McKenna's Paradox:
Charged Particle Exiting the Side of a Solenoid Magnet

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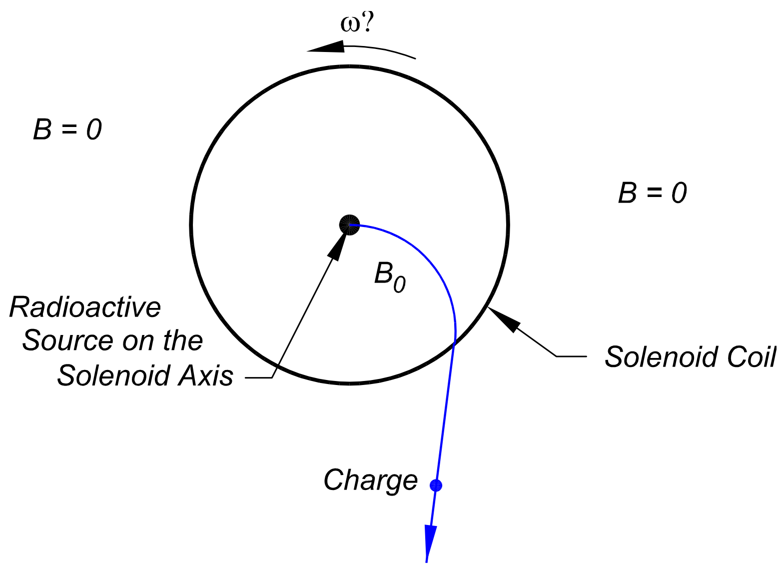
(April 12, 2006)

1 Problem

The "science fact" section of the March 1965 issue of Analog magazine featured an article by Richard P. McKenna that proposed a "motor" which would consist of a solenoid magnet that can rotate about its axis, plus a radioactive source mounted on that axis [1]. When a charged particle is emitted by a radioactive decay it traverses the solenoid field, which bends the particle's trajectory, giving it some angular momentum. By Newton's 3$^{\rm rd}$ law (argues McKenna), the reaction force on the magnet creates a torque that will cause the solenoid to spin in the opposite sense to that of the particle's trajectory.



McKenna seemed to be aware that this proposal could not actually work, but he lacked a consistent physical explanation as to why this would be so. Provide the missing explanation.

A side issue concerns the nature of the radioactive source, which must be charged before and/or after the emission of the moving charge. Here it suffices to consider the source to consist of a single charged particle that was introduced into the solenoid from infinity along the axis, experiencing no force in this process. The radioactive decays results in a neutral particle that can be ignored, plus the moving charge.

2 Solution

A direct calculation (sec. 2.1) of the torque about the z-axis of the solenoid due to the interaction of magnetic field of the moving charge with the currents in the solenoid shows that this torque vanishes. Hence, there is no tendency for the coil to rotate, and the system cannot be regarded as a motor.

However, this simple result is counterintuitive, because the solenoid coil does exert a torque on the charged particle, which causes its trajectory to be curved. The particle takes on (mechanical) angular momentum as a consequence. We expect that the total angular momentum of the system to be constant, since there are no external torques about the z-axis. Thus, it may seem that conservation of angular momentum is violated here.

But, properly considered, the charged particle has zero angular momentum at all times in this example. Therefore, we need not expect the solenoid to exhibit any "reaction angular momentum".[1]

The needed insight is that the system of a moving charged particle plus magnetic field contains both mechanical (angular) momentum as well as electromagnetic (angular) momentum. Because the system has azimuthal symmetry, the angular momentum of particle + field about the axis of the solenoid is constant in time. Also, as there are no external torques about the solenoid axis, the total angular momentum of the particle + field + solenoid about this axis is constant in time. Since the angular momentum of the particle + field does not change, neither does that of the solenoid.

The solenoid does NOT rotate in reaction to the deflection of the particle. Rather, the reaction appears in the electromagnetic fields of the system, which store a quantity of angular momentum equal and opposite to the mechanical angular momentum of the moving charge.

Section 2.2 considers electromagnetic (angular) momentum as part of the canonical angular) momentum of a charged particle in an external electromagnetic field. Section 2.3 calculates the electromagnetic (angular) momentum from the Poynting vector. A complication is that the approximation of an infinite solenoid is not sufficiently accurate when evaluating the electromagnetic angular momentum via the Poynting vector.[2]

The electromagnetic (angular) momentum is affected by possible shielding of the electric and magnetic fields by the materials from which the magnet is constructed. To simplify the discussion we consider two limiting cases:

- The magnet is made of nonconducting material, with relative dielectric constant and magnetic permeability both equal to unity, which permits the electric and magnetic fields to pass through the magnet without change. This could be arranged in principle by constructing the magnet out of two concentric cylindrical shells of opposite surface charge density that rotate in opposite senses, such that the magnet is electrically neutral everywhere.

- The magnetic is encased in a grounded, conducting shield with unit magnetic permeability. Then, charge e induces charges on the shell so as to eliminate the electric field

[1] The electric current that creates the magnetic field does have nonzero angular momentum. This angular momentum is also unchanged by the motion of the charged particle. The currents do not change, and the magnetic field due to these currents does not change.

