnetic dipole $\mathbf{m}$, whose scalar potential is $\Phi = \mu\,\mathbf{m}\cdot\mathbf{r}/4\pi r^3$, noting that the *magnetic force* $(X, Y, Z) = -\boldsymbol{\nabla}\Phi = \mathbf{B}$ on a unit magnetic pole p can also be written as $\boldsymbol{\nabla}\times\mathbf{A}$ (although Thomson did not assign a symbol to the vector $\mathbf{A}$), where $\mathbf{A} = (\alpha, \beta, \gamma) = \mu\,\mathbf{m}\times\mathbf{r}/4\pi r^3$, with $\boldsymbol{\nabla}\cdot\mathbf{A} = 0$, his eq. (2). This discussion is noteworthy for the sudden appearance of the vector potential of a magnetic dipole (with no reference to Neumann, whose 1845 paper [10] implied this result, and is generally credited with the invention of the vector potential although the relation $\mathbf{B} = \boldsymbol{\nabla}\times\mathbf{A}$ is not evident in this paper).

[36]Most contemporary discussions of magnetic field energy start from the relation and work towards eq. (46) and then (45).

[37]Maxwell seems to have made another sign error, in his discussion of the time rate of change of the field energy, such that his version of our eq. (47) had the correct sign.

[38]Other mention by Faraday of the electrotonic state include Art. 1661 of [6], Arts. 1729 and 1733 of [7], and Art. 3269 of [17].

where we use the symbol $\mathfrak{E} = (\alpha_2, \beta_2, \gamma_2)$ for Maxwell's vector *electromotive force*, which is not necessarily the same as the lab-frame electric field $\mathbf{E}$. Here, Maxwell claimed that the electric field experienced by a moving particle should be computed using the convective derivative of the vector potential, and not just the partial time derivative.

If Maxwell had persisted in following the consequences of this claim, he could have deduced, via a vector-calculus identity, that the *electro-motive force* experienced by a moving particle is,

$$\mathfrak{E} = -\left(\frac{\partial \mathbf{A}}{\partial t} + \boldsymbol{\nabla}(\mathbf{v} \cdot \mathbf{A}) + \mathbf{v} \times (\boldsymbol{\nabla} \times \mathbf{A})\right) = \mathbf{v} \times \mathbf{B} - \left(\frac{\partial \mathbf{A}}{\partial t} + \boldsymbol{\nabla}(\mathbf{v} \cdot \mathbf{A})\right). \tag{49}$$

If Maxwell had further considered that the electric field can have a term deducible from a scalar potential Ψ, then he might have claimed that the total *electro-motive force* on a moving particle is,

$$\mathfrak{E} = \frac{\mathbf{v}}{c} \times \mathbf{B} - \boldsymbol{\nabla}(\Psi + \mathbf{v} \cdot \mathbf{A}) - \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t}. \tag{50}$$

We will see below that Maxwell did not follow the path sketched above, but made important variants thereto, while others in the late 1800's (Helmholtz [36], Larmor [44], Watson [50], J.J. Thomson in his Appendix to Chap. IX of [55], p. 260) argued that he should have proceeded as above.

A.1.8 Faraday's Law

Faraday's Law was not formulated by Faraday himself, but by Maxwell (1856), p. 50 of [21]: *the electro-motive force depends on the change in the number of lines of inductive magnetic action which pass through the circuit,* as a summary of Faraday's comments in [16].[39] We express this as the equation (for a circuit at rest in the lab where $\mathfrak{E} = \mathbf{E}$),

$$\mathcal{E} = \oint \mathbf{E} \cdot d\mathbf{l} = -\frac{d\Phi_m}{dt} = -\frac{d}{dt}\int \mathbf{B} \cdot d\mathbf{S}. \tag{51}$$

A.2 In *On Physical Lines of Force* [24, 25, 26, 27]

A.2.1 The Magnetic Stress Tensor, Ampère's Law and the Biot-Savart Force Law

In [24] (1861), Maxwell considered a (linear) magnetic medium with uniform (relative) permeability μ to be analogous to a fluid filled with vortices, which led him, on p. 168, to deduce/invent the stress tensor $\mathsf{T}_{ij} = p_{ij}$ of the magnetic field,

$$\mathsf{T}_{ij} = \mu H_i H_j - \delta_{ij} 4\pi p_m, \tag{52}$$

where $\mathbf{H} = (\alpha, \beta, \gamma)$ is the magnetic field, and $p_m = p_1$ is a magnetic pressure. The volume force density $\mathbf{f}$ in the medium is then given by Maxwell's eq (3),

$$\mathbf{f} = \boldsymbol{\nabla} \cdot \mathsf{T}, \tag{53}$$

[39] The statement appears on a letter from Maxwell to W. Thomson, Nov. 23, 1854, p. 703 of [74].

and the x-component of this force, $f_x = X$, is given by Maxwell's eqs. (4)-(5),

$$\begin{aligned}
f_x &= \frac{\partial T_{xx}}{\partial x} + \frac{\partial T_{xy}}{\partial y} + \frac{\partial T_{xz}}{\partial z} \\
&= 2\mu H_x \frac{\partial H_x}{\partial x} - 4\pi \frac{\partial p_m}{\partial x} + \mu H_x \frac{\partial H_y}{\partial y} + H_y \frac{\partial H_x}{\partial y} + \mu H_x \frac{\partial H_z}{\partial z} + H_z \frac{\partial H_x}{\partial y} \\
&= H_x \boldsymbol{\nabla} \cdot \mu \mathbf{H} + \frac{\mu}{2} \frac{\partial H^2}{\partial x} - \mu H_y \left(\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} \right) - \mu H_z \left(\frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x} \right) - 4\pi \frac{\partial p_m}{\partial x} \\
&= H_x \boldsymbol{\nabla} \cdot \mathbf{B} + \frac{\mu}{2} \frac{\partial H^2}{\partial x} - B_y (\boldsymbol{\nabla} \times \mathbf{H})_z + B_z (\boldsymbol{\nabla} \times \mathbf{H})_y - 4\pi \frac{\partial p_m}{\partial x} \\
&= H_x \boldsymbol{\nabla} \cdot \mathbf{B} + \frac{\mu}{2} \frac{\partial H^2}{\partial x} - \mathbf{B} \times (\boldsymbol{\nabla} \times \mathbf{H}) - 4\pi \frac{\partial p_m}{\partial x},
\end{aligned} \tag{54}$$

where Maxwell introduced $\mathbf{B} = \mu \mathbf{H}$ as the magnetic induction field. In his eq. (6), p. 168, Maxwell stated that,

$$\boldsymbol{\nabla} \cdot \mathbf{B} = \mu \, \rho_m \qquad \left(\frac{d}{dx} \mu\alpha + \frac{d}{dy} \mu\beta + \frac{d}{dz} \mu\gamma = 4\pi \, m \right), \tag{55}$$

where ρ_m $(= m)$ is the density of *"imaginary magnetic matter"*. In his eq. (9), p. 171, Maxwell stated that,

$$\boldsymbol{\nabla} \times \mathbf{H} = \mathbf{J}_{\rm free} \qquad \left[\frac{1}{4\pi} \left(\frac{d\gamma}{dy} - \frac{d\beta}{dz} \right) = p, \; etc. \right], \tag{56}$$

where $\mathbf{J}_{\rm free} = (p, q, r)$ is the density of (free) electric current. He did not reference his discussion of our eq. (43) in his paper [21]), but arrived at eq. (56) via an argument that involved both magnetic poles and electric currents. Thus, the force density (54) can be rewritten as, Maxwell's eqs. (12)-(14), p. 172,

$$\mathbf{f} = \rho_m \mathbf{B} + \mathbf{J}_{\rm free} \times \mathbf{B} + \boldsymbol{\nabla} \frac{\mu H^2}{2} - 4\pi \boldsymbol{\nabla} p_m. \tag{57}$$

The second term of eq. (57) is (this author believes) the first statement of what is now commonly called the Biot-Savart force law for a free electric-current density,

$$\mathbf{F} = \int \mathbf{J}_{\rm free} \times \mathbf{B} \, d\text{Vol}, \tag{58}$$

in terms of a magnetic field (of which Biot and Savart [2] had no conception).[40]

[40] In 1845, Grassmann [8] argued that although Ampère claimed [3] that his force law was *uniquement déduite de l'expérience*, it included the assumption that it obeyed Newton's third law. He noted that Ampère's law (39) implies that the force is zero for parallel current elements whose lie of centers makes angle $\cos^{-1}\sqrt{2/3}$ to the direction of the currents, which seemed implausible to him. Grassmann claimed that, unlike Ampère, he would make no "arbitrary" assumptions, but in effect he assumed that there is no magnetic force between collinear current elements, which leads to a force law,

$$\mathbf{F}_{\text{on }1} = \oint_1 \oint_2 d^2\mathbf{F}_{\text{on }1}, \qquad d^2\mathbf{F}_{\text{on }1} = \frac{\mu}{4\pi} I_1 d\mathbf{l}_1 \times \frac{I_2 d\mathbf{l}_2 \times \hat{\mathbf{r}}}{r^2}, \tag{59}$$

However, Maxwell did not consider an electric current to be a flow of charged particles, so he did not immediately interpret eq. (57) as a derivation of the "Lorentz" force $q\mathbf{v} \times \mathbf{B}$ on a moving electric charge q.

Also, Maxwell did not note at this time that the third and fourth terms of eq. (57) cancel, in that $p_m = \mu H^2/2$ is the "magnetic pressure",[41] nor did he infer that $\boldsymbol{\nabla} \cdot \mathbf{B} = 0 = \mu\, \rho_m$.

A.2.2 Magnetic Field Energy

On p. 63 of [21], Maxwell had deduced that the energy stored in the magnetic field can be computed according to our eq. (45) via an argument that supposed the existence of magnetic charges (monopoles) and a corresponding magnetic scalar potential. In Prop VI, pp. 286-288 of [25], Maxwell used his model of molecular vortices to deduce the same result (given in his eqs. (45)-(46), p. 288, and again in his eq. (51), p. 289),

$$U_m = \int \frac{\mathbf{B} \cdot \mathbf{H}}{2}\, d\text{Vol} \qquad \left(E = \frac{1}{2}\mu(\alpha^2 + \beta^2 + \gamma^2)V \right). \tag{61}$$

A.2.3 Faraday's Law

In Prop. VII, pp. 288-291 of [25], Maxwell considered the time derivative of the magnetic field energy, written in his eq. (52), p. 289, as,

$$\frac{dU_m}{dt} = \int \mathbf{H} \cdot \frac{\partial \mathbf{B}}{\partial t}\, d\text{Vol} \qquad \left[\frac{dE}{dt} = \frac{1}{4\pi}\mu V \left(\alpha \frac{d\alpha}{dt} + \beta \frac{d\beta}{dt} + \gamma \frac{d\gamma}{dt} \right) \right], \tag{62}$$

He also introduced the *electromotive force* $\mathfrak{E} = (P, Q, R)$ on a unit electric charge, and argued, using his model of molecular vortices, that the (electro)magnetic field does work on an electric current density $\mathbf{J}_{\text{free}}$ at rate,[42]

$$\frac{dU_m}{dt} = -\int \mathbf{J}_{\text{free}} \cdot \mathbf{E}\, d\text{Vol} \qquad \left[\frac{dE}{dt} = -\frac{1}{4\pi} \sum (Pu + Qv + Rw) dS \right]. \tag{63}$$

In cases like the present where the *electromotive force* $\mathfrak{E}$ is that on a hypothetical unit charge at rest in the lab, it is the same as the electric field $\mathbf{E}$, whose symbol will be used in

---

in vector notation (which, of course, Grassmann did not use). While $d^2\mathbf{F}_{\text{on }1}$ is not equal and opposite to $d^2\mathbf{F}_{\text{on }2}$, Grassmann showed that the total force on circuit 1 is equal and opposite to that on circuit 2, $\mathbf{F}_{\text{on }1} = -\mathbf{F}_{\text{on }2}$.

Grassmann's result is now called the Biot-Savart force law,

$$\mathbf{F}_{\text{on }1} = \oint_1 I_1\, d\mathbf{l}_1 \times \mathbf{B}_2, \qquad \mathbf{B}_2 = \frac{\mu}{4\pi} \oint_2 \frac{I_2\, d\mathbf{l}_2 \times \hat{\mathbf{r}}}{r^2}, \tag{60}$$

although Grassmann did not identify the quantity $\mathbf{B}_2$ as the magnetic field.

[41] If the permeability μ is nonuniform, the third and fourth terms combine to yield the term $(H^2/2)\boldsymbol{\nabla}\mu$, as noted by Helmholtz (1881) [40].

[42] Maxwell first considered his dE/dt of eq. (47), p. 289, for a surface element, extending this to a volume in his eq. (50).

this section.[43] Maxwell next gave an argument in his eqs. (48)-(50) equivalent to using our eq. (56) to find,

$$\frac{dU_m}{dt} = -\int \mathbf{E} \cdot (\boldsymbol{\nabla} \times \mathbf{H})\, d\mathrm{Vol} = -\int \mathbf{H} \cdot (\boldsymbol{\nabla} \times \mathbf{E})\, d\mathrm{Vol}. \tag{64}$$

Comparing our eqs. (62) and (64), we infer,[44] as in Maxwell's eq. (54), p. 290, that,

$$\boldsymbol{\nabla} \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \qquad \left(\frac{dQ}{dz} - \frac{dR}{dy} = \mu \frac{d\alpha}{dt}\,, etc.\right). \tag{65}$$

This is the first statement of Faraday's law as a (vector) differential equation. Surprisingly, Maxwell did not give this differential form either in [28] or in his *Treatise* [55].

A.2.4 $\boldsymbol{\nabla} \cdot \mathbf{B} = 0$, $\boldsymbol{\nabla} \cdot \mathbf{A} = 0$

Also on p. 290 of [25], Maxwell discussed the relation $\mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A}$, his eq. (55), *subject to the conditions*, his eqs. (56) and (57),

$$\boldsymbol{\nabla} \cdot \mathbf{B} = 0, \qquad \text{and} \qquad \boldsymbol{\nabla} \cdot \mathbf{A} = 0. \tag{66}$$

Maxwell gave no justification for these conditions, which are the first statements by him of them.[45] While we now recognize that $\boldsymbol{\nabla} \cdot \mathbf{B} = 0$ holds in the absence of true magnetic charges, as apparently is the case in Nature (and that if $\mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A}$, then $\boldsymbol{\nabla} \cdot \mathbf{B} = 0$ follows from vector calculus), the relation $\boldsymbol{\nabla} \cdot \mathbf{A} = 0$ is a choice of "gauge" (in particular, the Coulomb gauge) and not a law of Nature.

A.2.5 Electric Field outside a Toroidal Coil with a Time-Varying Current

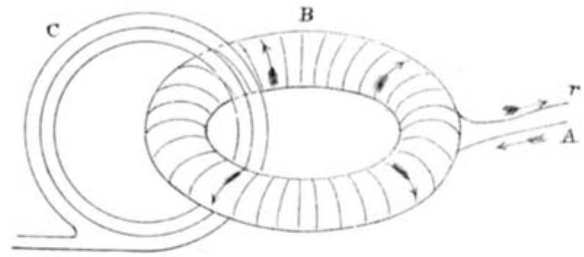
We digress slightly to note that on pp. 338-339 of [25], Maxwell considered a toroidal coil in his Fig. 3. If this coil carries an electric current, there is no exterior magnetic field even in the case of a time-dependent current (if one neglects electromagnetic radiation, whose existence Maxwell reported only in the next paper, [26], in his series *On Physical Lines of Force*). However, a changing current induces an external electric field, which seems like action at a distance. Maxwell noted that while the external magnetic field is zero, the external vector potential $\mathbf{A}$ (*electrotonic state*) is not, and the external electric field is related to the time derivative of $\mathbf{A}$. Is seems that the vector potential $\mathbf{A}$ had "physical reality" for Maxwell, which view was later extended to a quantum context by Aharonov and Bohm [78, 101].

[43] In sec. A.2.5, which concerns moving charges, we will use the symbol $\mathfrak{E}$ for Maxwell's *electromotive force* vector.

[44] This argument ignores a possible contribution to the field energy from the electric field.

[45] Maxwell likely followed Thomson, who considered that $\boldsymbol{\nabla} \cdot \mathbf{A} = 0$ in eqs. (2) and (3) of [12]. Thomson was inspired by Stokes' discussion [9] of incompressible fluid flow where the velocity vector $\mathbf{u}$ obeys $\boldsymbol{\nabla} \cdot \mathbf{u} = 0$, Stokes' eq. (13).

Fig: 3.



A.2.6 *Electromotive Force in a Moving Body*

In Prop. XI, pp. 340-341 of [25], Maxwell considered a body that might be deforming, translating, and/or rotating, and discussed the resulting changes in the magnetic field $\mathbf{H}$. On one hand, he stated in his eq. (70), p. 341, that,[46]

$$\delta\mathbf{H} = (\delta\mathbf{x}\cdot\boldsymbol{\nabla})\mathbf{H} + \delta t\frac{\partial\mathbf{H}}{\partial t} \qquad \left(\delta\alpha = \frac{d\alpha}{dx}\delta x + \frac{d\alpha}{dy}\delta y + \frac{d\alpha}{dz}\delta z + \frac{d\alpha}{dt}\delta t\right), \tag{67}$$

which uses the convective derivative. On the other hand, he stated before his eq. (69): *The variation of the velocity of the vortices in a moving element is due to two causes—the action of the electromotive forces, and the change of form and position of the element. The whole variation of α is therefore,*[47,48]

$$\delta\alpha = \frac{1}{\mu}\left(\frac{dQ}{dz} - \frac{dR}{dy}\right)\delta t + \alpha\frac{d}{dx}\delta x + \beta\frac{d}{dy}\delta x + \gamma\frac{d}{dz}\delta x \qquad \left(\delta\mathbf{H} = -\frac{1}{\mu}\boldsymbol{\nabla}\times\mathfrak{E}\,\delta t + (\mathbf{H}\cdot\boldsymbol{\nabla})\,\delta\mathbf{x}\right). \tag{68}$$

If we accept this relation, then we can follow Maxwell that for an incompressible medium, whose velocity field obeys $\boldsymbol{\nabla}\cdot\mathbf{v} = 0$, and if $\boldsymbol{\nabla}\cdot\mathbf{H} = 0$ (which Maxwell stated to hold *in the absence of free magnetism*, then his eqs. (69)-(70) do lead to his eq. (76), p. 342,

$$\frac{d}{dz}\left(Q + \mu\gamma\frac{dx}{dt} - \mu\alpha\frac{dz}{dt} - \frac{dG}{dt}\right) - \frac{d}{dy}\left(R + \mu\alpha\frac{dy}{dt} - \mu\beta\frac{dx}{dt} - \frac{dH}{dt}\right) = 0$$
$$\left[\boldsymbol{\nabla}\times\left(\mathfrak{E} - \mathbf{v}\times\mathbf{B} + \frac{\partial\mathbf{A}}{\partial t}\right) = 0\right]. \tag{69}$$

where here Maxwell wrote the vector potential (*electrotonic components*) as $\mathbf{A} = -(F, G, H)$. This leads to Maxwell's eq. (77),

$$\mathfrak{E} = \mathbf{v}\times\mathbf{B} - \boldsymbol{\nabla}\Psi - \frac{\partial\mathbf{A}}{\partial t} \qquad \left(P = \mu\gamma\frac{dy}{dt} - \mu\beta\frac{dz}{dt} + \frac{dF}{dt} - \frac{d\Psi}{dx},\ etc.\right) \tag{70}$$

where the "function of integration" Ψ was interpreted by Maxwell as the electric scalar potential (*electric tension*).