[2] See also [2].

outside the shell when the charge is inside, or to eliminate the electric field inside the shell when the charge is outside. In this case we say that the magnet coil is shielded from effects of the electric field, although the presence of the induced charge on the shield leads to additional interactions with the charge e that must be considered.

This problem was drawn to my attention by Romer [3, 4] who considered McKenna's paradox only from the point of view of Poynting. Related discussions are given in [2, 5, 6].

2.1 Direct Calculation of the Torque on the Solenoid Coil

The moving charge creates a magnetic field $\mathbf{B}_e$ that exerts a force on the currents in the solenoid coil. These currents are purely azimuthal in the approximation of a long solenoid, where the surface current density $\mathbf{K}$ in the coil is given by,

$$\mathbf{K} = K_\phi \hat{\boldsymbol{\phi}} = \frac{c}{4\pi} B_0 \hat{\boldsymbol{\phi}}. \tag{1}$$

The magnetic force on a surface element of area $r_0\, d\phi\, dz$, centered on the point $(\rho = r_0, \phi, z)$ in a cylindrical coordinate system whose axis coincides with a axis of the solenoid of radius r_0, is (in Gaussian units),

$$\begin{aligned} d\mathbf{F} &= \frac{1}{c}\mathbf{K} \times \mathbf{B}_e(r_0, \phi, z)\ r_0\, d\phi\, dz = \frac{B_0}{4\pi}\hat{\boldsymbol{\phi}} \times (B_{e,\rho}\hat{\boldsymbol{\rho}} + B_{e,\phi}\hat{\boldsymbol{\phi}} + B_{e,z}\hat{\mathbf{z}})\ r_0\, d\phi\, dz \\ &= \frac{B_0}{4\pi}(B_{e,z}\hat{\boldsymbol{\rho}} - B_{e,\rho}\hat{\mathbf{z}})\ r_0\, d\phi\, dz, \end{aligned} \tag{2}$$

which has no azimuthal component. However, there must be an azimuthal force for there to be a torque $\boldsymbol{\tau}$ about the z axis,

$$d\boldsymbol{\tau} = \mathbf{r} \times d\mathbf{F} = (\rho\hat{\boldsymbol{\rho}} + z\hat{\mathbf{z}}) \times (dF_\rho\hat{\boldsymbol{\rho}} + dF_\phi\hat{\boldsymbol{\phi}} + dF_z\hat{\mathbf{z}}) = -z\, dF_\phi\hat{\boldsymbol{\rho}} + (z\, dF_\rho - \rho\, dF_z)\hat{\boldsymbol{\phi}} + r\, dF_\phi\hat{\mathbf{z}}. \tag{3}$$

Thus, there is no torque on the coil about the z-axis in McKenna's example, and the solenoid coil does NOT rotate in reaction to the deflection of the charged particle.

The integral of the force (2) on the coil is nonzero, and if its axis is not restrained the coil will move in reaction to the deflection of the charged particle. See also [6].

2.2 Solution via the Canonical Angular Momentum

An alternative solution is based on the concept of canonical angular momentum.

We recall [7] that the canonical momentum $\mathbf{p}$ of a particle of charge e and rest mass m is (in rectangular coordinates and in Gaussian units),

$$\mathbf{p} = \mathbf{P} + \frac{e\mathbf{A}}{c}\,, \tag{4}$$

where $\mathbf{P} = m\mathbf{v}/\sqrt{1 - v^2/c^2}$ is the mechanical momentum of the particle, $\mathbf{v}$ is the particle's velocity, $\mathbf{A}$ is the vector potential of the magnetic field at the position of the particle, and c

is the speed of light. Following Faraday[3] and Maxwell [12], we identify,

$$\mathbf{P}_{\mathrm{EM_M}} = \frac{e\mathbf{A}}{c} \tag{5}$$

as the electromagnetic momentum associated with the interaction of charge e with the magnetic field whose vector potential is $\mathbf{A}$. The subscript M indicates that this interpretation is due to Maxwell.

We are then led to identify the angular momentum about the origin of the particle + field as,

$$\mathbf{l} = \mathbf{r} \times \mathbf{p}, \tag{6}$$

where $\mathbf{r}$ is the position vector of the particle. As shown in Appendix A.1, if the electromagnetic fields and their potentials have azimuthal symmetry, then the axial component of the canonical angular momentum (6) is a conserved quantity.

We adopt a cylindrical coordinate system (ρ, ϕ, z) with the axis of the solenoid along the z-axis. The radius of the solenoid is labeled r_0 and the magnetic field strength is $\mathbf{B} = B_0\hat{\mathbf{z}}$ for $\rho < r_0$ and zero for $\rho > r_0$ in the approximation of a long solenoid.

The magnetic field is related to the vector potential $\mathbf{A}$ according to $\mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A}$. Stokes' theorem, along with the symmetry of the solenoid, then tells us that the only nontrivial component of the vector potential is A_ϕ, as given by,

$$A_\phi = \begin{cases} \frac{\rho B_0}{2} & (\rho < r_0), \\ \frac{r_0^2 B_0}{2\rho} & (\rho > r_0). \end{cases} \tag{7}$$

In particular, the vector potential vanishes on the axis $\rho = 0$.

The z-component of the canonical angular momentum can be written as,

$$l_z = \rho\left(P_\phi + \frac{e}{c}A_\phi\right) = \rho P_\phi + l_{\mathrm{EM_M},z}, \tag{8}$$

where,

$$l_{\mathrm{EM_M},z} = \frac{e\rho A_\phi}{c} \tag{9}$$

is the electromagnetic part of the canonical angular momentum of charge e.

2.2.1 Nonconducting Solenoid

The charge e is the only nontrivial charge distribution in the case of a nonconducting solenoid.

When the charged particle is emitted from the axis, its initial mechanical momentum $\mathbf{P}$ is purely radial. So $P_{\phi,\mathrm{initial}} = 0$. As seen above, the initial vector potential on the axis is also zero. Hence, the initial canonical angular momentum of the particle + field is also zero.

Since the canonical angular momentum of the particle + field is a conserved quantity in this problem, it remains zero at all times during the particle's motion. We verify this explicitly in Appendix A.2.

[3]Electromagnetic momentum can be identified with the *electro-tonic* state, first discussed by Faraday in Art. 60 of [8]. Other mentions by Faraday of the electrotonic state include Art. 1661 of [9], Arts. 1729 and 1733 of [10], and Art. 3269 of [11].

The z-component of the total angular momentum of the particle + field + solenoid is also constant in time because there are no external torques about the z-axis. Hence, the z-component of the angular momentum of the solenoid is constant, and it experiences no rotation in reaction to the deflection of the particle.

The system does not act as a motor.

2.2.2 Grounded, Shielded Solenoid

If the solenoid coil is encased in a grounded, conducting shield, then charge $-e$ is induced on this shield at all times.

Since the shield is at radius r_0 and has negligible thickness, the electromagnetic angular momentum associated with the induced charge $-e$ is, from eq. (9),

$$l_{\mathrm{EM_M},z}(\mathrm{shield}) = -\frac{e r_0 A_\phi(r_0)}{c} = -\frac{e B_0 r_0^2}{2c}\,, \tag{10}$$

independent of the position of charge e. This constant angular momentum does not imply any additional torque in the problem, so the magnet again does not rotate.

The total field angular momentum when charge e is at radius ρ is,

$$l_{\mathrm{EM_M},z}(\rho) = \frac{e[\rho A_\phi(\rho) - r_0 A_\phi(r_0)]}{c} = \begin{cases} \frac{eB_0}{2c}(\rho^2 - r_0^2) & (\rho < r_0), \\ 0 & (\rho > r_0). \end{cases} \tag{11}$$

A particular feature of eq. (11) is that the field angular momentum vanishes when the charge e is outside a shielded solenoid.

The field angular momentum (11) for a grounded, shielded solenoid differs by a constant from eq. (10) as found for a nonconducting solenoid. Conservation of total angular momentum, mechanical plus electromagnetic, is unaffected by the presence of a constant term, so the solenoid does not rotate in reaction to the motion of charge e whether the solenoid is nonconducting or shielded.

We conclude this section with some remarks about electromagnetic momentum.

The charge induced on the shield of the solenoid exerts an attractive electric force on charge e, but this force is small compared to the magnetic force for any reasonable values of v and B_0. So, the motion of the charge is essentially the same for a nonconducting magnet or a shielded magnet.

However, the charge induced on the shield makes a significant change to the electromagnetic momentum $\mathbf{P}_{\mathrm{EM_M}}$, namely,

$$\begin{aligned} \mathbf{P}_{\mathrm{EM_M}}(\mathrm{shield}) &= \int \sigma(\phi, z)\frac{A_\phi(r_0)}{c}\hat{\boldsymbol{\phi}}\ d\mathrm{Area} = \frac{B_0 r_0}{2c}\int \sigma(\phi, z)(-\sin\phi\,\hat{\mathbf{x}} + \cos\phi\,\hat{\mathbf{y}})\ d\mathrm{Area} \\ &= \frac{B_0 r_0}{2c}\,\hat{\mathbf{y}}\int \sigma(\phi, z)\cos\phi\ d\mathrm{Area}, \end{aligned} \tag{12}$$

taking the charge e to be on the x axis, so that the induced charge distribution $\sigma(\phi, z)$ is symmetric in azimuthal angle ϕ.