[46] I believe that our eq. (67) holds for a body that has translated by $\delta\mathbf{x}$, without rotation or deformation.

[47] In this section, which considers the *electromotive force* on a moving, unit charge, we use the symbol $\mathfrak{E}$ for this, rather than the symbol $\mathbf{E}$.

[48] The term $(\mathbf{H}\cdot\boldsymbol{\nabla})\,\delta\mathbf{x}$ was motivated by Maxwell's Props. IX and X, pp. 340-341 of [25], but is not evident to this author.

The sense of Maxwell's derivation is that $q\mathfrak{E}$ would be the force experienced (in the lab frame) by an electric charge in the moving body, *i.e.*,

$$\mathbf{F} = q\mathfrak{E} = q\left(\mathbf{v}\times\mathbf{B} - \boldsymbol{\nabla}\Psi - \frac{\partial\mathbf{A}}{\partial t}\right) = q\,(\mathbf{E}+\mathbf{v}\times\mathbf{B}) \qquad (\text{“Lorentz”}), \tag{71}$$

using the relation $\mathbf{E} = -\boldsymbol{\nabla}\Psi - \partial\mathbf{A}/\partial t$ for the electric field in the lab frame. Then, eq. (71) is the first statement of the "Lorentz" force law. However, Maxwell's argument seemed to have had little impact, perhaps due to the doubtful character of his argument leading to our eq. (68).

If the body were in uniform motion with velocity $\mathbf{v}$, $\mathfrak{E}$ could be interpreted as the electric field $\mathbf{E}'$ experienced by an observer moving with the body. Then, (in Gaussian units),

$$\mathbf{E}' = \mathbf{E} + \frac{\mathbf{v}}{c}\times\mathbf{B}, \tag{72}$$

which is the low-velocity Lorentz transformation of the electric field $\mathbf{E}$.

A.2.7 Faraday's Law, Revisited

On p. 343 of [25], Maxwell considered a moving conductor, and moving circuit, in a magnetic field, with no electric field in the lab frame. In his eqs. (78)-(79) he applied his eq. (77) to a segment of a moving conductor, finding,

$$\mathfrak{E}'\cdot d\mathbf{l} = \mathbf{v}\times\mathbf{B}\cdot d\mathbf{l} = -\mathbf{B}\cdot\mathbf{v}\times d\mathbf{l} = -\mathbf{B}\cdot\frac{d\mathbf{S}}{dt} \tag{73}$$
$$\left[e = S(Pl+Qm+Rn) = S\mu\alpha\left(m\frac{dz}{dt} - n\frac{dy}{dt}\right)\right],$$

where $\mathfrak{E}$ is the *electromotive force* vector with respect to the moving conductor, $d\mathbf{S}/dt = d\mathbf{x}/dt\times d\mathbf{l}$ is the area swept out by the moving line element, $d\mathbf{l}$ of the conductor in unit time.

In the case of a moving, closed circuit, the total (scalar) electromotive force is then,

$$\mathcal{E} = \oint \mathfrak{E}\cdot d\mathbf{l} = -\frac{d}{dt}\int \mathbf{B}\cdot d\mathbf{S} = -\frac{d\Phi_m}{dt}\,, \tag{74}$$

i.e., the total electromotive force in a closed conductor is measured by the change of the number of lines of force which pass through it; and this is true whether the change be produced by the motion of the conductor or by any external cause.

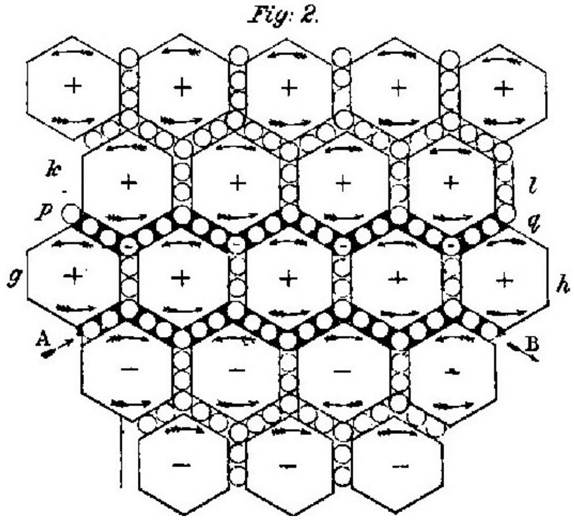
Since the above argument applies only to the case of a moving circuit, it does not demonstrate Maxwell's claim that our eq. (74) also holds for a circuit at rest with the *number of lines of force which pass through it* changing due to an *external cause*, *i.e.*, a time variation of the magnetic field $\mathbf{B}$. Presumably, Maxwell meant for the reader to recall his discussion on pp. 338-339 of [25] (sec. A.2.4 above): *This experiment shows that, in order to produce the electromotive force, it is not necessary that the conducting wire should be placed in a field of magnetic force, or that lines of magnetic force should pass through the substance of the wire or near it. All that is required is that lines of force should pass through the circuit of the conductor, and that these lines of force should vary in quantity during the experiment.*

A.2.8 Electric Currents in the Model of Molecular Vortices

On p. 13 of [26], Maxwell stated: *According to our theory, the particles which form the partitions between the cells constitute the matter of electricity. The motion of these particles constitutes an electric current; the tangential force with which the particles are pressed by the matter of the cells is electromotive force, and the pressure of the particles on each other corresponds to the tension or potential of the electricity.* Similarly, on p. 86 of [27], Maxwell stated: *in this paper I have regarded magnetism as a phenomenon of rotation, and electric currents as consisting of the actual translation of particles.*

Maxwell illustrated this vision in his Fig. 2, along with the description: *Let A B, P1. V. fig. 2, represent a current of electricity in the direction from A to B.*[49]

Fig: 2.



A contemporary version of this view of electric currents in magnetic matter is that there exist a "bound" current density therein, related by,

$$\mathbf{J}_{\text{bound}} = \boldsymbol{\nabla} \times \mathbf{M}, \tag{75}$$

where $\mathbf{M}$ is the density of magnetization (*i.e.*, of Ampèrian magnetic dipoles, which are "molecular" current loops). However, this relation does not appear in *On Physical Lines of Force*, where Maxwell seemed to have supposed that his vision applied to all media, including "vacuum", and not just to magnetic matter.

[49] On p. 283 of [24], Maxwell wrote: *"What is an electric current?"*

I have found great difficulty in conceiving of the existence of vortices in a medium, side by side, revolving in the same direction about parallel axes. The contiguous portions of consecutive vortices must be moving in opposite directions; and it is difficult to understand how the motion of one part of the medium can coexist with, and even produce, an opposite motion of a part in contact with it. The only conception which has at all aided me in conceiving of this kind of motion is that of the vortices being separated by a layer of particles, revolving each on its own axis in the opposite direction to that of the vortices, so that the contiguous surfaces of the particles and of the vortices have the same motion.

In mechanism, when two wheels are intended to revolve in the same direction, a wheel is placed between them so as to be in gear with both, and this wheel is called an "idle wheel". The hypothesis about the vortices which I have to suggest is that a layer of particles, acting as idle wheels, is interposed between each vortex and the next, so that each vortex has a tendency to make the neighbouring vortices revolve in the same direction with itself.

A.2.9 Displacement Current and Electromagnetic Waves

The most novel aspect of Maxwell's paper *On Physical Lines of Force* was his introduction of the "displacement current", and his deduction that the equations of electromagnetism then imply the existence of electromagnetic waves that propagate with the speed of light.

On p. 14 of [26], his discussion reads:

Electromotive force acting on a dielectric produces a state of polarization of its parts similar in distribution to the polarity of the particles of iron under the influence of a magnet, and, like the magnetic polarization, capable of being described as a state in which every particle has its poles in opposite conditions. In a dielectric under induction, we may conceive that the electricity in each molecule is so displaced that one side is rendered positively, and the other negatively electrical, but that the electricity remains entirely connected with the molecule, and does not pass from one molecule to another.

The effect of this action on the whole dielectric mass is to produce a general displacement of the electricity in a certain direction. This displacement does not amount to a current, because when it has attained a certain value it remains constant, but it is the commencement of a current, and its variations constitute currents in the positive or negative direction, according as the displacement is increasing or diminishing. The amount of the displacement depends on the nature of the body, and on the electromotive force; so that if h is the displacement (in the z-direction), *R the electromotive force, and E a coefficient depending on the nature of the dielectric,* $\quad R = -4\pi E^2 h$;
and if r is the value of the electric current (in the z-direction) *due to displacement,*

$$r_{\text{displacement}} = (-)\frac{dh}{dt} \quad \left(= \frac{1}{4\pi E^2}\frac{dR}{dt} \right), \tag{76}$$

where it seems to this author that a minus sign should be inserted in Maxwell's original version of our eq. (76).[50]

In the above, the *electromotive force* vector $(P, Q, R) = \mathbf{E}$ is the electric field, $E^2 = 1/\epsilon$ where ϵ is the relative permittivity (dielectric constant), the *displacement* vector $(f, g, h) = -\mathbf{D}/4\pi$ is proportional to our present electric field vector $\mathbf{D}$, and $(p, q, r) = \mathbf{J}_{\text{free}}$ is the free current density. Maxwell's relation $R = -4\pi E^2 h$ (repeated in his eq. (105), p. 18, of [24]) is equivalent to,

$$\mathbf{E} = \frac{\mathbf{D}}{\epsilon} . \tag{77}$$

Then, Maxwell's expression for the *electric current due to displacement* is equivalent to,

$$\mathbf{J}_{\text{displacement}} = \frac{d\mathbf{D}}{dt} . \tag{78}$$

On p. 19 of [26], Maxwell argued that *a variation of displacement is equivalent to a current, and this current* [$r_{\text{displacement}}$ of our eq. (76)] *must be taken into account in equations (9)* [our eq. (56)] *and added to r. The three equations then become,*

$$p = \frac{1}{4\pi}\left(\frac{d\gamma}{dy} - \frac{d\beta}{dz}\right) - \frac{1}{E^2}\frac{dP}{dt}, \ \textit{etc.} \qquad \left(\boldsymbol{\nabla} \times \mathbf{H} = \mathbf{J}_{\text{free}} + \frac{d\mathbf{D}}{dt}\right). \tag{79}$$

[50]For comments on reversals of signs in the relation between Maxwell's electric displacement and electric field, see [80].

This is the first statement of Maxwell's "fourth" equation as we know it today.

Maxwell next noted that the equation of continuity for free charge and current densities is, his eq. (113), p. 19 of [26],

$$\boldsymbol{\nabla}\cdot\mathbf{J}_{\rm free}+\frac{\partial\rho_{\rm free}}{\partial t}=0 \qquad \left(\frac{dp}{dx}+\frac{dq}{dy}+\frac{dr}{dz}+\frac{de}{dt}=0\right), \tag{80}$$

where $e=\rho_{\rm free}$ is the free charge density. On taking the divergence of eq. (79) and using eq. (80), we arrive at Maxwell's eqs. (114)-(115),

$$\frac{\partial}{\partial t}\boldsymbol{\nabla}\cdot\mathbf{D}=\frac{\partial\rho_{\rm free}}{\partial t}, \qquad \boldsymbol{\nabla}\cdot\mathbf{D}=\rho_{\rm free} \qquad \left[e=\frac{1}{4\pi E^2}\left(\frac{dP}{dx}+\frac{dQ}{dy}+\frac{dR}{dz}\right)\right], \tag{81}$$

which is the earliest statement of Maxwell's "first" equation.[51]

In Prop. XVI, p. 22 of [26], Maxwell considered *the rate of propagation of transverse vibrations through the elastic medium of which the cells are composed*, and found the constant E in our equation (79) to have a value remarkably close to the speed of light in vacuum, and concluded *that we can scarcely avoid the inference that light consists in the transverse undulations of the same medium which is the cause of electric and magnetic phenomena.*

A.2.10 *Electric Tension* and Poisson's Equation

A "sidelight" of Maxwell's discussion on p. 20 of [26] was his statement that in static electricity, the *electromotive force* (vector) can be related to the *electric tension* via his eq. (118), $\mathbf{E}=-\boldsymbol{\nabla}\Psi$. Further, with $\mathbf{E}=\mathbf{D}/\epsilon$, his eq. (119) (with a change of sign), and $\boldsymbol{\nabla}\cdot\mathbf{D}=\rho_{\rm free}$, his eq. (115), one has that the *tension* Ψ obeys Poisson's equation,

$$\nabla^2\Psi=-\frac{\rho_{\rm free}}{\epsilon}, \tag{82}$$

Maxwell's eq. (123) (with a change of sign).[52]

A.3 In *A Dynamical Theory of the Electromagnetic Field* [28]

A.3.1 Scalar Electromotive Force and the Integral Form of Faraday's Law

In sec. 24 of [28] Maxwell considered an electrical circuit A that carries current $I_{\rm A}=u$, and another circuit B that carries current $I_{\rm B}=v$, and noted that the magnetic flux, Φ_m, through circuit A is given by the first equation on p. 468,

$$\Phi_m=\int_{\rm A}\mathbf{B}\cdot d\mathbf{S}=LI_{\rm A}+MI_{\rm B} \qquad (Lu+Mv), \tag{83}$$

[51]Nowadays it is more common to argue that Maxwell's "first" and "fourth" equations together imply the continuity equation (80).

[52]As will be noted in Appendices A.3.10 and A.4.3 below, the relation (82) strictly holds only in the Coulomb gauge.

where $\mathbf{B}$ is the magnetic field, $d\mathbf{S}$ is an element of the area of a surface bounded by circuit A, L is the self inductance of circuit A, and M is the mutual inductance between circuits A and B.[53] Maxwell called this flux the *momentum*, or the *reduced momentum* of the circuit.

In sec. 50, Maxwell gave a verbal statement of Faraday's Law:
1st, If any closed curve be drawn in the field, the value of M for that curve will be expressed by the number of lines of force which pass through that closed curve.
2ndly. If this curve be a conducting circuit and be moved through the field, an electromotive force will act in it, represented by the rate of decrease of the number of lines passing through the curve.
We transcribe this into symbols as,

$$\mathcal{E} = -\frac{d\Phi_m}{dt}\,, \tag{84}$$

where $\mathcal{E}$ is the scalar *electromotive force.*

A.3.2 Vector *Electromagnetic Force* and Displacement Current

However, in sec. 56, Maxwell used the term *electromotive force* in a different way, to describe a vector, $\mathfrak{E} = (P, Q, R)$: *P represents the difference of potential per unit of length in a conductor placed in the direction of x at the given point.* This appears to mean that,

$$\mathfrak{E} = (P, Q, R) = -\boldsymbol{\nabla}\Psi, \tag{85}$$

where Ψ is Maxwell's symbol for the electric scalar potential. If so, this is the first mention of an aspect of the electric field $\mathbf{E}$ in [28], although he had introduced the *electrical displacement* $\mathbf{D} = (f, g, h)$ in sec. 55, along the with "displacement-current" (density), $(1/4\pi)\, d\mathbf{D}/dt$ in his eq. (A),

$$\mathbf{J}_{\text{total}} = \mathbf{J}_{\text{free}} + \frac{d\mathbf{D}}{dt} \qquad \left((p', q', r) = (p, q, r) + \frac{d(f, g, h)}{dt}\right), \tag{86}$$

where the free current density $\mathbf{J}_{\text{free}} = (p, q, r)$ was introduced in sec. 54, and the *total motion of electricity* is $\mathbf{J}_{\text{total}} = (p', q', r')$. Maxwell did not use separate symbols for partial and total derivatives, so that there can be some ambiguity as to his meaning when his equations describe moving systems.

A.3.3 Vector Potential aka *Electromagnetic Momentum*

In sec. 57, Maxwell introduced the vector potential $\mathbf{A} = (F, G, H)$, but called it the *electromagnetic momentum.* In his eq. (29), Maxwell identified $-d\mathbf{A}/dt$ with *the part of the electromotive force which depends on the motion of magnets or currents.* Thus, we might now presume that Maxwell's $\mathfrak{E} = (P, Q, R)$ of his sec. 56 is the electric field,

$$\mathbf{E} = -\boldsymbol{\nabla}\Psi - \frac{\partial\mathbf{A}}{\partial t}\,, \tag{87}$$

but this conclusion may be premature.

[53] This may be the first designation of L and M as self and mutual inductances.

A.3.4 Magnetic Flux aka Total *Electromagnetic Momentum of a Circuit*

In eq. (29) of sec. 58, Maxwell gave the relation for the magnetic flux Φ_m through a circuit, *the number of lines of magnetic force which pass through it*,

$$\left(\Phi_m = \int \mathbf{B} \cdot d\mathbf{S} = \right) \oint \mathbf{A} \cdot d\mathbf{l} \qquad \left[\int \left(F\frac{dx}{ds} + G\frac{dy}{ds} + H\frac{dz}{ds}\right) ds\right], \tag{88}$$

and called this the *total electromagnetic momentum* (which we must remember to distinguish from the *electromagnetic momentum* $\mathbf{A}$).

He also noted in sec. 58 that,

$$\left(\Phi_m = \oint \mathbf{A} \cdot d\mathbf{l} = \int\right) \boldsymbol{\nabla} \times \mathbf{A} \cdot d\mathbf{S} \qquad \left[\left(\frac{dH}{dy} - \frac{dG}{dz}\right) dy\, dz\right], \tag{89}$$

is the number of lines of magnetic force which pass through the area $dy\,dz$.

A.3.5 The Magnetic Fields H and B and Maxwell's Fourth Equation

In sec. 59, Maxwell introduced the magnetic field $\mathbf{H} = (\alpha, \beta, \gamma)$.

In sec. 60, Maxwell introduced the (relative) permeability μ, calling it the *coefficient of magnetic induction*.

In eq. (B) of sec. 61, Maxwell gave the *Equations for Magnetic Force*,

$$\mu\mathbf{H}\ (= \mathbf{B}) = \boldsymbol{\nabla} \times \mathbf{A} \qquad \left[\mu\alpha = \left(\frac{dH}{dy} - \frac{dG}{dz}\right),\ etc.\right]. \tag{90}$$

We use the symbol $\mathbf{B}$ for Maxwell's $\mu\mathbf{H}$.[54]

In eq. (C) of sec. 62, Maxwell gives a version of Ampère's Law,

$$\boldsymbol{\nabla} \times \mathbf{H} = \mathbf{J}_{\text{total}} \qquad \left(\frac{d\gamma}{dy} - \frac{d\beta}{dz} = 4\pi p',\ etc.\right), \tag{91}$$

recalling from our eq. (86) that Maxwell's vector (p', q', r') is the total current density $\mathbf{J}_{\text{total}} = \mathbf{J}_{\text{free}} + \mathbf{J}_{\text{displacement}}$.