To complete the calculation we need to know the ϕ dependence (but not the z dependence) of the induced charge distribution. Now, a line of charge parallel to the axis of the magnet leads to the same azimuthal distribution of induced charge as does the point charge e provided the line charge passes through the point charge. We recall that the electric field of a grounded, conducting cylinder of radius r_0 plus a line charge e at radius ρ can be described by an image method with an image line charge $-e$ at radius $\rho' = r_0^2/\rho$ [13]. Adding the electric fields from these two line charges at (r_0, ϕ) and dividing by 4π we obtain the azimuthal distribution $\sigma(\phi)$ of induced charge. Using this, we define $x = \rho/r_0$ if $\rho < r_0$ and $x = r_0/\rho$ if $\rho > r_0$ to obtain,

$$\mathbf{P}_{\mathrm{EM_M}}(\text{shield}) = -\frac{eB_0 r_0}{4\pi c}\,\hat{\mathbf{y}} \int_0^{2\pi} \cos\phi\, d\phi \frac{\sqrt{1 - 2x\cos\phi + x^2\cos^2\phi} + x\sqrt{x^2 - 2x\cos\phi + \cos^2\phi}}{1 - 2x\cos\phi + x^2}\,. \tag{13}$$

I believe that the integral in eq. (13) has the value $2\pi x$, which implies that the total electromagnetic momentum associated with a grounded, shielded solenoid is zero.

2.3 Electromagnetic Momentum via the Poynting Vector

Another well-known representation of electromagnetic momentum is via the Poynting vector [14], $c\mathbf{E} \times \mathbf{B}/4\pi$, following the insight of Poincaré [15] that this also corresponds the volume density of electromagnetic momentum when divided by c^2. Hence, the total momentum stored in the electromagnetic field is,

$$\mathbf{P}_{\mathrm{EM_P}} = \int \frac{\mathbf{E}(\mathbf{r}') \times \mathbf{B}(\mathbf{r}')}{4\pi c}\, d\mathrm{Vol}', \tag{14}$$

where the subscript P indicates that this interpretation follows Poynting.

The angular momentum stored in the electromagnetic field can then be deduced from the Poynting vector according to,

$$\mathbf{L}_{\mathrm{EM_P}} = \int \frac{\mathbf{r}' \times (\mathbf{E}(\mathbf{r}') \times \mathbf{B}(\mathbf{r}'))}{4\pi c}\, d\mathrm{Vol}'. \tag{15}$$

In Appendix A.3 we show that for static systems with charges and currents of finite extent the same values of momentum and angular momentum are obtained using either the canonical electromagnetic momentum or the Poynting vector. Here we verify this explicitly for McKenna's example, which will require care in dealing with a long solenoid.[4]

When the charge is still on the axis, and assuming that the solenoid is infinite, the Poynting vector of the system circulates about the z-axis in a negative sense, so that the electromagnetic angular momentum calculated according to eq. (15) has a negative z-component. This is in contrast to the electromagnetic angular momentum of eq. (9), which vanishes when the charge is on the axis. At the end of sec. 2.3.1 we show that no such discrepancy exists for a finite solenoid.

[4]See also [2].

The need for much more extensive discussion of electromagnetic momentum in static examples via the approach of Poynting compared to that using canonical momentum suggests to me that it is preferable to use the latter in these examples. The strength of Poynting's approach is realized in dynamical situations in which radiation is present and electromagnetic energy and momentum are not localized to the vicinity of the charges and currents.

2.3.1 Nonconducting Solenoid

In the case of a nonconducting solenoid the only electric field is that associated with the moving charge, $\mathbf{E}_e$. The electric fields due to the two shells of charge that comprise the solenoid cancel one another.

In the low-velocity limit the electric field of the charge e has no dependence on its velocity, so the integrals (14) and (15) depend only on the position of the charge, which we can take to be $\mathbf{r} = (\rho, 0, 0)$ without loss of generality. For an observation point $\mathbf{r}' = (\rho', \phi', z')$, whose rectangular coordinates are $[x', y', z'] = [\rho' \cos\phi', \rho' \sin\phi', z']$, the vector $\mathbf{R}$ from the charge to the observer is,

$$\mathbf{R} = \mathbf{r}' - \mathbf{r} = [x' - \rho, y', z'] \tag{16}$$

in rectangular coordinates. Then,

$$R = |\mathbf{R}| = \sqrt{(x' - \rho)^2 + y'^2 + z'^2} = \sqrt{\rho^2 + \rho'^2 - 2\rho\rho' \cos\phi' + z'^2}, \tag{17}$$

and the electric field at the observer is,

$$\mathbf{E}_e(\mathbf{r}') = e\frac{\mathbf{R}}{R^3} \tag{18}$$

The total magnetic field is the sum of that due to the moving charge plus that due to the solenoid, $\mathbf{B} = \mathbf{B}_e + B_0\hat{\mathbf{z}}$. However, the inclusion of the field $\mathbf{B}_e$ of the charge in the integrals (14) and (15) leads to infinite "self-momentum" terms, which we "renormalize" to be part of the finite "mechanical" momentum of the charge. Thus, we use only $\mathbf{B}(\mathbf{r}') = B_0\hat{\mathbf{z}}$ for $\rho' < r_0$ and $\mathbf{B} = 0$ for $\rho' > r_0$ when calculating the (interaction) electromagnetic momenta in this example.

In rectangular coordinates, we have that,

$$\mathbf{E}_e(\mathbf{r}') \times B_0\hat{\mathbf{z}} = \frac{eB_0}{R^3}[x' - \rho, y', z'] \times [0, 0, 1] = \frac{eB_0}{R^3}[y', \rho - x', 0]. \tag{19}$$

The x-component of eq. (19) is an odd function of y', so the x-component of integral (14) vanishes. All that remains is the y-component of integral (14), which we identify with the ϕ component of the electromagnetic momentum,

$$P_{\mathrm{EM_P},\phi} = \frac{eB_0}{4\pi c} \int \frac{\rho - x'}{R^3}\, d\mathrm{Vol}'. \tag{20}$$

Now,

$$\rho \int \frac{d\mathrm{Vol}'}{R^3} = \rho \int_0^{r_0} \rho'\, d\rho' \int_0^{2\pi} d\phi' \int_{-\infty}^{\infty} \frac{dz'}{(\rho^2 + \rho'^2 - 2\rho\rho' \cos\phi' + z'^2)^{3/2}}$$

$$
\begin{aligned}
&= \rho \int_0^{r_0^2} d\rho'^2 \int_0^{2\pi} \frac{d\phi'}{\rho^2 + \rho'^2 - 2\rho\rho' \cos\phi'} = 2\pi\rho \int_0^{r_0^2} \frac{d\rho'^2}{\left|\rho^2 - \rho'^2\right|} \\
&= 2\pi \left\{ \begin{array}{ll} \rho \ln \frac{\rho^2}{\rho^2 - r_0^2} & (\rho > r_0), \\ \rho \int_0^{\rho^2} \frac{d\rho'^2}{\rho^2 - \rho'^2} + \rho \int_{\rho^2}^{r_0^2} \frac{d\rho'^2}{\rho'^2 - \rho^2} & (\rho < r_0), \end{array} \right. \qquad (21)
\end{aligned}
$$

using Dwight 200.03 and 858.525 [16]. Also, using Dwight 858.536 we have,

$$
\begin{aligned}
\int \frac{x'\, d\mathrm{Vol}'}{R^3} &= \int_0^{r_0} \rho'^2\, d\rho' \int_0^{2\pi} \cos\phi'\, d\phi' \int_{-\infty}^{\infty} \frac{dz'}{(\rho^2 + \rho'^2 - 2\rho\rho' \cos\phi' + z'^2)^{3/2}} \\
&= \int_0^{r_0^2} \rho'\, d\rho'^2 \int_0^{2\pi} \frac{\cos\phi'\, d\phi'}{\rho^2 + \rho'^2 - 2\rho\rho' \cos\phi'} \\
&= 2\pi \left\{ \begin{array}{ll} \frac{1}{r\rho} \int_0^{r_0^2} \frac{\rho'^2\, d\rho'^2}{\rho^2 - \rho'^2} = \rho \ln \frac{\rho^2}{\rho^2 - r_0^2} - \frac{r_0^2}{\rho} & (\rho > r_0 > \rho'), \\ \frac{1}{\rho} \int_0^{\rho^2} \frac{\rho'^2\, d\rho'^2}{\rho^2 - \rho'^2} + \rho \int_{\rho^2}^{r_0^2} \frac{d\rho'^2}{\rho'^2 - \rho^2} & (\rho < r_0). \end{array} \right. \qquad (22)
\end{aligned}
$$

Inserting eqs. (21) and (22) into (20) and recalling eq. (7), we find,

$$
P_{\mathrm{EM_P},\phi} = \frac{eB_0}{2c} \left\{ \begin{array}{ll} \frac{r_0^2}{\rho} & (\rho > r_0) \\ \int_0^{\rho^2} \frac{\rho - \rho'^2/\rho}{\rho^2 - \rho'^2}\, d\rho'^2 = \rho & (\rho < r_0) \end{array} \right\} = \frac{eA_\phi(\rho)}{c}\,. \qquad (23)
$$

Thus, the electromagnetic momentum (23) calculated according to Poynting is equal to the canonical electromagnetic momentum (5) for the vector potential (7).

We now evaluate the electromagnetic angular momentum (15) using eq. (19),

$$
\begin{aligned}
\mathbf{L}_{\mathrm{EM_P}} &= \int \frac{\mathbf{r}' \times (\mathbf{E}_e \times B_0 \hat{\mathbf{z}})}{4\pi c}\, d\mathrm{Vol}' = \frac{eB_0}{4\pi c} \int \frac{[x', y', z'] \times [y', \rho - x', 0]}{R^3}\, d\mathrm{Vol}' \\
&= \frac{eB_0}{4\pi c} \int \frac{(x' - r)z'\hat{\mathbf{x}} + y'z'\hat{\mathbf{y}} + (x'\rho - \rho'^2)\hat{\mathbf{z}}}{R^3}\, d\mathrm{Vol}' = \frac{eB_0}{4\pi c} \hat{\mathbf{z}} \int \frac{x'\rho - \rho'^2}{R^3}\, d\mathrm{Vol}'. \qquad (24)
\end{aligned}
$$