A.3.6 *Electromotive Force in a Circuit* at Rest

In eq. (32), sec. 63 of [28], Maxwell stated that the *electromotive force acting round* an electrical circuit is related by,[55]

$$\mathcal{E} = \oint \mathfrak{E} \cdot d\mathbf{l} \qquad \left[\xi = \int \left(P\frac{dx}{ds} + Q\frac{dy}{ds} + R\frac{dz}{ds}\right) ds\right]. \tag{92}$$

[54] The symbol $\mathbf{B}$ for the quantity $\mu_0\mathbf{H} + \mathbf{M}$, where $\mathbf{M}$ is the magnetization density, was introduced by W. Thomson in 1871, eq. (r), p. 401 of [45], and appears in Art. 399 of Maxwell's *Treatise* [55].

[55] This meaning of the term *electromotive force* is still in use today. However, Maxwell also used the term *electromotive force* in sec. 65 of [28] to describe the force $\mathbf{v} \times \mu\mathbf{H}$ on a moving, unit charge in a magnetic field, referring to his eq. (D).

Supposing this circuit, A, carries current $I_{\rm A} = u$, and another circuit, B, carries current $I_{\rm B} = v$, Maxwell, in his eq. (33), reminded us the magnetic flux through circuit A is given by our eq. (83),

$$\Phi_m = \oint_{\rm A} \mathbf{A} \cdot d\mathbf{l} = LI_{\rm A} + MI_{\rm B} \qquad \left[\int \left(F\frac{dx}{ds} + G\frac{dy}{ds} + H\frac{dz}{ds}\right) ds = Lu + Mv\right], \tag{93}$$

as he had previously discussed in sec. 24. Then, his eq. (34) states that,

$$\mathcal{E} = -\frac{d}{dt}(LI_{\rm A} + MI_{\rm B}) \quad \left(= -\int \frac{d\mathbf{A}}{dt} \cdot d\mathbf{l}\right), \tag{94}$$

so comparison with our eq. (92) leads to the inference, Maxwell's eq. (35), that *if there is no motion of the circuit A*,

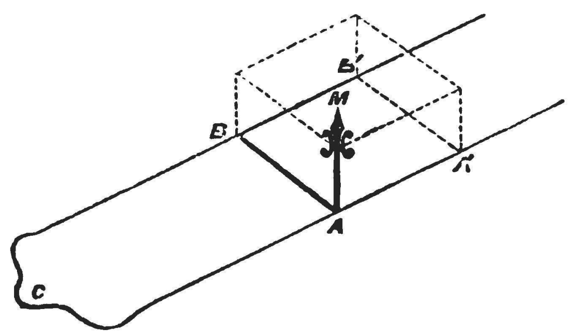
$$\mathfrak{E} = -\frac{\partial \mathbf{A}}{\partial t} - \boldsymbol{\nabla}\Psi \qquad \left(P = -\frac{dF}{dt} - \frac{d\Psi}{dx},\ etc.\right), \tag{95}$$

where Ψ could be any scalar function. But, the discussion in his sec. 56 led Maxwell to identify Ψ of eq. (95) as the electrical scalar potential.

In eq. (95), we have written $\partial\mathbf{A}/\partial t$, while Maxwell wrote $d\mathbf{A}/dt$, in that for an observer (at rest) of a circuit at rest, use of a convective derivative is not appropriate.

A.3.7 The Vector *Electromotive Force on a Moving Conductor*

In sec. 64, Maxwell deduced the force on a bar the slides on a U-shaped rail, while carrying a current, with the entire system in an external magnetic field. He gave no figure in [28], but the figure below is associated with his discussion of this example in Arts. 594-597 of [55]. C represents a battery that drives the current in the circuit.



Maxwell desired a very general discussion in sec. 64, so he considered a circuit whose plane was not perpendicular to any of the x, y or z axes, which makes his description rather intricate. Here, we suppose the circuit lies in the x-z plane, with the sliding piece, AB, of length a parallel to the x-axis, and the long arms of the U-shaped rail parallel to the z-axis, at, say $x = 0$ and a. The velocity $v_z = dz/dt$ of the sliding bar is in the z-direction, and the uniform, external magnetic field is in the y-direction.

As in sec. 63, Maxwell considered changes in the magnetic flux through the circuit, $\oint \mathbf{A} \cdot d\mathbf{l}$, our eq. (88), to infer the strength of his vector $\mathfrak{E}$. The part of the line integral over the sliding bar changes at rate,[56]

[56]In eqs.(96)-(99), the quantities A_x, A_z and B_y are evaluated at the location of the sliding bar.

$$a\frac{dA_x}{dz}\frac{dz}{dt}\,, \tag{96}$$

as indicated in the first equation on p. 485. Because the length of the circuit in z is increasing, the line integral also changes at rate,

$$\frac{dz}{dt}[A_z(x=0) - A_z(x=a)] = -\frac{dz}{dt}\frac{dA_z}{dx}a, \tag{97}$$

as given in the second equation on p. 485. Hence, the total rate of change of magnetic flux, given in the third and fourth equations on p. 485, is,[57]

$$\frac{d\Phi_m}{dt} = av_z\left(\frac{dA_x}{dz} - \frac{dA_z}{dx}\right) = av_z B_y = -\mathcal{E} = -\oint \mathfrak{E}\cdot d\mathbf{l}. \tag{98}$$

Maxwell considered that eq. (98) describes a contribution to the *electromotive force* $\mathfrak{E}$ beyond that in eq. (95), which additional contribution would be localized to the component $\mathfrak{E}_x$ along the sliding bar (of length a), *i.e.*, $\oint \mathfrak{E}\cdot d\mathbf{l} = a\mathfrak{E}_x$. Hence, he concluded in his eq. (36) that,

$$\mathfrak{E}_x = -v_z B_y \qquad \left(P = -\mu\beta\frac{dz}{dt}\right), \qquad i.e., \qquad \mathfrak{E} = \mathbf{v}\times\mathbf{B}, \tag{99}$$

is the part of $\mathfrak{E}$ due to the motion of the sliding bar.

Finally, in sec. 65, Maxwell stated that the total *electromotive force on a moving conductor* is his eq. (D),

$$\mathfrak{E} = \mathbf{v}\times\mathbf{B} - \frac{\partial\mathbf{A}}{\partial t} - \boldsymbol{\nabla}\Psi \qquad \left[P = \mu\left(\gamma\frac{dy}{dt} - \beta\frac{dz}{dt}\right) - \frac{dF}{dt} - \frac{d\Psi}{dx},\ etc.\right], \tag{100}$$

recalling his eq. (35), our eq. (95). Again, we have written $\partial\mathbf{A}/\partial t$ where Maxwell wrote $d\mathbf{A}/dt$.

Note that Maxwell's argument in sec. 65 does not address the mechanical force on the sliding bar, $I\mathbf{a}\times\mathbf{B}$, *which is the subject of most present discussions of this example.*

A.3.8 Other *General Equations of the Electromagnetic Field*

For completeness, we briefly record other general considerations by Maxwell in [28].

In sec. 66, Maxwell's eq. (66), $P = kf$, *etc.*, would seem to be the equivalent of $\mathbf{D} = \epsilon\mathbf{E}$, for the displacement field $\mathbf{D} = (f, g, h)$, the electric field $\mathbf{E} = (P, Q, R)$, and the (relative) permittivity $\epsilon = 1/k$.

However, in sec. 67 we read that Ohm's law can be written as $P = -\varrho p$, *etc.*, which would seem to imply that $\mathbf{E} = -\varrho\mathbf{J}$, where ϱ is the electrical resistivity (= 1 / conductivity) and $\mathbf{J} = (p, q, r)$ is the electric current density.[58] Then in sec. 68 we read that,

$$e + \frac{df}{dx} + \frac{dg}{dy} + \frac{dh}{dz} = 0 \qquad (\boldsymbol{\nabla}\cdot\mathbf{D} = -\rho), \tag{101}$$

[57]On p. 485, *the equations of Magnetic Force (8)* should read: *the equations of Magnetic Force (B)*.

[58]There seems to be a minus sign "loose" here. In a draft of [28], p. 160 of [89], Ohm's law was written as $P = \varrho p$, *etc.*, and in Art. 609 of [55], eq. (G) reads $\mathfrak{J} = C\mathfrak{E}$, ($\mathbf{J} = \sigma\mathbf{E}$).

and in sec. 69 that the equation of continuity is,

$$\frac{de}{dt} + \frac{dp}{dx} + \frac{dq}{dy} + \frac{dr}{dz} = 0 \qquad \left(\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot \mathbf{J} = 0\right), \tag{102}$$

where $e = \rho$ is the electric charge density. It appears that either $\mathbf{E} = -(P, Q, R)$ and $\mathbf{D} = -(f, g, h)$, or $\rho = -e$.[59]

Fortunately, in Arts. 604-619 of his Treatise [55], Maxwell presented these relations with signs that match our present usage.

A.3.9 *Electromagnetic Theory of Light*

Also for completeness, we include remarks on secs 91-100 of [28], where Maxwell presented his electromagnetic theory of light.

He considered plane waves in a linear, nonconducting medium with permittivity ϵ and permeability μ, such that $\mathbf{D} = \epsilon \mathbf{E}$ and $\mathbf{B} = \mu \mathbf{H}$.[60] The waves propagated in direction $\hat{\mathbf{n}}$ with speed v, such that the wave fields were only functions of the single scalar $\varphi = \hat{\mathbf{n}} \cdot \mathbf{x} - vt$, as noted at the beginning of sec. 92.

For these waves, the relation between the magnetic field $\mathbf{B}$ and the vector potential $\mathbf{A}$ can be written as,

$$\mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A} = \hat{\mathbf{n}} \times \frac{\partial \mathbf{A}}{\partial \varphi}, \tag{103}$$

Consequently, $\hat{\mathbf{n}} \cdot \mathbf{B} = 0$, and the magnetic field vector is transverse to the direction of propagation of the wave, as noted in eq. (62), sec. 92.

In sec. 93, Maxwell used his equation $\boldsymbol{\nabla} \times \mathbf{H} = \mathbf{J}_{\rm total}$, noting that in the nonconducting medium the only current is the displacement current $\partial \mathbf{D}/\partial t = \epsilon \partial \mathbf{E}/\partial t$, and that the electric field can be written in terms of potentials as $\mathbf{E} = -\partial \mathbf{A}/\partial t - \boldsymbol{\nabla}\Psi$. Then, he wrote,

$$\begin{aligned} \boldsymbol{\nabla} \times \mathbf{H} &= \frac{\boldsymbol{\nabla} \times \mathbf{B}}{\mu} = \frac{\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \mathbf{A})}{\mu} = \frac{\boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \mathbf{A}) - \nabla^2 \mathbf{A}}{\mu} \\ &= \mathbf{J}_{\rm total} = \frac{\partial \mathbf{D}}{\partial t} = \epsilon \frac{\partial \mathbf{E}}{\partial t} = -\epsilon \frac{\partial}{\partial t}\left(\frac{\partial \mathbf{A}}{\partial t} + \boldsymbol{\nabla}\Psi\right), \end{aligned} \tag{104}$$

which corresponds to Maxwell's eq. (68) of sec. 94. On taking the curl of this, and replacing $\boldsymbol{\nabla} \times \mathbf{A}$ by $\mathbf{B}$, he found,

$$\nabla^2 \mathbf{B} = \epsilon\mu \frac{\partial^2 \mathbf{B}}{\partial t^2} \qquad \left[k \nabla^2 \mathbf{B} = 4\pi\mu \frac{d^2 \mathbf{B}}{dt^2}\right], \tag{105}$$

his eq. (69). Hence, the wavespeed is $v = 1/\sqrt{\epsilon\mu}$ $\quad [V = \sqrt{k/4\pi\mu}]$, which in air/vacuum is extremely close to the speed of light.

[59]This issue here is likely related to Maxwell's vision of electric charge as an aspect of the displacement field $\mathbf{D}$, which leads to a concept of charge density that is the negative of our present view.

[60]In this and the following section we use SI units.

Maxwell did not comment on the electric field of the wave, saying (sec. 95) instead that *this wave consists entirely of magnetic disturbances*.[61,62]

A.3.10 Waves of the Potentials, Coulomb Gauge

In secs. 98-99 of [28], Maxwell considers waves of the potentials. He reminded the reader that his discussion in sec. 94, our eq. (104), could emphasize the potentials rather than the magnetic field, resulting in the wave equation,

$$\boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\mathbf{A}) - \nabla^2\mathbf{A} = -\epsilon\mu\left(\frac{\partial^2\mathbf{A}}{\partial t^2} + \boldsymbol{\nabla}\frac{\partial\Psi}{\partial t}\right). \tag{108}$$

Perhaps taking inspiration from sec. 82 of [14] by W. Thomson (1950), Maxwell noted that one can make a (gauge) transformation of the vector potential according to his eq. (74), p. 500,

$$\mathbf{A}' = \mathbf{A} - \boldsymbol{\nabla}\chi, \tag{109}$$

such that $\mathbf{B} = \boldsymbol{\nabla}\times\mathbf{A} = \boldsymbol{\nabla}\times\mathbf{A}'$, and the scalar potential should be transformed to his eq. (77), where our Ψ' is Maxwell's ϕ,

$$\Psi' = \Psi + \frac{\partial\chi}{\partial t}, \tag{110}$$

such that $\mathbf{E} = -\partial\mathbf{A}/\partial t - \boldsymbol{\nabla}\Psi = -\partial\mathbf{A}'/\partial t - \boldsymbol{\nabla}\Psi'$.

In particular, Maxwell noted that since $\boldsymbol{\nabla}\cdot\mathbf{A}' = \boldsymbol{\nabla}\cdot\mathbf{A} - \nabla^2\chi$, we can have $\boldsymbol{\nabla}\cdot\mathbf{A}' = 0$, his eq. (75), by taking $\nabla^2\chi = \boldsymbol{\nabla}\cdot\mathbf{A}$, his eq. (73). With this choice of the gauge function χ, the potentials $\mathbf{A}'$ and Ψ' are in the Coulomb gauge (which Maxwell had already favored in 1861, p. 290 of [25]). The wave equation for the vector potential in the Coulomb gauge follows from eq. (108) as,

$$\nabla^2\mathbf{A}' - \epsilon\mu\frac{\partial^2\mathbf{A}'}{\partial t^2} - = \epsilon\mu\boldsymbol{\nabla}\frac{\partial\Psi'}{\partial t} \qquad \text{(Coulomb gauge)}. \tag{111}$$

However, Maxwell stated in his eq. (78) that *we find* the right side of eq. (111) to be zero.

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[61] Maxwell could we have added an argument for the electric field by noting that,

$$\boldsymbol{\nabla}\times\mathbf{H} = \hat{\mathbf{n}}\times\frac{\partial\mathbf{H}}{\partial\varphi} = \epsilon\frac{\partial\mathbf{E}}{\partial t} = -\epsilon v\frac{\partial\mathbf{E}}{\partial\varphi}, \quad \Rightarrow \quad \hat{\mathbf{n}}\times\mathbf{H} = \epsilon v\mathbf{E}, \quad \text{and} \quad \hat{\mathbf{n}}\cdot\frac{\partial\mathbf{E}}{\partial\varphi} = \boldsymbol{\nabla}\cdot\mathbf{E} = 0, \tag{106}$$

and then taking the curl of Faraday's law to find,

$$\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\mathbf{E}) = \boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\mathbf{E}) - \nabla^2\mathbf{E} = -\nabla^2\mathbf{E} = -\frac{\partial}{\partial t}\boldsymbol{\nabla}\times\mu\mathbf{H} = -\epsilon\mu\frac{\partial^2\mathbf{E}}{\partial t^2}, \tag{107}$$

such that $\mathbf{E}$ is transverse to $\hat{\mathbf{n}}$ and $\mathbf{B}$, and has wave propagation at speed $v = 1/\sqrt{\epsilon\mu}$.

[62] Since Maxwell's $k/4\pi$ (our ϵ) and μ are constants determined by laboratory experiments on static systems, he could have remarked that the speed of light is the same to any (inertial) observer.

To see that $\Psi' = 0$ (and hence that $\boldsymbol{\nabla}\partial\Psi/\partial t = 0$) in Maxwell's example, one can transcribe Maxwell's eq. (G) of sec. 65, $\boldsymbol{\nabla}\cdot\mathbf{D} = \rho_{\rm free}$, together with eq. (E) of sec. 66 that $\mathbf{D} = \epsilon\mathbf{E}$, and eq. (35) of sec. 63 that $\mathbf{E} = -\boldsymbol{\nabla}\Psi - \partial\mathbf{A}/\partial t$, as,[63]

$$-\frac{\boldsymbol{\nabla}\cdot\mathbf{D}}{\epsilon} = -\boldsymbol{\nabla}\cdot\mathbf{E} = \nabla^2\Psi + \frac{\partial}{\partial t}\boldsymbol{\nabla}\cdot\mathbf{A} = -\frac{\rho_{\rm free}}{\epsilon}\,, \tag{112}$$

such that in the Coulomb gauge, where $\boldsymbol{\nabla}\cdot\mathbf{A}' = 0$, we have that

$$\nabla^2\Psi' = -\frac{\rho_{\rm free}}{\epsilon} \qquad \text{(Coulomb gauge)}. \tag{113}$$

For Maxwell's example of plane waves, $\rho_{\rm free} = 0$, such that $\Psi' = \int(\rho_{\rm free}/\epsilon r)\,d\text{Vol} = 0$. Maxwell gave the reader little clue of this lore, although one infers that he was aware of it.

In sec. 99 Maxwell argued for a stronger result, which we now report as that for plane waves, $\mathbf{E} = -\partial\mathbf{A}'/\partial t$ (and $\mathbf{B} = \boldsymbol{\nabla}\times\mathbf{A}'$) in any gauge (where the potentials are $\mathbf{A} = \mathbf{A}' + \boldsymbol{\nabla}\chi$ and $\Psi = \Psi' - \partial\chi/\partial t$). To this author, Maxwell derivation was not convincing.[64] However, the result follows from the condition that $\boldsymbol{\nabla}\cdot\mathbf{E} = 0$ for plane waves, and Helmholtz' theorem (Appendix A.1.5 above), such that $\mathbf{E} = \mathbf{E}_{\rm rot} = -\partial\mathbf{A}_{\rm rot}/\partial t$, where $\boldsymbol{\nabla}\cdot\mathbf{A}_{\rm rot} = 0$, *i.e.*, $\mathbf{A}_{\rm rot} = \mathbf{A}'$ = the Coulomb-gauge vector potential. Then, $0 = \mathbf{E}_{\rm irr} = -\partial\mathbf{A}_{\rm irr}/\partial t - \boldsymbol{\nabla}\Psi$, so while in general the potentials $\mathbf{A}_{\rm irr}$ and Ψ can be nonzero, they do not contribute to the plane wave.[65]

A.4 In Maxwell's *Treatise* [34, 35, 54, 55]

Maxwell's discussion of the "Lorentz" force law in Arts. 598-601 of his *Treatise* has been reviewed in sec. 1 above. Maxwell's presentation in his earlier papers of this topic (and of other aspects of his novel vision of electrodynamics) is perhaps superior to that in his *Treatise*.[66]

Here, we add some comments on the electric potential, on Art. 619, and on Maxwell's electromagnetic theory of light.