We have,

$$
\begin{aligned}
\int \frac{\rho'^2\ d\mathrm{Vol}'}{R^3} &= \int_0^{r_0} \rho'^3\, d\rho' \int_0^{2\pi} d\phi' \int_{-\infty}^{\infty} \frac{dz'}{(\rho^2 + \rho'^2 - 2\rho\rho' \cos\phi' + z'^2)^{3/2}} \\
&= \int_0^{r_0^2} \rho'^2\, d\rho'^2 \int_0^{2\pi} \frac{d\phi'}{\rho^2 + \rho'^2 - 2\rho\rho' \cos\phi'} = 2\pi \int_0^{r_0^2} \frac{\rho'^2\, d\rho'^2}{\left|\rho^2 - \rho'^2\right|} \\
&= 2\pi \left\{ \begin{array}{ll} \rho^2 \ln \frac{\rho^2}{\rho^2 - r_0^2} - r_0^2 & (\rho > r_0), \\ \int_0^{\rho^2} \frac{\rho'^2\, d\rho'^2}{\rho^2 - \rho'^2} + \int_{\rho^2}^{r_0^2} \frac{\rho'^2\, d\rho'^2}{\rho'^2 - \rho^2} & (\rho < r_0). \end{array} \right. \qquad (25)
\end{aligned}
$$

Inserting eqs. (22) and (25) into (24) and recalling eq. (7) we find,

$$
\begin{aligned}
\mathbf{L}_{\mathrm{EM_P}}(\mathbf{r}) &= \frac{eB_0}{2c} \hat{\mathbf{z}} \left\{ \begin{array}{ll} 0 & (\rho > r_0) \\ \int_{\rho^2}^{r_0^2} \frac{\rho'^2 - \rho^2}{\rho^2 - \rho'^2}\, d\rho'^2 = \rho^2 - r_0^2 & (\rho < r_0) \end{array} \right\} = \rho \frac{eA_\phi(\rho)}{c} \hat{\mathbf{z}} + \mathbf{L}_{\mathrm{EM_P}}(0) \\
&= l_{\mathrm{EM_M}}(\mathbf{r})\, \hat{\mathbf{z}} + \mathbf{L}_{\mathrm{EM_P}}(0) \qquad \text{(infinite solenoid)}, \qquad (26)
\end{aligned}
$$

where,

$$\mathbf{L}_{\mathrm{EM_P}}(0) = -\frac{eB_0 r_0^2}{2c}\hat{\mathbf{z}} \qquad \text{(infinite solenoid)}, \tag{27}$$

is the electromagnetic angular momentum of an infinite solenoid when charge e is at the origin. When the charge e is outside an infinite solenoid, the electromagnetic momentum density is nonzero only inside the solenoid, where it has roughly the same direction everywhere; there is no moment of electromagnetic momentum (*i.e.*, no electromagnetic angular momentum) about the solenoid axis when $\rho > r_0$.

The electromagnetic angular momentum (26) in McKenna's example differs by a constant from the electromagnetic part, (9), of the canonical angular momentum. Both forms imply the same changes in angular momentum as the position of the charge changes, so both forms are adequate to verify conservation of angular momentum during the particle's motion.

McKenna's example approximates a real, finite-length solenoid by an infinite solenoid. As was remarked by Romer [4], and more explicitly by Johnson *et al.* [6], the weak magnetic field outside a long, but finite solenoid contributes in significant measure to the electromagnetic angular momentum, but not to the electromagnetic momentum, when an additional charge is present. We now give a model of the magnetic field outside a long, but finite solenoid which indicates that the electromagnetic angular momentum of a long solenoid plus charge e is the same whether calculated by eq. (9) or by eq. (15).

A useful approximation to the field outside a solenoid of length $L \gg r_0$ is to suppose that a pair of magnetic charges of strength $\pm p$, where,

$$p = \frac{\pi r_0^2 B_0}{4\pi} = \frac{B_0 r_0^2}{4}, \tag{28}$$

are located on the axis of the solenoid at $z = \pm L/2$. The magnetic flux, $4\pi p$, due to each of the magnetic charges is equal to the flux contained within the solenoid, $\pi r_0^2 B_0$, which flux then spreads out over all space outside the solenoid. We thereby obtain an estimate for the magnetic field near the plane $z = 0$ outside the solenoid,

$$B_z(\rho > r_0, z \approx 0) \approx -\frac{p}{L^2(1 + 4\rho^2/L^2)^{3/2}} = -\frac{r_0^2 B_0}{4L^2(1 + 4\rho^2/L^2)^{3/2}}. \tag{29}$$