[63]Our eq. (112) is Maxwell's eq. (79), sec. 99, except that he had Ψ' (his ϕ) in place of Ψ (and $J = \boldsymbol{\nabla}\cdot\mathbf{A}$).

[64]Last-minute corrections by Maxwell to sec. 99 of [28] are discussed in [77]. See also p. 203 of [81].

[65]In any region where $\boldsymbol{\nabla}\cdot\mathbf{E} \approx 0$, such as far from all sources of the electromagnetic fields, these fields can be deduced only from a Coulomb-gauge vector potential to a good approximation.

[66]On p. 300 of Whittaker's *History of the Theories of Aether and Electricity* [73], one reads about Maxwell: *In 1871 he returned to Cambridge as Professor of Experimental Physics; and two years later published his* Treatise on Electricity and Magnetism. *In this celebrated work is comprehended almost every branch of electric and magnetic theory; but the intention of the writer was to discuss the whole as far as possible from a single point of view, namely, that of Faraday; so that little or no account was given of the hypotheses which had been propounded in the two preceding decades by the great German electricians. So far as Maxwell's purpose was to disseminate the ideas of Faraday, it was undoubtedly fulfilled; but the* Treatise *was less successful when considered as the exposition of its author's own views. The doctrines peculiar to Maxwell—the existence of displacement-currents, and of electromagnetic vibrations identical with light were not introduced in the first volume, or in the first half of the second volume; and the account which was given of them was scarcely more complete, and was perhaps less attractive, than that which had been furnished in the original memoirs.*

A.4.1 Articles 70-77, *On Potential Functions*

At the end of Art. 70 of [54], Maxwell wrote: *Definition of Potential. The Potential at a Point is the work which would be done on a unit of positive electricity by the electric forces if it were placed at the point without disturbing the electric distribution, and carried from that point to an infinite distance; or, what comes to the same thing, the work which must be done by an external agent in order to bring the unit of positive electricity from an infinite distance (or from any place where the potential is zero) to the given point.*

This statement is in Vol. 1 of Maxwell's *Treatise*, which deals only with static phenomenon. However, even in Vol. 2, Maxwell never acknowledged that this definition of potential is ill defined when time-dependent magnetic fields are involved, such that the work done depends on the path.

In Art. 73, *Potential due to any Electrical System*, Maxwell stated that the electric potential V can be computed from the *electric density* ρ (tacitly, of free charge) according to,

$$V = \int \frac{\rho_{\rm free}}{4\pi\epsilon\, r}\, d{\rm Vol}. \tag{114}$$

In Art. 77, *On the Equations of Laplace and Poisson*, Maxwell stated that the potential V obeys Poisson's equation,

$$\nabla^2 V = -\frac{\rho_{\rm free}}{\epsilon}\,, \tag{115}$$

in present notation (Maxwell defined his symbol ∇^2 to be the negative of ours).

Equations (114)-(115) can be taken as a matter of definition in time-dependent examples, which corresponds to use of the Coulomb gauge. The wording of the *Treatise* is consistent throughout with these equations, *i.e.*, with the choice of the Coulomb gauge.[67]

A.4.2 Article 619, *Quaternion Expressions for the Electromagnetic Equations*

Article 619 of [35] is meant as a summary of Maxwell's theory, but the transcription of material in Arts. 598-603 was awkward. This was noticed by FitzGerald [43], who attempted to improve the story, but perhaps did not succeed. FitzGerald's comments were incorporated in Art. 619 of the 3rd edition of the *Treatise*, but some typos were also introduced.

Article 619 mentions that the vector potential is subject to the condition $\boldsymbol{\nabla}\cdot\mathbf{A} = 0$, which is a choice, not a requirement, and corresponds to the use of the Coulomb gauge by Maxwell.[68] We have previously remarked how with this choice the scalar potential Ψ obeys the instantaneous Poisson equation $\nabla^2\Psi = -e/\epsilon$, where e is the (free) electric-charge density, as acknowledged by Maxwell in Art. 783.

The results of Arts. 598-599, Maxwell's eq. (B) and our eq. (8), are then reproduced in Art. 619.

[67] It may be that when Maxwell wrote in Art. 598 that Ψ *represents, according to a certain definition, the electric potential*, he had in mind the definitions of our eqs. (114)-(115) rather than that of Art. 70.

[68] Maxwell's preference that $\boldsymbol{\nabla}\cdot\mathbf{A} = 0$ was indicated already in eq. (57), p. 290 of [25], but without justification. He expressed this preference again in Arts. 616-617, where the context is magnetostatics.

However, the next sentence in Art. 619 is problematic: *The equations (C) of mechanical force (*Art. 603*), of which the first is,*

$$X = cv - bw - e\frac{d\Psi}{dx} - m\frac{d\Omega}{dx}\,, \tag{116}$$

become,

$$\mathfrak{F} = \mathfrak{J} \times \mathfrak{B} - e\boldsymbol{\nabla}\Psi - m\boldsymbol{\nabla}\Omega. \tag{117}$$

Article 603 presented only that the force density on a conduction current $\mathbf{J}_{\text{cond}}$ ($\mathfrak{K}$) is $\mathbf{f} = \mathbf{J}_{\text{cond}} \times \mathbf{B}$ ($\mathfrak{F} = \mathfrak{J} \times \mathfrak{B}$), but in Art. 619 Maxwell indicates that $\mathfrak{J} = \mathfrak{K} + \dot{\mathfrak{D}}$, ($\mathbf{J}_{\text{total}} = \mathbf{J}_{\text{cond}} + \mathbf{J}_{\text{displacement}}$).[69] This is problematic in that there is no mechanical force on the electric-field part of displacement $\mathbf{D} = \mathbf{E} + \epsilon_0 \mathbf{P}$, where $\mathbf{P}$ is the volume density of electric dipoles (a concept not recognized by Maxwell, and only introduced much later by Lorentz).[70]

Maxwell added a term $-m\boldsymbol{\nabla}\Omega$ to his expression for the *mechanical force*. From the last sentence in Art. 619, we infer he had in mind the special case of permanent magnetism, described by a volume density $\mathfrak{I}$ ($\mathbf{M}$) of magnetic dipoles, with a corresponding volume density $m = \boldsymbol{\nabla} \cdot \mathfrak{I}$ ($= \boldsymbol{\nabla} \cdot \mathbf{M}$), such that the magnetic field $\mathfrak{H}$ ($\mathbf{H}$) $= -\boldsymbol{\nabla}\Omega$ can be deduced from a magnetic scalar potential Ω. This special case is not strictly compatible with the existence of an electric current density $\mathfrak{J}$ in the term $\mathfrak{J} \times \mathfrak{B}$.[71]

Maxwell also included a term $-e\boldsymbol{\nabla}\Psi$ in his expression for the *mechanical force*. This is clearly not the general case for electric mechanical forces on an electric charge density e, but would apply if the charge density were static, and the corresponding electric field related to a scalar potential, $\mathbf{E} = -\boldsymbol{\nabla}\Psi$. FitzGerald (1883) [43] noted that in general the electric force on charge density e is due to the total electric field $-\boldsymbol{\nabla}\Psi - \partial\mathbf{A}/\partial t$, and so suggested that $-e\boldsymbol{\nabla}\Psi$ be replaced by $e\mathfrak{E}$. This would be valid if Maxwell's $\mathfrak{E}$ represented the (lab-frame) electric field. However, the point of Arts. 598-599 was that the symbol $\mathfrak{E}$ does not represent the (lab-frame) electric field, but rather the total electric force in case of moving charge. Nonetheless, FitzGerald's suggestion was implemented in Art. 619 of the 3rd

[69] In the 3rd edition of the *Treatise*, the relation $\mathfrak{J} = \mathfrak{K} + \dot{\mathfrak{D}}$ was mistyped as $\mathfrak{E} = \mathfrak{K} + \mathfrak{D}$.

[70] Maxwell only regarded the relation between $\mathbf{D}$ and $\mathbf{E}$ as $\mathbf{D} = \epsilon\mathbf{E}$, where ϵ is now called the (relative) dielectric constant and/or the (relative) permittivity. See Art. 111 of [54] for Maxwell's use of the term *polarization*.

In 1885, Heaviside introduced the concept of an electret as the electrical analog of a permanent magnet [47], and proposed that the electrical analog of magnetization (density) be called electrization. He did not propose a symbol for this, nor did he write an equation such as $\mathbf{D} = \mathbf{E} + \epsilon_0\mathbf{P}$.

The density of electric dipoles was called the polarization by Lorentz (1892) in sec. 102, p. 465 of [56], and assigned the symbol $\mathbf{M}$.

Larmor (1895), p. 738 of [58], introduced the vector (f', g', h') for what is now written as the polarization density $\mathbf{P}$, and related it to the electric field $\mathbf{E} = (P, Q, R)$ as $(f', g', h') = (K - 1)(P, Q, R)/4\pi$, *i.e.*, $\mathbf{P} = (\epsilon - 1)\mathbf{E}/4\pi = (\mathbf{D} - \mathbf{E})/4\pi$. Larmor's notation was mentioned briefly on p. 91 of [61] (1898).

The symbol $\mathbf{M}$ for dielectric polarization was changed to $\mathbf{P}$ by Lorentz on p. 263 of [65] (1902), and a relation equivalent to $\mathbf{D} = \mathbf{E} + \epsilon_0\mathbf{P}$ was given in eq. (22), p. 265. See also p. 224, and eq. (147), p. 240 of [66] (1903), which latter subsequently appeared as eq. (142), p. 155 of the textbook [67] (1904) by Abraham.

[71] The computation of mechanical forces due to magnetism is subject to ambiguities that persist in the literature to this day. For further comments by the author on this topic, see [94].

edition of Maxwell's *Treatise* [55], which has the effect that the revised expression for the *mechanical force* includes $2\mathfrak{J} \times \mathfrak{B}$.[72]

A.4.3 Articles 781-805: *Electromagnetic Theory of Light*

Article 783

In Art. 783 of [35], Maxwell set the stage for discussion of electromagnetic waves other than plane waves, and made a slight generalization of his discussion of the electromagnetic theory of light in secs. 91-99 of [28], by considering currents in a medium with electrical conductivity σ. Then, eqs. (1) and (2) of Art. 783 combine to give the second line of our eq. (104) becomes,[73]

$$\mathbf{J}_{\text{total}} = \sigma\mathbf{E} + \frac{\partial\mathbf{D}}{\partial t} = \sigma\mathbf{E} + \epsilon\frac{\partial\mathbf{E}}{\partial t} = -\left(\sigma + \epsilon\frac{\partial}{\partial t}\right)\left(\frac{\partial\mathbf{A}}{\partial t} + \boldsymbol{\nabla}\Psi\right), \tag{118}$$

which generalizes the second line of our eq. (104), and our wave equation (108) becomes,

$$\boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\mathbf{A}) - \nabla^2\mathbf{A} = -\mu\left(\sigma + \epsilon\frac{\partial}{\partial t}\right)\left(\frac{\partial\mathbf{A}}{\partial t} + \boldsymbol{\nabla}\Psi\right), \tag{119}$$

which is Maxwell's eq. (6) of Art. 783, noting that in the *Treatise*, his symbol ∇^2 is the negative of ours. Maxwell had also deduced this relation, but for the case of a nonconducting medium, as eq. (68), sec. 94 of [28].

In [28], Maxwell then took to the curl of eq. (119) (with $\sigma = 0$) to find a wave equation for the magnetic field, with wavespeed $1/\sqrt{\epsilon\mu}$.

In Art. 783, Maxwell took the divergence of his eq. (6), our eq. (119), to find his eq. (8), which we write as,

$$\mu\left(\sigma + \epsilon\frac{\partial}{\partial t}\right)\left(\frac{\partial}{\partial t}(\boldsymbol{\nabla}\cdot\mathbf{A}) + \nabla^2\Psi\right) = 0. \tag{120}$$

Maxwell's next sentence included the phrase: $\nabla^2\Psi$ *which is proportional to the volume-density of free electricity*, as holds in electrostatics. Here, Maxwell supposes that even in time-dependent examples, the scalar potential obeys $\nabla^2\Psi = -\rho_{\text{free}}/\epsilon_0$, as he discussed in Art. 77 for the static case. We could say that this assumption presumes use of the Coulomb gauge ($\boldsymbol{\nabla}\cdot\mathbf{A} = 0$), but in Art. 783 Maxwell appeared to deduce the Coulomb-gauge condition from his assumption. That is, he stated that in a nonconducting medium any free electricity is at rest, such that $\nabla^2\Psi$ *in independent of t*, and *hence J (= $\boldsymbol{\nabla}\cdot\mathbf{A}$) must be a linear function of t, or constant, or zero.*[74]

[72] This unfortunate "improvement" must have contributed to the impression in the late 1800's that Maxwell's theory of electromagnetism was hard to follow.

[73] We use SI units in this section.

[74] The case of a vector potential that is a linear function of time finds application in electrostatics, where one can set $\Psi = 0$ and $\mathbf{A} = -\mathbf{E}\,t$, which is the so-called Gibb's gauge [60, 100]. Of course, there is no wave propagation in this case.

Maxwell concluded Art. 783 with the statement: *we may leave* J $(= \boldsymbol{\nabla} \cdot \mathbf{A})$ *and* Ψ *out of account when considering periodic disturbances*. This claim happens to be true for plane waves, as noted at the end of Appendix A.3.10 above, but is not so in general.[75]

In Art. 783, Maxwell appears to wish to show that waves of the vector potential also propagate with speed $1/\sqrt{\epsilon\mu}$. In a sense, eq. (119) already shows this, if we rewrite it as,

$$\nabla^2 \mathbf{A} - \epsilon\mu \frac{\partial^2 \mathbf{A}}{\partial t^2} = \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \mathbf{A}) + \epsilon\mu \frac{\partial \boldsymbol{\nabla}\Psi}{\partial t} + \sigma \frac{\partial}{\partial t}\left(\frac{\partial \mathbf{A}}{\partial t} + \boldsymbol{\nabla}\Psi\right), \tag{123}$$

which linear differential equation indicates that at least part of the vector potential propagates at speed $1/\sqrt{\epsilon\mu}$. But, it seems that Maxwell wished to show that there could be no other part of the vector potential that propagates with a different speed.[76]

We now know that this cannot be shown, in that one can adopt the so-called velocity gauge (see, for example, sec. 2.3.1 of [104]) in which the scalar potential Ψ propagates with any specified speed v, and the corresponding vector potential $\mathbf{A}$ has a term that propagates at speed $1/\sqrt{\epsilon\mu}$, and another of the form $\boldsymbol{\nabla}\phi$ which propagates at speed v, such that the electric field $\mathbf{E} = -\partial\mathbf{A}/\partial t - \boldsymbol{\nabla}\Psi$ and the magnetic field $\mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A}$ propagate only with speed $1/\sqrt{\epsilon\mu}$.[77] Of course, in the velocity gauge, $\nabla^2\Psi$ is not proportional to $\rho_{\rm free}$, so Maxwell's apparent assumption in Art. 783 of this relation excluded use of a velocity gauge, except for the Coulomb gauge (with $v = \infty$).

Article 784

In Art. 783, Maxwell's eq. (9) gives the wave equation for the vector potential in the Coulomb gauge (or the physically trivial variants) and in a nonconducting medium.

Article 785

Article 785 is interesting in that Maxwell considered spherical waves from a localized source, and noted that a distant observer detects wave associated with earlier behavior at

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[75]See, for example, Prob. 2 of [92], where eqs. (91)-(92) give the Coulomb-gauge potentials for an oscillating (Hertzian) electric dipole $\mathbf{p} = \mathbf{p}_0\, e^{-i\omega t}$ at the origin as,

$$\Psi = \mathbf{p}_0 \cdot \hat{\mathbf{r}} \frac{e^{-i\omega t}}{4\pi\epsilon_0\, r^2} \qquad \text{(Coulomb gauge)}, \tag{121}$$

$$\begin{aligned} \mathbf{A} &= -\frac{i}{k}\mathbf{E} - \frac{i}{k}\boldsymbol{\nabla}\Psi \qquad \text{(Coulomb gauge)} \qquad (122) \\ &= -ik\,\hat{\mathbf{r}} \times (\mathbf{p}_0 \times \hat{\mathbf{r}}) \frac{e^{i(kr-\omega t)}}{4\pi\epsilon_0\, r} + [\mathbf{p}_0 - 3(\mathbf{p}_0 \cdot \hat{\mathbf{r}})\hat{\mathbf{r}}] \left[\frac{e^{i(kr-\omega t)}}{4\pi\epsilon_0\, r^2} + i\frac{e^{i(kr-\omega t)} - e^{-i\omega t}}{4\pi\epsilon_0\, kr^3}\right]. \end{aligned}$$

Note how the (periodic) vector potential (122) consists of a part that propagates with speed $\omega/k = c$, and a part that propagates "instantaneously".

The first line of eq. (122) holds for any Coulomb-gauge vector potential of angular frequency ω.

An unpublished manuscript by Maxwell from 1873 [33], probably inspired by Sellmeier [32], contained the statement: *The vibrations of molecules which have definite periods, and which produce emission and absorption of particular kinds of light, are due to forces between the parts of the molecule...* Unfortunately, Maxwell did not relate this phenomenon to oscillating electric dipoles in his electromagnetic theory.

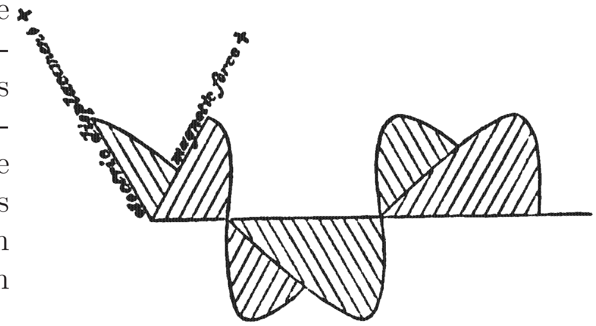
[76]Apparently, Maxwell did not consider instantaneous action at a distance as wave propagation.

[77]The Lorenz gauge, where $\boldsymbol{\nabla} \cdot \mathbf{A} = -\epsilon\mu\, \partial\Psi/\partial t$, is the velocity gauge with $v = 1/\sqrt{\epsilon\mu}$.

the source. However, Maxwell did not relate this behavior to the retarded potentials of Lorenz [30] (1867), to which Maxwell was averse.[78]

Articles 790-791

In sec. 95 of [28], Maxwell may have left the reader with the impression that only the magnetic field, and not the electric field, participates in waves. This possible misimpression was corrected in Arts. 790-791 of [35], which included the Fig. 66 below, illustrating the in-phase oscillations of **E** and **B** for a linearly polarized plane wave (in Gaussian units, where $E = B$ for a plane wave in vacuum).



Article 798

In Art. 798, Maxwell considered a conducting medium, but (tacitly) with no free charge. Then, (in the Coulomb gauge) $\boldsymbol{\nabla} \cdot \mathbf{A} = 0$, and $\partial\Psi/\partial t = 0$, so Maxwell's eq. (6), Art. 783, our eq. (119) becomes eq. (2) of Art. 798,

$$\nabla^2 \mathbf{A} - \epsilon\mu \frac{\partial^2 \mathbf{A}}{\partial t^2} - \mu\sigma \frac{\partial \mathbf{A}}{\partial t} = 0, \tag{124}$$

Maxwell noted that this wave equation implies waves that are exponentially damped in space over a characteristic distance $1/p$, now called the skin depth.

A.5 In *Note on the Electromagnetic Theory of Light* [31]

In 1868, Maxwell published a paper whose second part was titled *Note on the Electromagnetic Theory of Light* [31]. Although this paper did not bear on the issue of special relativity, we include a few remarks for completeness.[79]

A.5.1 Retarded Potentials

The 1868 paper is the only place where Maxwell mentioned the retarded potentials of Riemann [29] and Lorenz [30], to which he objected that they lead to violations of Newton's third law in electromagnetism (as does Maxwell's theory as well; see, for example, [76]), and also to nonconservation of energy. The latter objection was based on a misunderstanding, as reviewed by the author in [97].

A.5.2 Displacement Current

In 1882, FitzGerald closed his paper [41] with: *It may be worth while remarking, that no effect except light has ever yet been traced to the displacement-currents assumed by*

[78]See, for example, [97], and Appendix A.5.1 below.

[79]Maxwell also wrote *An Elementary Treatise on Electricity*, published posthumously in 1881 [49], which considered only electro-and magnetostatics. For discussion of how this work illustrates Maxwell's thinking on electromagnetism, see [81].

Maxwell in order to be able to assume all currents to flow in closed circuits. It has not, as far as I am aware, been ever actually demonstrated that open circuits, such as Leyden-jar discharges, produce exactly the same effects as closed circuits; and until some such effect of displacement-currents is observed, the whole theory of them will be open to question.

Indeed, in [28, 35, 26], Maxwell discussed his concept of displacement current primarily in relation to his theory of light, so it is noteworthy that in [31] he included mention of its effect in circuits with capacitors:

Theorem D.—When the electric displacement increases or diminishes, the effect is equivalent to that of an electric current in the positive or negative direction.

Thus, if the two conductors in the last case are now joined by a wire, there will be a current in the wire from A to B.

At the same time since the electric displacement in the dielectric is diminishing, there will be an action electromagnetically equivalent to that of an electric current from B to A through the dielectric.

According to this view, the current produced in discharging a condenser is a complete circuit, and might be traced within the dielectric itself by a galvanometer properly constructed. I am not aware that this has been done, so that this part of the theory, though apparently a natural consequence of the former, has not been verified by direct experiment. The experiment would certainly be a very delicate and difficult one.

A.5.3 Wave Equations

Perhaps his discussion of perceived difficulties with retarded potentials sensitized Maxwell to the desirability of a deduction of a wave equation for electromagnetism that did not invoke potentials. This was provided for the magnetic field $\mathbf{B}$ in the latter part of [31].

B Appendix: J.J. Thomson (1880)

While Arts. 599 and 769-770 of Maxwell's *Treatise* are consistent with the low-velocity limit of special relativity, this was not evident at the time, when electromagnetism was generally interpreted in an æther theory. For example, in his first research paper, J.J. Thomson [38] used Arts. 598-599 of Maxwell's *Treatise* to reach a "peculiar" conclusion as to the speed of light in a dielectric medium that has velocity $\mathbf{v}$ with respect to the frame of the æther.

In the present section, quantities in the ether frame will be unprimed, while a quantity in the frame of the moving dielectric will be denoted with a $'$.[80]

Thomson began with Maxwell's eq. (B) of Art. 598 of [35], $\mathbf{E}' = \mathbf{E} + \mathbf{v}/c \times \mathbf{B}$ (in Gaussian units), and noted that $\mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A}$ and hence $\boldsymbol{\nabla} \cdot \mathbf{B} = 0$ (or *vice versa*).

Thomson's eqs. (1)-(3) correspond to relating the electric displacement field in the moving frame by,[81]

$$\mathbf{D}' = \epsilon \mathbf{E}'. \tag{125}$$

[80] The dielectric medium could be vacuum.

[81] Thomson include a term $-\boldsymbol{\nabla}\phi$ in his eqs. (1)-(3), calling this the *potential due to polarization*. As he later took the curl of these equations, the somewhat mysterious term had no effect on the results of the paper.

His eq. (4) is more properly then,

$$\boldsymbol{\nabla}' \cdot \mathbf{D}' = 0. \tag{126}$$

Thomson's goal was to deduce a wave equation for $\mathbf{D}'$, and to infer from this the speed of light in the moving frame. To this end, he took the curl of eq. (125), assuming that $\boldsymbol{\nabla}' \times \mathbf{P}' = 0$ and that the dielectric medium is in uniform motion such that derivatives of the velocity $\mathbf{v}$ are zero,

$$\boldsymbol{\nabla}' \times \frac{\mathbf{D}'}{\epsilon} = \boldsymbol{\nabla}' \times \mathbf{E}' = \boldsymbol{\nabla}' \times \left(\mathbf{E} + \frac{\mathbf{v}}{c} \times \mathbf{B}\right) = \boldsymbol{\nabla}' \times \mathbf{E} + \frac{\mathbf{v}}{c}(\boldsymbol{\nabla}' \cdot \mathbf{B}) - \left(\frac{\mathbf{v}}{c} \cdot \boldsymbol{\nabla}'\right)\mathbf{B}. \tag{127}$$

Thomson then supposed that the effect of taking derivatives with respect to spacetime coordinates x, y, z and t is the same in the ether frame and in the moving frame. That is, he assumed that Galilean relativity relates these coordinates in the two frames. Note that this assumption was not needed in the interpretation of Maxwell's Arts. 599 and 769-770 (although Maxwell did use this assumption in his Arts. 600-601).

With this tacit assumption of Galilean relativity for (x, y, z, t), eq. (127) can be written,

$$\boldsymbol{\nabla} \times \frac{\mathbf{D}'}{\epsilon} = \boldsymbol{\nabla} \times \mathbf{E} + \frac{\mathbf{v}}{c}(\boldsymbol{\nabla} \cdot \mathbf{B}) - \left(\frac{\mathbf{v}}{c} \cdot \boldsymbol{\nabla}\right)\mathbf{B} = -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} - \left(\frac{\mathbf{v}}{c} \cdot \boldsymbol{\nabla}\right)\mathbf{B}, \tag{128}$$

in that $\boldsymbol{\nabla} \cdot \mathbf{B} = 0$ (while in special relativity, $\boldsymbol{\nabla}' \cdot \mathbf{B} \neq 0$).

It was not appreciated in 1880, and perhaps not until the work of LeBellac and Levy-Leblond in 1973 [83], that Maxwell's equations are not consistent with Galilean relativity. When Maxwell's eq. (12) applies, the appropriate modifications to Maxwell's equations to be compatible with Galilean relativity are those of so-called magnetic Galilean relativity (sec. 2.3 of [83]),

$$\boldsymbol{\nabla} \cdot \mathbf{D}_m = 4\pi\rho_m, \qquad \boldsymbol{\nabla} \cdot \mathbf{B}_m = 0, \qquad \boldsymbol{\nabla} \times \mathbf{E}_m = -\frac{1}{c}\frac{\partial \mathbf{B}_m}{\partial t}, \qquad \boldsymbol{\nabla} \times \mathbf{H}_m = \frac{4\pi}{c}\mathbf{J}_m \tag{129}$$

where ρ_m and $\mathbf{J}_m$ and the volume densities of "free" charge and currents, and there is no "displacement current" and no electromagnetic waves. While the velocity c has a value equal to the speed of light in vacuum it is to be deduced from static experiments and is not related to (nonexistent) wave propagation in Galilean relativity.

Thomson considered a nonmagnetic dielectric medium in which $\mathbf{B} = \mathbf{H}$, with no free charge or current densities, and supposed that in this case,

$$\boldsymbol{\nabla} \times \mathbf{B} = \frac{1}{c}\frac{\partial \mathbf{D}}{\partial t}, \tag{130}$$

whereas in a consistent Galilean view of this case, $\boldsymbol{\nabla} \times \mathbf{B} = 0$. He then took the curl of eq. (128), writing (his eq. (10)),

$$\nabla^2 \frac{\mathbf{D}'}{\epsilon} = \frac{1}{c^2}\frac{\partial \mathbf{D}}{\partial t^2} + \left(\frac{\mathbf{v}}{c^2} \cdot \boldsymbol{\nabla}\right)\frac{\partial \mathbf{D}}{\partial t}, \tag{131}$$

In the magnetic Galilean relativity of [83] this would be just $\nabla^2 \mathbf{D}' = 0$, corresponding to instantaneous propagation of electromagnetic effects.

At this point Thomson seems to have assumed that $\mathbf{D}' = \mathbf{D}$ even though his argument began with Maxwell's relation (12) in which $\mathbf{E}' \neq \mathbf{E}$. Assuming a wavefunction $\mathbf{D}' = \mathbf{D} = \mathbf{D}_0\, e^{i(kx-\omega t)}$ and $\mathbf{v} = v\,\hat{\mathbf{x}}$, the wave equation (131) leads to the dispersion relation,

$$\omega^2 - vk\omega - \frac{k^2c^2}{\epsilon} = 0, \qquad v' = \frac{\omega}{k} = \frac{1}{2}\left(v \pm 2c\sqrt{\frac{1}{\epsilon} + \frac{v^2}{4c^2}}\right). \tag{132}$$

Only the positive root could make sense, leading to,

$$v' \approx \frac{c}{\sqrt{\epsilon}} + \frac{v}{2} \qquad (v \ll c), \tag{133}$$

which was interpreted as the speed of the waves in the moving dielectric (even in the limit that $\epsilon \to 0$, where v' would exceed c).

Thomson claimed that his result agreed with an experiment by Fizeau [15, 23] with moving water ($\epsilon \approx 1.75$), although the result was actually closer to,

$$v' \approx \frac{c}{\sqrt{\epsilon}} + v\left(1 - \frac{1}{\epsilon}\right) \qquad (v \ll c). \tag{134}$$

The form (134) was "predicted" by Lorentz, eq. (91) of [59], and also holds in the theory of special relativity [70] (in the approximation that ϵ is independent of frequency). Only for the special case that $\epsilon = 2$ would Thomson's calculation agree with experiment, so his analysis illustrates how naïve assumptions about relativity and Maxwell's equations can lead to "peculiar" conclusions for general values of the dielectric constant.

C Appendix: From Faraday's Law to the Lorentz Transformation of the Electromagnetic Fields

This section is based on [84] (1979), and employs Gaussian units.

C.1 Force on a Moving Circuit

For a circuit at rest, the integral form of Faraday's law can be written as,

$$\mathcal{E} = \oint \mathbf{E}\cdot d\mathbf{l} = -\frac{1}{c}\frac{d}{dt}\int \mathbf{B}\cdot d\mathbf{S} = -\frac{1}{c}\frac{d\Phi_{\mathbf{B}}}{dt} \quad \left(= -\frac{1}{c}\frac{d}{dt}\oint \mathbf{A}\cdot d\mathbf{l}\right), \tag{135}$$

where $\mathcal{E}$ is the electromotive force in the circuit, $\mathbf{E}$ is the electric field, $d\mathbf{l}$ is an element of length along the circuit, $d\mathbf{S}$ is an element of area of a surface bounded by the current loop that generates magnetic field $\mathbf{B} = \boldsymbol{\nabla}\times\mathbf{A}$, $\mathbf{A}$ is the vector potential (Faraday's electrotonic state[82]), and $\Phi_{\mathbf{B}} = \int \mathbf{B}\cdot d\mathbf{S}$ is the magnetic flux through the circuit.[83,84]

[82]Faraday introduced his electrotonic state in Art. 60 of [5].

[83]Equation (135) also implies the relation,

$$\mathbf{E}_{\text{induced}} = -\frac{1}{c}\frac{\partial\mathbf{A}}{\partial t}, \tag{136}$$

for the electric field induced by a changing magnetic field (due to a changing current in the circuit).

[84]Faraday's Law was not formulated by Faraday himself, but by Maxwell (1856), p. 50 of [21]: *the electromotive force depends on the change in the number of lines of inductive magnetic action which pass through*

We now consider a circuit that moves with velocity $\mathbf{v}$ in the lab. An inference from Galilean relativity (which implies Newton's First Law[85]) is that an observer moving with velocity $\mathbf{v}$ measures fields $\mathbf{E}'$ and $\mathbf{B}'$, and the integral form of Faraday's law for this observer would be,

$$\mathcal{E}' = \oint \mathbf{E}' \cdot d\mathbf{l}' = \oint \mathbf{E}' \cdot d\mathbf{l} = -\frac{1}{c'}\frac{d}{dt'}\int \mathbf{B}' \cdot d\mathbf{S}' = -\frac{1}{c}\frac{d}{dt}\int \mathbf{B}' \cdot d\mathbf{S}\,, \tag{137}$$

where the integrals are performed over the circuit in the moving frame, where it is at rest. That is, according to Galilean relativity, the moving observer measures length and time to be the same as for an observer at rest, and the constant c, which is determined by experiments performed at rest in any frame, has the same value in any (inertial) frame.

We next suppose that $\mathbf{B}' = \mathbf{B}$, *i.e.*, observers in the lab frame and in the moving frame assign the same value to the magnetic field. We now consider how an observer at rest in the lab frame might describe the moving observer's calculation, of $(d/dt)\int \mathbf{B}' \cdot d\mathbf{S}$ for a circuit at rest in his frame, as,

$$\int \frac{D\mathbf{B}}{Dt} \cdot d\mathbf{S}, \tag{138}$$

---

the circuit, as a summary of Faraday's comments in [16].

Maxwell did not give the mathematical form (135) in [21]), but he did give (the equivalent of) eq. (136) on p. 64 ($\alpha_2 = -(1/4\pi)\, d\alpha_0/dt$, *etc.*). Then, on p. 66 he stated this equation in words as: *Law VI. The electro-motive force on any element of a conductor is measured by the instantaneous rate of change of the electro-tonic intensity on that element, whether in magnitude or direction.*

Maxwell deduced (via an energy argument!) the differential form of Faraday's law, $\boldsymbol{\nabla}\times\mathbf{E} = -(1/c)\,\partial\mathbf{B}/\partial t$, in eq. (54), p. 290, of [25] (although he used $\mu\mathbf{H}$ rather than $\mathbf{B}$ for the magnetic field as the latter was only invented by W. Thomson in 1871, eq. (r), p. 401 of [45]). Surprisingly, Maxwell did not give this differential form either in [28] or in his *Treatise* [55].

Digression: On p. 64 of [21], Maxwell deduced that the electric field induced by changing currents at a point at rest in the lab is that given in eq. (136) above. Then, he stated that for a moving (charged) "particle" with velocity $\mathbf{v}$, the field it experiences should be computed using the convective derivative, $-c\,\mathbf{E}_{\text{on moving charge}} = D\mathbf{A}/Dt = \partial\mathbf{A}/\partial t + (\mathbf{v}\cdot\boldsymbol{\nabla})\mathbf{A}$.

While we now consider electric charge to be a phenomenon separate from the electromagnetic field, Maxwell considered charge (density) to be an aspect of "displacement" in the æther, $\rho = \boldsymbol{\nabla}\cdot\mathbf{D}$. In this view, it seems natural to suppose that a moving charge samples an electromagnetic field in a manner analogous to a particle moving through a fluid, where the convective derivative describes the time dependence of, for example, pressure and density its experiences.

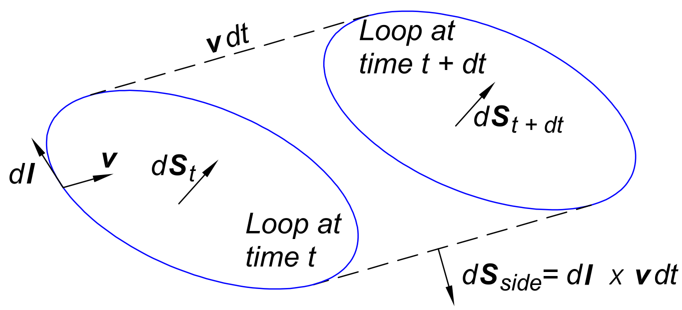
Later, in eq. (D), sec. 65, p. 485, of [28] and eq. (B), Art. 598, p. 239 (and also eq. (10), Art. 599, p. 241) of [55], Maxwell realized that the force on a moving charge should include the term $\mathbf{v}/c\times\mathbf{B}$ (and not $\mathbf{H}$!), and managed to arrive at the correct "Lorentz" force law despite his use of the convective derivative (in eq. (2) of Art. 598 of [55]). However, Helmholtz, eq. (5^d), p. 309 of [36], argued that the term $\mathbf{v}/c\times\mathbf{B}$ should be accompanied by the additional term $-\boldsymbol{\nabla}(\mathbf{A}\cdot\mathbf{v}/c)$, which claim was seconded on p. 12 of [44] (1884) and on p. 273 of [50] (1888). These claims may have had the effect that Maxwell's derivation of the force on a moving "particle", when corrected/clarified, was not considered to yield the Lorentz force law. For example, when J.J. Thomson edited the 3rd edition of Maxwell's *Treatise* he added a comment, p. 260 of [55], casting doubt Maxwell's analysis of the force on a moving charge. This is unfortunate in that Lorentz, eq. (V), sec. 12 of [59] and eq. (23) of [71], wrote the force law as $\mathbf{F} = q(\mathbf{D} + \mathbf{v}/c\times\mathbf{H})$, which is not correct, although this was clarified only in 1944 by experiments [75] on the motion of high-energy particles penetrating magnetized steel.