We can use this approximation to obtain corrections to eqs. (14) and (15) for the electromagnetic momentum and angular momentum. However, it is quicker to use the insight of J.J. Thomson [17, 18, 19], that the electromagnetic momentum vanishes for a pair of electric and magnetic charges, e and p respectively, but their electromagnetic angular momentum according to eq. (15) is,

$$\mathbf{L}_{\mathrm{EM_P}} = \frac{ep}{c}\hat{\mathbf{r}}_{ep}, \tag{30}$$

which is directed along the line from the electric charge to the magnetic charge. Then, the electromagnetic angular momentum associated with the exterior magnetic field of the solenoid, as represented by two magnetic charges $\pm p$ at $z = \pm L/2$ when charge e is at position $(\rho, 0, 0)$, is,

$$\mathbf{L}_{\mathrm{EM_P}} = 2\frac{ep}{c}\frac{L/2}{\sqrt{\rho^2 + L^2/4}}\,\hat{\mathbf{z}} \approx 2\frac{ep}{c}\,\hat{\mathbf{z}} = \frac{eB_0 r_0^2}{2c}\,\hat{\mathbf{z}} \quad \text{(in exterior field of a long solenoid)}, \tag{31}$$

where the approximation holds for $\rho \ll L$. This result is the negative of eq. (27), so eq. (26) is corrected to become

$$\mathbf{L}_{\mathrm{EM_P}}(\mathbf{r}) = \rho \frac{eA_\phi(\rho)}{c} \hat{\mathbf{z}} = l_{\mathrm{EM_M}}(\mathbf{r})\, \hat{\mathbf{z}} \qquad \text{(long, but finite solenoid).} \tag{32}$$

2.3.2 Grounded, Shielded Solenoid

If the solenoid coil is encased in a grounded, conducting shield then charge $-e$ is induced on this shield independent of the position of charge e. We suppose that the shield has negligible thickness, so the radial coordinate of the induced charge is r_0. The induced charge has a distribution in coordinates ϕ and z, and the extent of this distribution in z is of order $|\rho - r_0|$ when charge e is at radius ρ.

The calculation of $\mathbf{L}_{\mathrm{EM_P}}$ for charge e did not depend on its position in ϕ or z, so the contribution to the electromagnetic momentum from the charge $-e$ induced at radius $\rho = r_0$ can be found by appropriate use of the calculations for charge e in sec. 2.3.1.

When charge e is inside a shielded solenoid, an electric field exists only for $\rho < r_0$, so there is no contribution to the electromagnetic angular momentum from the region outside the magnet. Therefore, the result of eq. (26) for $\rho < r_0$ correctly describes the electromagnetic angular momentum due to charge e for both an infinite shielded solenoid as well as a long, but finite shielded solenoid. The electromagnetic angular momentum associated with the charge $-e$ that resides on the inner surface of the shield is also described by eq. (26) on setting ρ to r_0 (and changing e to $-e$), and hence this contribution is zero.

When the charge e is outside the shielded magnet, an electric field exists only for $\rho > r_0$. In the approximation of an infinite solenoid there is no electromagnetic angular momentum. For a long, but finite solenoid we use eq. (31), but since the total charge is zero, the electromagnetic angular momentum is zero.

In sum, for charge e in the vicinity of a grounded, shielded, finite solenoid the electromagnetic angular momentum is,

$$L_{\mathrm{EM_P},z}(\rho) = \begin{cases} \frac{eB_0}{2c}(\rho^2 - r_0^2) & (\rho < r_0), \\ 0 & (\rho > r_0). \end{cases} = \frac{e[\rho A_\phi(\rho) - r_0 A_\phi(r_0)]}{c} = l_{\mathrm{EM_M},z}(\rho), \tag{33}$$

in agreement with eq. (11) obtained using the canonical electromagnetic momentum.

Turning to the electromagnetic momentum in the case of a shielded solenoid, we immediately see that this is zero when charge e is outside the solenoid. Here, an electric field exists only outside the solenoid, so eq. (23) does not apply when $\rho > r_0$. Any electromagnetic momentum would be due to the weak magnetic field exterior to the solenoid, but this momentum is zero as argued at the end of sec. 2.3.1 [17, 18].

When charge e is inside a shielded solenoid the electromagnetic momentum is due to the fields of this charge plus those of the charge $-e$ induced on the inner surface of the shield.

The electromagnetic momentum associated with charge e is correctly given by eq. (23) for $\rho < r_0$, namely $P_{\mathrm{EM_P},\phi}(e) = eB_0\rho/2c$.

The electromagnetic momentum associated with the induced charge does not depend on the location of this charge in z, so we can evaluate this momentum by supposing that charge

e is spread over a line at radius ρ. Then, the electric field due to the induced charge on a shielded solenoid is the same as the field inside a nonconducting solenoid from a line charge $-e$ at radius r_0^2/ρ [13]. We can eq. (23) for $\rho' = r_0^2/\rho > r_0$ and $e' = -e$, since this evaluates the electromagnetic momentum due only to fields inside the solenoid even when the charge is outside. Thus, $P_{\mathrm{EM_P},\phi}(\text{induced}) = -eB_0 r_0^2/2c\rho' = -eB_0\rho/2c = -P_{\mathrm{EM_P},\phi}(e)$.

The total electromagnetic momentum is zero for charge e either inside or outside of a grounded, shielded, finite solenoid.