[85] *Law I. Every body perseveres in its state of rest, or of uniform motion in a right line, unless it is compelled to change that state by forces impressed thereon*; p. 83 of [1].

over the circuit at time t in the lab frame. Of course, $D\mathbf{B}/Dt$ does not equal $\partial\mathbf{B}/\partial t$ as it must incorporate effects of the motion of the circuit in the lab frame.[86]

Referring to the figure on the next page, which is in the lab frame, and where the direction of a surface element $d\mathbf{S}$ is related to the line element $d\mathbf{l}$ by the right-hand rule, we can express the time derivative of the magnetic flux through the moving circuit in lab-frame quantities as,

$$\frac{d}{dt}\int_{\text{moving circuit}} \mathbf{B}\cdot d\mathbf{S} \approx \frac{1}{dt}\left[\int_{t+dt} \mathbf{B}_{t+dt}\cdot d\mathbf{S}_{t+dt} - \int_t \mathbf{B}_t\cdot d\mathbf{S}_t\right]. \tag{139}$$



We have, approximately, that,

$$\mathbf{B}_{t+dt} = \mathbf{B}_t + \frac{\partial\mathbf{B}_t}{\partial t}\,dt, \tag{140}$$

so,

$$\frac{d}{dt}\int_{\text{moving circuit}} \mathbf{B}\cdot d\mathbf{S} \approx \int_{t+dt}\frac{\partial\mathbf{B}_t}{\partial t}\cdot d\mathbf{S}_{t+dt} + \frac{1}{dt}\left[\int_{t+dt} \mathbf{B}_t\cdot d\mathbf{S}_{t+dt} - \int_t \mathbf{B}_t\cdot d\mathbf{S}_t\right]. \tag{141}$$

We now play a famous trick, and consider the integral over the entire surface of the volume swept out by the circuit during time interval dt, taking $d\mathbf{S}$ to be directed out of this volume,

$$\int_{\text{entire surface}} \mathbf{B}_t\cdot d\mathbf{S} = \int_{t+dt} \mathbf{B}_t\cdot d\mathbf{S}_{t+dt} - \int_t \mathbf{B}_t\cdot d\mathbf{S}_t + \int_{\text{side}} \mathbf{B}_t\cdot d\mathbf{S}_{\text{side}}. \tag{142}$$

By Gauss' theorem,

$$\int \mathbf{B}_t\cdot d\mathbf{S} = \int \boldsymbol{\nabla}\cdot\mathbf{B}_t\,d\text{Vol} = \int (\boldsymbol{\nabla}\cdot\mathbf{B}_t)\,\mathbf{v}\,dt\cdot d\mathbf{S}_t, \tag{143}$$

and an area element on the "sides" of the surface is related by $d\mathbf{S}_{\text{side}} = d\mathbf{l}\times\mathbf{v}\,dt$, such that,

$$\int_{\text{side}} \mathbf{B}_t\cdot d\mathbf{S}_{\text{side}} = \oint \mathbf{B}_t\cdot d\mathbf{l}\times\mathbf{v}\,dt = -dt\oint \mathbf{B}_t\times\mathbf{v}\cdot d\mathbf{l} = dt\int \boldsymbol{\nabla}\times(\mathbf{B}_t\times\mathbf{v})\cdot d\mathbf{S}_t. \tag{144}$$

We can now rewrite eq. (142) as We now play a famous trick, and consider the integral over the entire surface of the volume swept out by the circuit during time interval dt, taking $d\mathbf{S}$ to be directed out of this volume,

$$\int_{t+dt} \mathbf{B}_t\cdot d\mathbf{S}_{t+dt} - \int_t \mathbf{B}_t\cdot d\mathbf{S}_t = dt\int (\boldsymbol{\nabla}\cdot\mathbf{B}_t)\,\mathbf{v}\cdot d\mathbf{S}_t + dt\int \boldsymbol{\nabla}\times(\mathbf{B}_t\times\mathbf{v})\cdot d\mathbf{S}_t. \tag{145}$$

[86]Such issues occur frequently in fluid dynamics.

Then, recalling eq. (141), and taking the limit as $dt \to 0$, we have that,

$$\frac{d}{dt}\int_{\text{moving circuit}} \mathbf{B}\cdot d\mathbf{S} = \int \frac{\partial \mathbf{B}}{\partial t}\cdot d\mathbf{S} + \int (\boldsymbol{\nabla}\cdot\mathbf{B})\,\mathbf{v}\cdot d\mathbf{S} + \int \boldsymbol{\nabla}\times(\mathbf{B}\times\mathbf{v})\cdot d\mathbf{S} \equiv \int \frac{D\mathbf{B}}{Dt}\cdot d\mathbf{S}, \tag{146}$$

where,

$$\frac{D\mathbf{B}}{Dt} = \frac{\partial \mathbf{B}}{\partial t} + (\boldsymbol{\nabla}\cdot\mathbf{B})\,\mathbf{v} + \boldsymbol{\nabla}\times(\mathbf{B}\times\mathbf{v}) = \frac{\partial \mathbf{B}}{\partial t} + (\mathbf{v}\cdot\boldsymbol{\nabla})\mathbf{B}, \tag{147}$$

is the convective derivative.

Faraday's Law for the moving circuit, eq. (137), can now be written as,

$$\begin{aligned}\mathcal{E}' &= \oint_{\text{moving circuit}} \mathbf{E}'\cdot d\mathbf{l} = \int_{\text{moving circuit}} \boldsymbol{\nabla}\times\mathbf{E}'\cdot d\mathbf{S} = -\frac{1}{c}\int_{\text{fixed circuit}} \frac{D\mathbf{B}}{Dt}\cdot d\mathbf{S} \\ &= -\frac{1}{c}\int_{\text{fixed circuit}} \left[\frac{\partial \mathbf{B}}{\partial t} + \boldsymbol{\nabla}\times(\mathbf{B}\times\mathbf{v})\right]\cdot d\mathbf{S},\end{aligned} \tag{148}$$

noting that $\boldsymbol{\nabla}\cdot\mathbf{B} = 0$. This holds at any fixed time t, so we infer that,

$$\boldsymbol{\nabla}\times\left(\mathbf{E}' - \frac{\mathbf{v}}{c}\times\mathbf{B}\right) = -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} = \boldsymbol{\nabla}\times\mathbf{E}, \tag{149}$$

and hence,

$$\mathbf{E} = \mathbf{E}' - \frac{\mathbf{v}}{c}\times\mathbf{B}, \qquad \mathbf{E}' = \mathbf{E} + \frac{\mathbf{v}}{c}\times\mathbf{B}. \tag{150}$$

That is, the force on the moving circuit is given by the Lorentz force law.

This argument, which did not use potentials, but did involve a convective derivative, is perhaps what Maxwell's Art. 598 of [55] could/should have been. Instead, this argument may have been first given in sec. 86, p. 398 of [67] (1904). See also sec. 9-3, p 160 of [79].

C.2 Magnetic Field According to a Moving Observer

The preceding section was based on the assumption that the magnetic field is the same for an observer on the moving circuit as for one at rest in the lab. However, we could consider a moving, mathematical loop (not associated with a physical electric current) in a region of vacuum, away from the sources of laboratory fields $\mathbf{E}$ and $\mathbf{B}$. Instead of emphasizing Faraday's Law, and supposing that $\mathbf{B}' = \mathbf{B}$ according to a moving observer, we could emphasize Maxwell's extension of Ampère's Law, which in vacuum and in the lab frame reads,[87]

$$\boldsymbol{\nabla}\times\mathbf{B} = \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t}, \qquad \int_{\text{fixed loop}} \boldsymbol{\nabla}\times\mathbf{B}\cdot d\mathbf{S} = \oint_{\text{fixed loop}} \mathbf{B}\cdot d\mathbf{l} = \frac{1}{c}\frac{d}{dt}\int_{\text{fixed loop}} \mathbf{E}\cdot d\mathbf{S}, \tag{152}$$

[87] Ampère's law in the form,

$$\boldsymbol{\nabla}\times\mathbf{H} = \frac{4\pi}{c}\mathbf{J}_{\text{free}}, \tag{151}$$

is not due to Ampère himself, but was first given by Maxwell in eq. (9), p. 171, of [24] (1861).

and suppose that for a moving loop, $\mathbf{E}' = \mathbf{E}$. Then, an argument parallel to that of the preceding section would imply that the magnetic field according to the moving observer is,

$$\mathbf{B}' = \mathbf{B} - \frac{\mathbf{v}}{c} \times \mathbf{E}, \tag{153}$$

where the change of sign compared to eq. (150) is due to the difference in signs between the Faraday's Law and Maxwell version of Ampère's law.

C.3 Both E and B are Different for Observers in the Lab and in a Moving Frame

The previous two sections have assumed that either $\mathbf{E}$ or $\mathbf{B}$ is the same for observers in the lab and in a moving frame. But, it is much more plausible that both $\mathbf{E}$ and $\mathbf{B}$ have different values in different frames (for the same physical configuration).

If eqs. (150) and (153) were the correct general relations for the fields $\mathbf{E}'$ and $\mathbf{B}'$ according to an observer with velocity $\mathbf{v}$ in the lab, we would expect that the transformation from fields in the moving frame to the lab frame, which latter has velocity $\mathbf{v}' = -\mathbf{v}$ with respect to the former, would be obtained by exchanging primed and unprimed quantities,

$$\mathbf{E} = \mathbf{E}' - \frac{\mathbf{v}}{c} \times \mathbf{B}', \tag{154}$$

$$\mathbf{B} = \mathbf{B}' + \frac{\mathbf{v}}{c} \times \mathbf{E}'. \tag{155}$$

We could then check for consistency of these transformations by, form example, starting from eq. (155), slightly rearranged, and then using eq. (154), also slightly rearranged,

$$\begin{aligned} \mathbf{B}' &= \mathbf{B} - \frac{\mathbf{v}}{c} \times \mathbf{E}' = \mathbf{B} - \frac{\mathbf{v}}{c} \times \left(\mathbf{E} + \frac{\mathbf{v}}{c} \times \mathbf{B}'\right) \\ \left(1 - \frac{v^2}{c^2}\right) \mathbf{B}' &= \mathbf{B} - \frac{\mathbf{v}}{c} \times \mathbf{E} - \left(\frac{\mathbf{v}}{c} \cdot \mathbf{B}'\right) \frac{\mathbf{v}}{c}. \end{aligned} \tag{156}$$

Similarly, we would find,

$$\left(1 - \frac{v^2}{c^2}\right) \mathbf{E}' = \mathbf{E} + \frac{\mathbf{v}}{c} \times \mathbf{B} - \left(\frac{\mathbf{v}}{c} \cdot \mathbf{E}'\right) \frac{\mathbf{v}}{c}. \tag{157}$$

Thus our inferences in secs. B.1-2 for the transformations of the fields to a moving frame lead to inconsistencies at order v^2/c^2 (although they are valid at order v/c).

A partial remedy would be to "split" the factor $1 - v^2/c^2$ between the transformations and their inverses,

$$\mathbf{E}' = \frac{1}{\sqrt{1 - v^2/c^2}} \left(\mathbf{E} + \frac{\mathbf{v}}{c} \times \mathbf{B}\right) + ?, \qquad \mathbf{E} = \frac{1}{\sqrt{1 - v^2/c^2}} \left(\mathbf{E}' - \frac{\mathbf{v}}{c} \times \mathbf{B}'\right) + ?, \tag{158}$$

$$\mathbf{B}' = \frac{1}{\sqrt{1 - v^2/c^2}} \left(\mathbf{B} - \frac{\mathbf{v}}{c} \times \mathbf{E}\right) + ?, \qquad \mathbf{B} = \frac{1}{\sqrt{1 - v^2/c^2}} \left(\mathbf{B}' + \frac{\mathbf{v}}{c} \times \mathbf{E}'\right) + ?, \tag{159}$$

but it is less obvious how to deal with the "extra" terms $(\mathbf{v}/c\cdot\mathbf{E})\,\mathbf{v}/c$ and $(\mathbf{v}/c\cdot\mathbf{B})\,\mathbf{v}/c$ in eqs. (156)-(157). An inspired "guess" would be to include these forms in the transformations, but with an as-yet-undetermined coefficient α. Introducing the notation,

$$\gamma = \frac{1}{\sqrt{1-v^2/c^2}}\,, \tag{160}$$

we consider the transformations,

$$\mathbf{E}' = \gamma\left(\mathbf{E}+\frac{\mathbf{v}}{c}\times\mathbf{B}\right) - \alpha\left(\frac{\mathbf{v}}{c}\cdot\mathbf{E}\right)\frac{\mathbf{v}}{c}\,, \tag{161}$$

$$\mathbf{B}' = \gamma\left(\mathbf{B}-\frac{\mathbf{v}}{c}\times\mathbf{E}\right) - \alpha\left(\frac{\mathbf{v}}{c}\cdot\mathbf{B}\right)\frac{\mathbf{v}}{c}\,. \tag{162}$$

The inverse transformations are again obtained by swapping primed and unprimed quantities, and changing $\mathbf{v}$ to $-\mathbf{v}$,

$$\mathbf{E} = \gamma\left(\mathbf{E}'-\frac{\mathbf{v}}{c}\times\mathbf{B}'\right) - \alpha\left(\frac{\mathbf{v}}{c}\cdot\mathbf{E}'\right)\frac{\mathbf{v}}{c}\,, \tag{163}$$

$$\mathbf{B} = \gamma\left(\mathbf{B}'+\frac{\mathbf{v}}{c}\times\mathbf{E}'\right) - \alpha\left(\frac{\mathbf{v}}{c}\cdot\mathbf{B}'\right)\frac{\mathbf{v}}{c}\,. \tag{164}$$

To determine α, we rearrange eq. (163)-(164),

$$\mathbf{E}' = \frac{\mathbf{E}}{\gamma}+\frac{\mathbf{v}}{c}\times\mathbf{B}'+\frac{\alpha}{\gamma}\left(\frac{\mathbf{v}}{c}\cdot\mathbf{E}'\right)\frac{\mathbf{v}}{c}\,, \tag{165}$$

$$\mathbf{B}' = \frac{\mathbf{B}}{\gamma}-\frac{\mathbf{v}}{c}\times\mathbf{E}'+\frac{\alpha}{\gamma}\left(\frac{\mathbf{v}}{c}\cdot\mathbf{B}'\right)\frac{\mathbf{v}}{c}\,, \tag{166}$$

and then use eq. (166) in (165) to find,

$$\frac{\mathbf{E}'}{\gamma^2} = \frac{1}{\gamma}\left(\mathbf{E}+\frac{\mathbf{v}}{c}\times\mathbf{B}\right)+\left(\frac{\alpha}{\gamma}-1\right)\left(\frac{\mathbf{v}}{c}\cdot\mathbf{E}'\right)\frac{\mathbf{v}}{c}\,. \tag{167}$$

We also have from eq. (161) that,

$$\frac{\mathbf{v}}{c}\cdot\mathbf{E}' = \left(\gamma-\alpha\frac{v^2}{c^2}\right)\frac{\mathbf{v}}{c}\cdot\mathbf{E}, \tag{168}$$

so that eq. (167) can be rewritten as,

$$\mathbf{E}' = \gamma\left(\mathbf{E}+\frac{\mathbf{v}}{c}\times\mathbf{B}\right)+\gamma^2\left(\frac{\alpha}{\gamma}-1\right)\left(\gamma-\alpha\frac{v^2}{c^2}\right)\left(\frac{\mathbf{v}}{c}\cdot\mathbf{E}'\right)\frac{\mathbf{v}}{c}\,. \tag{169}$$

This should be the same as eq. (163), which gives a quadratic equation for α,

$$\alpha = \gamma^2\left(\frac{\alpha}{\gamma}-1\right)\left(\gamma-\alpha\frac{v^2}{c^2}\right), \qquad \alpha^2-2\gamma\frac{c^2}{v^2}\alpha+\gamma^2\frac{c^2}{v^2}=0, \qquad \alpha=\frac{c^2}{v^2}(\gamma\pm1). \tag{170}$$

For the transformations to have the trivial form when $\mathbf{v}=0$, we take the negative root, $\alpha=(c^2/v^2)(\gamma-1)$.

The self-consistent transformations of the electromagnetic fields from the lab frame to a moving frame, deduced from use of Faraday's and Ampère's Laws (as formulated by Maxwell), are,

$$\mathbf{E}' = \gamma\left(\mathbf{E} + \frac{\mathbf{v}}{c} \times \mathbf{B}\right) - (\gamma - 1)(\hat{\mathbf{v}} \cdot \mathbf{E})\,\hat{\mathbf{v}}\,, \qquad \mathbf{B}' = \gamma\left(\mathbf{B} - \frac{\mathbf{v}}{c} \times \mathbf{E}\right) - (\gamma - 1)(\hat{\mathbf{v}} \cdot \mathbf{B})\,\hat{\mathbf{v}}, \tag{171}$$

$$\mathbf{E} = \gamma\left(\mathbf{E}' - \frac{\mathbf{v}}{c} \times \mathbf{B}'\right) - (\gamma - 1)(\hat{\mathbf{v}} \cdot \mathbf{E}')\,\hat{\mathbf{v}}\,, \qquad \mathbf{B} = \gamma\left(\mathbf{B}' + \frac{\mathbf{v}}{c} \times \mathbf{E}'\right) - (\gamma - 1)(\hat{\mathbf{v}} \cdot \mathbf{B}')\,\hat{\mathbf{v}}, \tag{172}$$

as deduced by Einstein, sec. 6 of [69].[88,89] Of course, the present analysis does not yield the insight that there also is a transformation of spacetime coordinates between the two frames.

If we decompose the field vectors into components parallel and perpendicular to velocity $\mathbf{v}$, $\mathbf{E} = \mathbf{E}_{\parallel} + \mathbf{E}_{\perp}$ where $\mathbf{E}_{\parallel} = (\hat{\mathbf{v}} \cdot \mathbf{E})\,\hat{\mathbf{v}}$, then,

$$\mathbf{E}'_{\parallel} = \mathbf{E}_{\parallel}, \qquad \mathbf{E}'_{\perp} = \gamma\left(\mathbf{E}_{\perp} + \frac{\mathbf{v}}{c} \times \mathbf{B}\right), \qquad \mathbf{B}'_{\parallel} = \mathbf{B}_{\parallel}, \qquad \mathbf{B}'_{\perp} = \gamma\left(\mathbf{B}_{\perp} - \frac{\mathbf{v}}{c} \times \mathbf{E}\right), \tag{174}$$

$$\mathbf{E}_{\parallel} = \mathbf{E}'_{\parallel}, \qquad \mathbf{E}_{\perp} = \gamma\left(\mathbf{E}'_{\perp} - \frac{\mathbf{v}}{c} \times \mathbf{B}'\right), \qquad \mathbf{B}_{\parallel} = \mathbf{B}'_{\parallel}, \qquad \mathbf{B}_{\perp} = \gamma\left(\mathbf{B}'_{\perp} + \frac{\mathbf{v}}{c} \times \mathbf{E}'\right). \tag{175}$$

The low-velocity approximations to these transformations are,

$$\mathbf{E}' \approx \mathbf{E} + \frac{\mathbf{v}}{c} \times \mathbf{B}, \qquad \mathbf{B}' \approx \mathbf{B}_{\perp} - \frac{\mathbf{v}}{c} \times \mathbf{E}, \tag{176}$$

$$\mathbf{E} \approx \mathbf{E}' - \frac{\mathbf{v}}{c} \times \mathbf{B}', \qquad \mathbf{B} \approx \mathbf{B}'_{\perp} + \frac{\mathbf{v}}{c} \times \mathbf{E}', \tag{177}$$

as previously found in secs. B.1-2.

C.4 Comments

This Appendix shows that arguments using convective (time) derivatives can, with considerable effort, lead to the full, relativistic transformations of fields from the lab frame to a moving frame, although most straightforward use of the convective derivative only yields the low-velocity transformation.

C.4.1 Use of Potentials to Compute the Fields

As also remarked in footnote 50 above, Maxwell made an argument based on potentials, rather than fields, in Art. 598 of [55], and while he arrived at the correct low-velocity approximation to the electric field in a moving frame via discussion of the convective derivative

[88] Einstein's derivation was based on the assumptions that Maxwell's equations have the same form in both the lab frame and a moving (inertial) frame. In particular, he used Faraday's and Ampère's Laws in empty space,

$$\boldsymbol{\nabla} \times \mathbf{E} = -\partial \mathbf{B}/\partial ct, \qquad \boldsymbol{\nabla} \times \mathbf{B} = \partial \mathbf{E}/\partial ct, \qquad \text{and} \qquad \boldsymbol{\nabla}' \times \mathbf{E}' = -\partial \mathbf{B}'/\partial ct', \qquad \boldsymbol{\nabla}' \times \mathbf{B}' = \partial \mathbf{E}'/\partial ct' \tag{173}$$

together with the transformation of coordinates $(ct, \mathbf{x})$ to $(ct', \mathbf{x}')$ to deduce eqs. (174)-(175).

[89] The transformation of the fields $\mathbf{D}$ and $\mathbf{H}$ was given by Lorentz (1899), eq. ($\mathrm{V_c}$), p. 430, of [63], and later in eq. (6), p. 812, of [68] (1904), where $k = \gamma$ (Lorentz' eq. (3)) and $l = 1$ (p. 824). Discussion of the field transformations was given by Larmor (1897) in [62], and more clearly on p. 168 of [64] (1900).

of the vector potential, there was an ambiguity as to whether his symbol Ψ was the electrical scalar potential, as he claimed it to be. It is felicitous that there is no ambiguity if one considers the fields rather than the potentials (sec. 5.1 above), as perhaps first done by Abraham, sec. 86, p. 398 of [67] (1904).

We elaborate on this topic by deducing how the transform of potentials for low-velocity, and the resulting transform of the fields if they are then deduced from the transformed potentials.

In any (inertial) frame the fields $\mathbf{E}$ and $\mathbf{B}$ are related the scalar potential Ψ and the vector potential $\mathbf{A}$ (in some gauge) by,

$$\mathbf{E} = -\boldsymbol{\nabla}\Psi - \frac{\partial \mathbf{A}}{\partial ct}, \qquad \text{and} \qquad \mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A}, \tag{178}$$

provided the derivatives are taken with respect to the coordinates of that frame.

The electromagnetic potentials comprise a 4-vector $(V, \mathbf{A})$, so the Lorentz transformations of the potentials from the lab frame to the primed frame that has velocity $\mathbf{v}$ with respect to the lab frame are,

$$\Psi' = \gamma\left(\Psi - \frac{\mathbf{v}}{c} \cdot \mathbf{A}\right) \approx \Psi - \frac{\mathbf{v}}{c} \cdot \mathbf{A}, \tag{179}$$

$$\mathbf{A}' = \mathbf{A} + (\gamma - 1)(\mathbf{A} \cdot \hat{\mathbf{v}})\, \hat{\mathbf{v}} - \gamma\frac{\mathbf{v}}{c}\Psi \approx \mathbf{A} - \frac{\mathbf{v}}{c}\Psi, \tag{180}$$

where the approximations hold for low velocity.

For the derivatives, we note that $(\partial/\partial ct, -\boldsymbol{\nabla})$ is a 4-vector,[90] so its transform is,

$$\frac{\partial}{\partial ct'} = \gamma\left(\frac{\partial}{\partial ct} - \frac{\mathbf{v}}{c} \cdot (-\boldsymbol{\nabla})\right) \approx \frac{\partial}{\partial ct} + \frac{\mathbf{v}}{c} \cdot \boldsymbol{\nabla}, \qquad \frac{\partial}{\partial t'} \approx \frac{\partial}{\partial t} + \mathbf{v} \cdot \boldsymbol{\nabla}, \tag{181}$$

$$-\boldsymbol{\nabla}' = -\boldsymbol{\nabla} + (\gamma - 1)(-\boldsymbol{\nabla} \cdot \hat{\mathbf{v}})\, \hat{\mathbf{v}} - \gamma\frac{\mathbf{v}}{c}\frac{\partial}{\partial ct} \approx -\boldsymbol{\nabla}, \tag{182}$$

where we neglect terms of order $1/c^2$. Note that the low-velocity approximation to the time derivative in the moving frame is the convective derivative in terms of lab-frame quantities,

The fields in the moving frame can now be computed as,

$$\begin{aligned}
\mathbf{E}' &= -\boldsymbol{\nabla}'\Psi' - \frac{\partial \mathbf{A}'}{\partial ct'} \approx -\boldsymbol{\nabla}\left(\Psi - \frac{\mathbf{v}}{c} \cdot \mathbf{A}\right) - \left(\frac{\partial}{\partial ct} + \frac{\mathbf{v}}{c} \cdot \boldsymbol{\nabla}\right)\left(\mathbf{A} - \frac{\mathbf{v}}{c}V\right) \\
&\approx -\Psi - \frac{\partial \mathbf{A}}{\partial ct} + \boldsymbol{\nabla}\left(\frac{\mathbf{v}}{c} \cdot \mathbf{A}\right) - \left(\frac{\mathbf{v}}{c} \cdot \boldsymbol{\nabla}\right)\mathbf{A} \\
&= \mathbf{E} + \left(\frac{\mathbf{v}}{c} \cdot \boldsymbol{\nabla}\right)\mathbf{A} + \frac{\mathbf{v}}{c} \times (\boldsymbol{\nabla} \times \mathbf{A}) - \left(\frac{\mathbf{v}}{c} \cdot \boldsymbol{\nabla}\right)\mathbf{A} = \mathbf{E} + \frac{\mathbf{v}}{c} \times \mathbf{B},
\end{aligned} \tag{183}$$

$$\begin{aligned}
\mathbf{B}' &= \boldsymbol{\nabla}' \times \mathbf{A}' \approx \boldsymbol{\nabla} \times \left(\mathbf{A} - \frac{\mathbf{v}}{c}V\right) = \mathbf{B} + \boldsymbol{\nabla} \times \left(\frac{\mathbf{v}}{c}(-V)\right) = \mathbf{B} - \frac{\mathbf{v}}{c} \times (-\boldsymbol{\nabla}V) \\
&= \mathbf{B} - \frac{\mathbf{v}}{c} \times \mathbf{E} - \frac{\mathbf{v}}{c} \times \frac{\partial \mathbf{A}}{\partial ct} \approx \mathbf{B} - \frac{\mathbf{v}}{c} \times \mathbf{E}.
\end{aligned} \tag{184}$$

[90] Strictly, $(\Psi, \mathbf{A})$ is a covariant 4-vector and $(\partial/\partial ct, -\boldsymbol{\nabla})$ is a contravariant 4-vector. These distinctions are unimportant is special relativity for inertial frames, but are significant when considering accelerated frames. See, for example, [96].

Thus, using the potentials and the various low-velocity Lorentz transformations, we recover the low-velocity forms (176)-(177) for the electromagnetic fields.[91]

C.4.2 Lorentz Force

Another felicitous result is that the low-velocity transform of the electric field from lab frame to a moving frame has the same form as the lab-frame Lorentz force (per unit charge), $\mathbf{F}/q = \mathbf{E} + \mathbf{v}/c \times \mathbf{B}$, for arbitrary velocity. Of course, neither Maxwell (1873) nor Lorentz (1892) were aware of this happy result of the Lorentz transformation of 4-force (Minkowski force).

References

[1] I. Newton, *Philosophiæ Naturalis Principia Mathematica* (1686),
http://kirkmcd.princeton.edu/examples/mechanics/newton_principia.pdf

[2] J.-B. Biot and F. Savart, *Note sur la Magnétisme de la pile de Volta*, Ann. Chem. Phys. **15**, 222 (1820), http://kirkmcd.princeton.edu/examples/EM/biot_acp_15_222_20.pdf
English translation on p. 118 of R.A.R. Tricker, *Early Electrodynamics, the First Law of Circulation* (Pergamon, 1965),
http://kirkmcd.princeton.edu/examples/EM/tricker_early_em.pdf

[3] A.M. Ampère, *Théorie mathématique des Phénomènes électro-dynamiques uniquement déduite de l'Expérience*, 2nd ed. (Paris, 1883),
http://kirkmcd.princeton.edu/examples/EM/ampere_theorie_26_83.pdf
English translation in A.K.T. Assis and J.P.M.C. Chahib, *Ampère's Electrodynamics* (Aperion, 2015), http://kirkmcd.princeton.edu/examples/EM/assis_ampere_15.pdf

[4] G. Green, *Mathematical Papers* (Macmillan, 1871),
http://kirkmcd.princeton.edu/examples/EM/green_papers.pdf

[5] M. Faraday, *Experimental Researches in Electricity*, Phil. Trans. Roy. Soc. London **122**, 125 (1832), http://kirkmcd.princeton.edu/examples/EM/faraday1_ptrsl_122_125_32_1-139.pdf

[91]Hence, it seems to this author that Maxwell's correct expression in sec. (65) of [28] and Art. 598 of [55], for the low-velocity field experienced by a moving circuit could well have been deduced by a valid argument, despite the doubts cast on this by Helmholtz, and Thomson.

Thomson felt that his objection was validated by the example of a rotating, conducting sphere is in uniform external magnetic field, in his note on p. 260 of [55]. However, this example involves and accelerated frame, for which the magnetic field is considered to be the same by a rotating observer and one at rest (see, for example, [96]).

If we had supposed that the magnetic field, and the vector potential were that same in the lab frame and in the moving frame, then eq. (183) above would read $\mathbf{E}' \approx \mathbf{E} + \mathbf{v}/c \times \mathbf{B} + (\mathbf{v} \cdot \boldsymbol{\nabla})\mathbf{A}/c$. So, while the assumption that $\mathbf{B}' = \mathbf{B}$ permits one to deduce the correct, low-velocity electric-field transformation via consideration of Faraday's Law for moving circuit (as in sec. B.1 above), use of this assumption in an argument based on potentials does lead to an erroneous result.

[6] M. Faraday, *Experimental Researches in Electricity—Thirteenth Series*, Phil. Trans. Roy. Soc. London **128**, 125 (1838), Art. 1644,
http://kirkmcd.princeton.edu/examples/EM/faraday13_ptrsl_128_125_38_1480-1666.pdf

[7] M. Faraday, *Experimental Researches in Electricity—Fourteenth Series*, Phil. Trans. Roy. Soc. London **128**, 265 (1838),
http://kirkmcd.princeton.edu/examples/EM/faraday14_ptrsl_128_265_38_1667-1748.pdf

[8] H. Grassmann, *Neue Theorie der Elektrodynamik*, Ann. d. Phys. **64**, 1 (1845),
http://kirkmcd.princeton.edu/examples/EM/grassmann_ap_64_1_45.pdf
http://kirkmcd.princeton.edu/examples/EM/grassmann_ap_64_1_45_english.pdf

[9] G.G. Stokes, *On the Theories of the Internal Friction of Fluids in Motion, and of the Equilibrium and Motion of Elastic Solids*, Trans. Camb. Phil. Soc. **8**, 287 (1845),
http://kirkmcd.princeton.edu/examples/fluids/stokes_tcps_8_287_45.pdf

[10] F.E. Neumann, *Allgemeine Gesetze der inducirten elektrischen Ströme*, Abh. König. Akad. Wiss. Berlin **1**, 45 (1845), http://kirkmcd.princeton.edu/examples/EM/neumann_akawb_1_45.pdf

[11] W. Weber, *Elektrodynamishe Maassbestimmungen*, Abh. König. "achs. Gesell. Wissen. p. 209 (1846), http://kirkmcd.princeton.edu/examples/EM/weber_aksgw_209_46.pdf
Translated as *Determinations of Electrodynamic Measure*,
http://kirkmcd.princeton.edu/examples/EM/weber_aksgw_209_46_english.pdf

[12] W. Thomson, *On a Mechanical Representation of Electric, Magnetic, and Galvanic Forces*, Cam. Dublin Math. J. **2**, 61 (1847),
http://kirkmcd.princeton.edu/examples/EM/thomson_cdmj_2_61_47.pdf

[13] G. Kirchhoff, *Ueber eine Anleitung der Ohm'schen, Gesetze, welche sich an die Theorie der Elektrostatik anschließt*, Ann. d. Phys. **18**, 506 (1849),
http://kirkmcd.princeton.edu/examples/EM/kirchhoff_apc_154_506_49.pdf
On a Deduction of Ohm's Laws, in connexion with the Theory of Electro-statics, Phil. Mag. **37**, 463 (1850), http://kirkmcd.princeton.edu/examples/EM/kirchhoff_pm_37_463_50.pdf
Démonstration des lois de Ohm fondée sur les principes ordinaires de l'Électricité statique, Ann. Chim. Phys. **41**, 496 (1854),
http://kirkmcd.princeton.edu/examples/EM/kirchhoff_acp_41_496_54.pdf

[14] W. Thomson, *A Mathematical Theory of Magnetism*, Phil. Trans. Roy. Soc. London **141**, 269 (1851), http://kirkmcd.princeton.edu/examples/EM/thomson_ptrsl_141_269_51.pdf

[15] H, Fizeau, *Sur les hypothèses relatives à l'éther lumineux, et sur une expérience qui parait démontrer que le mouvement des corps change la vitesse avec laquelle la lumière se propage dans leur intérieur*, Compt. Rend. Acad. Sci. **33**, 349 (1851),
http://kirkmcd.princeton.edu/examples/GR/fizeau_cras_33_349_51.pdf
The Hypotheses Relating to the Luminous Aether, and an Experiment which Appears to Demonstrate that the Motion of Bodies Alters the Velocity with which Light Propagates itself in their Interior, Phil. Mag. **2**, 568 (1851),
http://kirkmcd.princeton.edu/examples/GR/fizeau_pm_2_568_51.pdf

[16] M. Faraday, *Experimental Researches in Electricity—Twenty-eighth Series*, Phil. Trans. Roy. Soc. London **142**, 25 (1852),
http://kirkmcd.princeton.edu/examples/EM/faraday28_ptrsl_142_25_52_3070_3176.pdf

[17] M. Faraday, *On the Physical Character of the Lines of Magnetic Force*, Phil. Mag. **3**, 401 (1852), http://kirkmcd.princeton.edu/examples/EM/faraday29a_pm_3_401_52_3243-3299.pdf

[18] G. Quincke, *Ueber die Verbreitung eines elektrischen Stromes in Metallplatten*, Ann. d. Phys. **7**, 382 (1856), http://kirkmcd.princeton.edu/examples/EM/quincke_ap_7_382_56.pdf
Mémoire sur la propagation de l'Électricité dans les plaques métallique, Ann. Chim. Phys. **47**, 203 (1856), http://kirkmcd.princeton.edu/examples/EM/quincke_acp_47_203_56.pdf

[19] R. Kohlsrausch and W. Weber, *Elektrodynamische Maassbestimmungen inbesondere Zurückführung der Stromintensitäts-Messungen auf Mechansishes Maass*, Abh. König. Sächs. Gesell. Wiss. 219 (1856), http://kirkmcd.princeton.edu/examples/EM/weber_aksgw_219_56.pdf

[20] W. Weber and R. Kohlsrausch, *Ueber die Elektricitätsmenge, welche bei galvanischen Strömen durch den Querschnitt der Kette fliesst*, Ann. d. Phys. **99**, 10 (1856),
http://kirkmcd.princeton.edu/examples/EM/weber_ap_99_10_56.pdf
http://kirkmcd.princeton.edu/examples/EM/weber_ap_99_10_56_english.pdf

[21] J.C. Maxwell, *On Faraday's Lines of Force*, Trans. Camb. Phil. Soc. **10**, 27 (1858),
http://kirkmcd.princeton.edu/examples/EM/maxwell_tcps_10_27_58.pdf

[22] H. Helmholtz, *Über Integrale der hydrodynamischen Gleichungen, welche den Wirbelbewegungen entsprechen*, Crelles J. **55**, 25 (1858),
http://kirkmcd.princeton.edu/examples/fluids/helmholtz_jram_55_25_58.pdf

[23] H, Fizeau, *Sur les hypothèses relatives à l'éther lumineux, et sur une expérience qui parait démontrer que le mouvement des corps change la vitesse avec laquelle la lumière se propage dans leur intérieur*, Ann. Chim. Phys. **57**, 385 (1859),
http://kirkmcd.princeton.edu/examples/GR/fizeau_acp_57_385_59.pdf
On the Effect of the Motion of a Body upon the Velocity with which it is traversed by Light, Phil. Mag. **19**, 245 (1860), http://kirkmcd.princeton.edu/examples/GR/fizeau_pm_19_245_60.pdf

[24] J.C. Maxwell, *On Physical Lines of Force. Part I.—The Theory of Molecular Vortices applied to Magnetic Phenomena*, Phil. Mag. **21**, 161 (1861),
http://kirkmcd.princeton.edu/examples/EM/maxwell_pm_21_161_61.pdf

[25] J.C. Maxwell, *On Physical Lines of Force. Part II.—The Theory of Molecular Vortices applied to Electric Currents*, Phil. Mag. **21**, 281, 338 (1861),
http://kirkmcd.princeton.edu/examples/EM/maxwell_pm_21_281_61.pdf

[26] J.C. Maxwell, *On Physical Lines of Force. Part III.—The Theory of Molecular Vortices applied to Statical Electricity*, Phil. Mag. **23**, 12 (1862),
http://kirkmcd.princeton.edu/examples/EM/maxwell_pm_23_12_62.pdf

[27] J.C. Maxwell, *On Physical Lines of Force. Part IV.—The Theory of Molecular Vortices applied to the Action of Magnetism on Polarized Light*, Phil. Mag. **23**, 85 (1862), http://kirkmcd.princeton.edu/examples/EM/maxwell_pm_23_85_62.pdf

[28] J.C. Maxwell, *A Dynamical Theory of the Electromagnetic Field*, Phil. Trans. Roy. Soc. London **155**, 459 (1865), http://kirkmcd.princeton.edu/examples/EM/maxwell_ptrsl_155_459_65.pdf

[29] B. Riemann, *Ein Beitrag zur Elektrodynamik*, Ann. d. Phys. **207**, 237 (1867), http://kirkmcd.princeton.edu/examples/EM/riemann_ap_207_237_67.pdf
A Contribution to Electrodynamics, Phil. Mag. **34**, 368 (1867), http://kirkmcd.princeton.edu/examples/EM/riemann_pm_34_368_67.pdf

[30] L. Lorenz, *Ueber die Identität der Schwingungen des Lichts mit den elektrischen Strömen*, Ann. d. Phys. **207**, 243 (1867), http://kirkmcd.princeton.edu/examples/EM/lorenz_ap_207_243_67.pdf
On the Identity of the Vibration of Light with Electrical Currents, Phil. Mag. **34**, 287 (1867), http://kirkmcd.princeton.edu/examples/EM/lorenz_pm_34_287_67.pdf

[31] J.C. Maxwell, *On a Method of making a Direct Comparison of Electrostatic with Electromagnetic Force; with a Note on the Electromagnetic Theory of Light*, Phil. Trans. Roy. Soc. London **158**, 643 (1868), http://kirkmcd.princeton.edu/examples/EM/maxwell_ptrsl_158_643_68.pdf

[32] W. Sellmeier, *Ueber die durch die Aetherschwingungen erregten Mitschwingungen der Körpertheilchen und deren Rückwirkung auf die erstern, besonders zur Erklärung der Dispersion und ihrer Anomalien*, Ann. d. Phys. **145**, 399 (1872), http://kirkmcd.princeton.edu/examples/optics/sellmeier_ap_145_399_72.pdf

[33] J.C. Maxwell, *Manuscript on the Theory of Anomalous Dispersion, circa 1873*, in *The Scientific Letters and Papers of James Clerk Maxwell*, Vol. II, Part II, ed. by P.M. Harman (Cambridge U. Press, 1994), p. 864, http://kirkmcd.princeton.edu/examples/optics/maxwell_ms461_73.pdf

[34] J.C. Maxwell, *A Treatise on Electricity and Magnetism*, Vol. 1 (Clarendon Press, 1873), http://kirkmcd.princeton.edu/examples/EM/maxwell_treatise_v1_73.pdf

[35] J.C. Maxwell, *A Treatise on Electricity and Magnetism*, Vol. 2 (Clarendon Press, 1873), http://kirkmcd.princeton.edu/examples/EM/maxwell_treatise_v2_73.pdf

[36] H. von Helmholtz, *Ueber die Theorie der Elektrodynamik. Dritte Abhandlung. Die Elektrodynamische Kräfte in bewegten Leitern*, J. Reine Angew. Math. **78**, 273 (1874), p. 309, http://kirkmcd.princeton.edu/examples/EM/helmholtz_jram_78_273_74.pdf

[37] H.A. Rowland, *On the Magnetic Effect of Electric Convection*, Am. J. Sci. **15**, 30 (1878), http://kirkmcd.princeton.edu/examples/EM/rowland_ajs_15_30_78.pdf

[38] J.J. Thomson, *On Maxwell's Theory of Light*, Phil. Mag. **9**, 284 (1880), http://kirkmcd.princeton.edu/examples/EM/thomson_pm_9_284_80.pdf

[39] J.J. Thomson, *On the Electric and Magnetic Effects produced by the Motion of Electrified Bodies*, Phil. Mag. **11**, 229 (1881),
http://kirkmcd.princeton.edu/examples/EM/thomson_pm_11_229_81.pdf

[40] H. Helmholtz, *Ueber die auf das Innere magnetische oder dielectrisch polarisirter Körper wirkenden Kräfte*, Ann. d. Phys. **13**, 385 (1881),
http://kirkmcd.princeton.edu/examples/EM/helmholtz_ap_13_385_81.pdf

[41] G.F. FitzGerald, *Note on Mr. J.J. Thomson's Investigation of the Electromagnetic Action of a Moving Electrified Sphere*, Phil. Mag. **13**, 302 (1882),
http://kirkmcd.princeton.edu/examples/EM/fitzgerald_pm_13_302_82.pdf

[42] G.F. FitzGerald, *On Electromagnetic Effects Due to the Motion of the Earth*, Trans. Roy. Dublin Soc. **1**, 319 (1882), http://kirkmcd.princeton.edu/examples/EM/fitzgerald_trds_82.pdf

[43] G.F. FitzGerald, *On Maxwell's Equations for the Electromagnetic Action of Moving Electricity*, Brit. Assoc. Rep. (1883),
http://kirkmcd.princeton.edu/examples/EM/fitzgerald_bar_83.pdf

[44] J. Larmor, *Electromagnetic Induction in Conducting Sheets and Solid Bodies*, Phil. Mag. **17**, 1 (1884), http://kirkmcd.princeton.edu/examples/EM/larmor_pm_17_1_84.pdf

[45] W. Thomson, *Reprints of Papers on Electrostatics and Magnetism* (Macmillan, 1884),
http://kirkmcd.princeton.edu/examples/EM/thomson_electrostatics_magnetism.pdf

[46] O. Heaviside, *Electromagnetic Induction and Its Propagation*, part 4, Electrician **14**, 306 (1885), http://kirkmcd.princeton.edu/examples/EM/heaviside_eip4_electrician_14_306_85.pdf
Also p. 447 of [57].

[47] O. Heaviside, *Electromagnetic Induction and Its Propagation*, part 12, Electrician **15**, 230 (1885); also p. 488 of [57].
http://kirkmcd.princeton.edu/examples/EM/heaviside_eip12_electrician_15_230_85.pdf

[48] O. Heaviside, *Electromagnetic Induction and Its Propagation*, part 22, Electrician **16**, 146 (1886), http://kirkmcd.princeton.edu/examples/EM/heaviside_electrician_16_146_86.pdf
Also p. 542 of [57].

[49] J.C. Maxwell, *An Elementary Treatise on Electricity*, 2nd ed. (Clarendon Press, 1888),
http://kirkmcd.princeton.edu/examples/EM/maxwell_elementary_treatise_88.pdf

[50] H.W. Watson, *Note on the Electromotive Force in Moving Conductors*, Phil. Mag. **25**, 271 (1888), http://kirkmcd.princeton.edu/examples/EM/watson_pm_25_271_88.pdf

[51] O. Heaviside, *On the Electromagnetic Effects due to the Motion of Electrification through a Dielectric*, Phil. Mag. **27**, 324 (1889),
http://kirkmcd.princeton.edu/examples/EM/heaviside_pm_27_324_89.pdf

[52] H.A. Rowland and C.T. Hutchinson, *On the Electromagnetic Effect of Convection-Currents*, Phil. Mag. **27**, 445 (1889),
http://kirkmcd.princeton.edu/examples/EM/rowland_pm_27_445_89.pdf

[53] J.J. Thomson, *On the Magnetic Effects produced by Motion in the Electric Field*, Phil. Mag. **28**, 1 (1889), http://kirkmcd.princeton.edu/examples/EM/thomson_pm_28_1_89.pdf

[54] J.C. Maxwell, *A Treatise on Electricity and Magnetism*, Vol. 1, 3rd ed. (Clarendon Press, 1892, 1904), http://kirkmcd.princeton.edu/examples/EM/maxwell_treatise_v1_04.pdf

[55] J.C. Maxwell, *A Treatise on Electricity and Magnetism*, Vol. 2, 3rd ed. (Clarendon Press, 1892), http://kirkmcd.princeton.edu/examples/EM/maxwell_treatise_v2_92.pdf

[56] H.A. Lorentz, *La Théorie Électromagnétique de Maxwell et son Application aux Corps Mouvants*, Arch. Neérl. **25**, 363 (1892), http://kirkmcd.princeton.edu/examples/EM/lorentz_ansen_25_363_92.pdf

[57] O. Heaviside, *Electrical Papers*, Vol. 1 (Macmillan, 1894), http://kirkmcd.princeton.edu/examples/EM/heaviside_electrical_papers_1.pdf

[58] J. Larmor, *A Dynamical Theory of the Electric and Luminiferous Medium Part II. Theory of Electrons*, Phil. Trans. Roy. Soc. London A **186**, 695 (1895), http://kirkmcd.princeton.edu/examples/EM/larmor_ptrsla_186_695_95.pdf

[59] H.A. Lorentz, *Versuch einer Theorie der Electrischen und Optischen Erscheinungen Bewegten Körpern* (E.J. Brill, 1895), http://kirkmcd.princeton.edu/examples/EM/lorentz_electrical_theory_95.pdf http://kirkmcd.princeton.edu/examples/EM/lorentz_electrical_theory_95_english.pdf

[60] J.W. Gibbs, *Velocity of Propagation of Electrostatic Forces*, Nature **53**, 509 (1896), http://kirkmcd.princeton.edu/examples/EM/gibbs_nature_53_509_96.pdf

[61] J.G. Leathem, *On the theory of the Magneto-Optic phenomena of Iron, Nickel, and Cobalt*, Phil. Trans. Roy. Soc. London A **190**, 89 (1897), http://kirkmcd.princeton.edu/examples/EM/leathem_ptrsla_190_89_97.pdf

[62] J. Larmor, *A Dynamical Theory of the Electric and Luminiferous Medium.—Part III. Relations with Material Media*, Phil. Trans. Roy. Soc. London A **190**, 205 (1897), http://kirkmcd.princeton.edu/examples/EM/larmor_ptrsla_190_205_97.pdf

[63] H.A. Lorentz, *Simplified Theory of Electrical and Optical Phenomena in Movings Systems*, Proc. Roy. Acad. Amsterdam **1**, 427 (1899), http://kirkmcd.princeton.edu/examples/EM/lorentz_pknaw_1_427_99.pdf

[64] J. Larmor, *Æther and Matter* (Cambridge U. Press, 1900), http://kirkmcd.princeton.edu/examples/EM/larmor_aether_matter_00.pdf

[65] H.A. Lorentz, *The fundamental equations for electromagnetic phenomena in ponderable bodies, deduced from the theory of electrons*. Proc. Roy. Acad. Amsterdam **5**, 254 (1902), http://kirkmcd.princeton.edu/examples/EM/lorentz_pknaw_5_254_02.pdf

[66] H.A. Lorentz, *Weiterbildung der Maxwellschen Theorie. Elektronentheorie*, Enzykl. Math. Wiss. **5**, part II, 145 (1904), http://kirkmcd.princeton.edu/examples/EM/lorentz_emw_5_2_145_04.pdf

[67] M. Abraham and A. Föppl, *Theorie der Elektizität*, Vol. 1, 2nd ed. (Teubner, 1904), http://kirkmcd.princeton.edu/examples/EM/abraham_foppl_elektrizitat_v1_04.pdf

[68] H.A. Lorentz, *Electromagnetic phenomena in a system moving with any velocity smaller than that of light*, Proc. Roy. Acad. Amsterdam **6**, 809 (1904), http://kirkmcd.princeton.edu/examples/EM/lorentz_pknaw_6_809_04.pdf

[69] A. Einstein, *Zur Elektrodynamik bewegter Körper*, Ann. d. Phys. **17**, 891 (1905), http://kirkmcd.princeton.edu/examples/EM/einstein_ap_17_891_05.pdf
http://kirkmcd.princeton.edu/examples/EM/einstein_ap_17_891_05_english.pdf

[70] M. Laue, *Die Mitführung des Lichtes durch bewegte Körper nach dem Relativitätsprinzip*, Ann. d. Phys. **23**, 989 (1907), http://kirkmcd.princeton.edu/examples/GR/vonlaue_ap_23_989_07.pdf

[71] H.A. Lorentz, *The Theory of Electrons* (B.G. Teubner, 1909), http://kirkmcd.princeton.edu/examples/EM/lorentz_theory_of_electrons_09.pdf

[72] P. Frank and H. Rothe, *Über die Transformation der Raumzeitkoordinaten von ruhenden auf bewegte Systeme*, Ann. d. Phys. **34**, 825 (1911), http://kirkmcd.princeton.edu/examples/GR/frank_ap_34_825_11.pdf

[73] E.T. Whittaker, *A History of the Theories of Aether and Electricity from the Age of Descartes to the Close of the Nineteenth Century* (Longmans, Green, 1910), http://kirkmcd.princeton.edu/examples/EM/whittaker_history.pdf

[74] J. Larmor, *The Origins of Clerk Maxwell's Electric Ideas, as Described in Familiar Letters to W. Thomson*, Proc. Camb. Phil. Soc. **32**, 695 (1936), http://kirkmcd.princeton.edu/examples/EM/larmor_pcps_32_695_36.pdf

[75] F. Rasetti, *Deflection of mesons in magnetized iron*, Phys. Rev. **66**, 1 (1944), http://kirkmcd.princeton.edu/examples/EM/rasetti_pr_66_1_44.pdf
This paper mentions the earlier history of erratic results on this topic.

[76] L. Page and N.I. Adams, Jr, *Action and Reaction Between Moving Charges*, Am. J. Phys. **13**, 141 (1945), http://kirkmcd.princeton.edu/examples/EM/page_ajp_13_141_45.pdf

[77] P.F. Cranefield, *Clerk Maxwell's Corrections to the Page Proofs of "A Dynamical Theory of the Electromagnetic Field"*, Ann. Sci. **10**, 359 (1954), http://kirkmcd.princeton.edu/examples/EM/cranefield_as_10_359_54.pdf

[78] Y. Aharonov and D. Bohm, *Significance of Electromagnetic Potentials in Quantum Theory*, Phys. Rev. **115**, 485 (1959), http://kirkmcd.princeton.edu/examples/QM/aharonov_pr_115_485_59.pdf

[79] W.K.H. Panofsky and M. Phillips, *Classical Electricity and Magnetism*, 2nd ed. (Addison-Wesley, 1962), http://kirkmcd.princeton.edu/examples/EM/panofsky-phillips.pdf

[80] J. Bromberg, *Maxwell's Electrostatics*, Am. J. Phys. **36**, 142 (1968), http://kirkmcd.princeton.edu/examples/EM/bromberg_ajp_36_142_68.pdf

[81] P.M. Heimann, *Maxwell and the Modes of Consistent Representation*, Arch. Hist. Exact Sci. **6**, 171 (1970), http://kirkmcd.princeton.edu/examples/EM/heimann_ahes_6_171_70.pdf

[82] P.H. Heimann, *Maxwell, Hertz, and the Nature of Electricity*, Isis **62**, 149 (1971), http://kirkmcd.princeton.edu/examples/EM/heimann_isis_62_149_71.pdf

[83] M. LeBellac and J.-M. Levy-Léblond, *Galilean Electromagnetism*, Nuovo Cim. **14B**, 217 (1973), http://kirkmcd.princeton.edu/examples/EM/lebellac_nc_14b_217_73.pdf

[84] K.T. McDonald, *Ph501 Lecture 9: Faraday's Law* (Oct. 1979), http://kirkmcd.princeton.edu/examples/ph501/ph501lecture9.pdf

[85] O. Knudsen, *19th Century Views on Induction in Moving Conductors*, Centaurus **24**, 346 (1980), http://kirkmcd.princeton.edu/examples/EM/knudsen_centaurus_24_346_80.pdf

[86] G.H. Keswani and C.W. Kilmister, *Intimations of Relativity Before Einstein*, Brit. J. Phil. Sci. **34**, 343 (1983), http://kirkmcd.princeton.edu/examples/GR/keswani_bjps_34_343_83.pdf

[87] U. Bartocci and M.M. Capria, *Some remarks on classical electromagnetism and the principle of relativity*, Am. J. Phys. **59**, 1030 (1991), http://kirkmcd.princeton.edu/examples/EM/bartocci_ajp_59_1030_91.pdf

[88] O. Darrigol, *The Electrodynamics of Moving Bodies from Faraday to Hertz*, Centaurus **36**, 245 (1993), http://kirkmcd.princeton.edu/examples/EM/darrigol_centaurus_36_245_93.pdf

[89] *The scientific letters and papers of James Clerk Maxwell*, Vol. II, Part I, ed. by P.M. Harman (Cambridge U. Press, 1995).

[90] M.G. Doncel and J.A. de Lorenzo, *The electrotonic state, a metaphysical device for Maxwell too?* Eur. J. Phys. **17**, 6 (1996), http://kirkmcd.princeton.edu/examples/EM/doncel_ejp_17_6_96.pdf

[91] O. Darrigol, *Electrodynamics from Ampère to Einstein* (Oxford U. Press, 2000), http://kirkmcd.princeton.edu/examples/EM/darrigol_em_00.pdf

[92] K.T. McDonald, *Electrodynamics Problem Set 8* (2001), http://www.kirkmcd.princeton.edu/examples/ph501set8.pdf

[93] D. Dameron, *An Electronic Attempt to Measure the Time Delay Calculated from the Force* $-q(\mathbf{v}\cdot\mathbf{grad})\mathbf{A}$, Phys. Essays **15**, 327 (2002), http://kirkmcd.princeton.edu/examples/EM/dameron_pe_15_327_02.pdf

[94] K.T. McDonald, *Methods of Calculating Forces on Rigid, Linear Magnetic Media* (Mar. 18, 2002), http://kirkmcd.princeton.edu/examples/magnetic_force.pdf

[95] D.J. Dunstan, *Derivation of special relativity from Maxwell and Newton*, Proc. Roy. Soc. London A **366**, 1861 (2008),
http://kirkmcd.princeton.edu/examples/EM/dunstan_ptrsla_366_1861_08.pdf

[96] K.T. McDonald, *Electrodynamics of Rotating Systems* (Aug. 6, 2008),
http://kirkmcd.princeton.edu/examples/rotating_EM.pdf

[97] K.T. McDonald, *Maxwell's Objection to Lorenz' Retarded Potentials* (Oct. 26, 2009),
http://kirkmcd.princeton.edu/examples/maxwell.pdf

[98] K.T. McDonald, *Displacement Current of a Uniformly Moving Charge* (June 15, 2010),
http://kirkmcd.princeton.edu/examples/dedt.pdf

[99] K.T. McDonald, *Charge Density in a Current-Carrying Wire* (Dec. 23, 2010),
http://kirkmcd.princeton.edu/examples/wire.pdf

[100] K.T. McDonald, *Potentials of a Hertzian Dipole in the Gibbs Gauge* (Aug. 23, 2012),
http://kirkmcd.princeton.edu/examples/gibbs.pdf

[101] K.T. McDonald, *Classical Aspects of the Aharonov-Bohm Effect* (Nov. 29, 2013),
http://kirkmcd.princeton.edu/examples/aharonov.pdf

[102] K.T. McDonald, *Force between Two Comoving Electric Charges* (May 24, 2014),
http://kirkmcd.princeton.edu/examples/comoving.pdf

[103] D.V. Redžić, *Force exerted by a moving electric current on a stationary or co-moving charge: Maxwell's theory versus relativistic electrodynamics*, Eur. J. Phys. **35**, 045011 (2014), http://kirkmcd.princeton.edu/examples/EM/redzic_ejp_35_045011_14.pdf

[104] K.-H. Yang and K.T. McDonald, *Formal Expressions for the Electromagnetic Potentials in Any Gauge* (Feb. 25, 2015), http://kirkmcd.princeton.edu/examples/gauge.pdf

[105] D.V. Redžić, *On an episode in the life of* $\mathfrak{E}$, Eur. J. Phys. **39**, 055206 (2018),
http://kirkmcd.princeton.edu/examples/EM/redzic_ejp_39_055206_18.pdf

[106] K.T. McDonald, *Is Faraday's Disk Dynamo a Flux-Rule Exception?* (July 27, 2019),
http://kirkmcd.princeton.edu/examples/faradaydisk.pdf