A Appendices

A.1 Azimuthal Symmetry Implies that $\mathbf{l}_z$ Is Conserved

One way to deduce the conserved quantities for the particle's motion is to consider its Lagrangian or Hamiltonian. If an external electromagnetic field is present as well, with electric potential V and vector potential $\mathbf{A}$, the Lagrangian $\mathcal{L}$ of the particle can be written as,

$$\mathcal{L} = -\frac{mc^2}{\gamma} + \frac{e\mathbf{A}\cdot\mathbf{v}}{c} - eV, \tag{34}$$

where $\mathbf{v} = d\mathbf{r}/dt$ is the particle's velocity [7]. The canonical momentum associated with a rectangular coordinate x_i is therefore $p_i = \partial\mathcal{L}/\partial\dot{x}_i$, leading to eq. (4). Then, the Hamiltonian $\mathcal{H}$ of the system is,

$$\mathcal{H} = \sqrt{m^2c^4 + \left(\mathbf{p} - \frac{e\mathbf{A}}{c}\right)^2} + eV. \tag{35}$$

If the external electromagnetic fields have azimuthal symmetry, then the potentials V and $\mathbf{A}$ do also. We consider a cylindrical coordinate system (r, ϕ, z) with the z axis being the axis of symmetry of the fields. Then, both the Lagrangian and the Hamiltonian have no azimuthal dependence,

$$\frac{\partial\mathcal{L}}{\partial\phi} = \frac{\partial\mathcal{H}}{\partial\phi} = 0, \tag{36}$$

so the equations of motion (and the identities $\mathbf{r} = \rho\hat{\boldsymbol{\rho}} + z\hat{\mathbf{z}}$, $\dot{\mathbf{r}} = \mathbf{v} = \dot{\rho}\hat{\boldsymbol{\rho}} + \rho\dot{\phi}\hat{\boldsymbol{\phi}} + \dot{z}\hat{\mathbf{z}}$) tell us that the canonical momentum p_ϕ is a constant of the motion (even for time-dependent fields, so long as they are azimuthally symmetric),[5]

$$p_\phi = \frac{\partial\mathcal{L}}{\partial\dot{\phi}} = \rho\left(\gamma m\rho\dot{\phi} + \frac{eA_\phi}{c}\right) = \rho(\mathbf{p})_\phi = l_z. \tag{37}$$

We also see that the canonical momentum p_ϕ can be interpreted as the z component of the canonical angular momentum (6), so l_z is also a constant of the motion.

For completeness, we verify that $dl_z/dt = 0$ using the Lorentz force law,

$$\frac{d\mathbf{P}}{dt} = e\left(\mathbf{E} + \frac{\mathbf{v}}{c}\times\mathbf{B}\right) = e\left(-\nabla V - \frac{1}{c}\frac{\partial\mathbf{A}}{\partial t} + \frac{\mathbf{v}}{c}\times(\nabla\times\mathbf{A})\right). \tag{38}$$

[5] Note that the definition (37) of the canonical momentum p_ϕ leads to the awkwardness that $p_\phi = \rho(\mathbf{p})_\phi$, where $(\mathbf{p})_\phi$ is the ϕ component of the canonical momentum vector $\mathbf{p}$ of eq. (4).

We begin with the ordinary angular momentum $\mathbf{L} = \mathbf{r} \times \mathbf{P}$, and consider the z component of its time derivative,

$$\frac{dL_z}{dt} = \frac{d(\mathbf{r} \times \mathbf{P})_z}{dt} = \left(\mathbf{r} \times \frac{d\mathbf{P}}{dt}\right)_z = \rho \left(\frac{d\mathbf{P}}{dt}\right)_\phi . \tag{39}$$

From eq. (38) we have, since $\partial V/\partial\phi = \partial A_r/\partial\phi = \partial A_z/\partial\phi = 0$,

$$\begin{aligned} \left(\frac{d\mathbf{P}}{dt}\right)_\phi &= -\frac{e}{c}\left(\frac{\partial A_\phi}{\partial t} + \frac{\dot\rho}{\rho}\frac{\partial(\rho A_\phi)}{\partial \rho} + \dot z \frac{\partial A_\phi}{\partial z}\right) = -\frac{e}{c\rho}\left(\frac{\partial(\rho A_\phi)}{\partial t} + \dot\rho \frac{\partial(\rho A_\phi)}{\partial \rho} + \dot z \frac{\partial(\rho A_\phi)}{\partial z}\right) \\ &= -\frac{e}{c\rho}\frac{d(\rho A_\phi)}{dt}, \end{aligned} \tag{40}$$

where $\frac{d}{dt}$ when applied to a field such as the vector potential $\mathbf{A}$ is the convective derivative associated with the moving particle. Noting that $\mathbf{P} = \gamma m(\dot\rho \hat{\boldsymbol\rho} + \rho\dot\phi\hat{\boldsymbol\phi} + \dot z \hat{\mathbf{z}})$ and $\dot{\hat{\boldsymbol\rho}} = \dot\phi \hat{\boldsymbol\phi}$, we also find,

$$\left(\frac{d\mathbf{P}}{dt}\right)_\phi = \frac{dP_\phi}{dt} + \dot\phi P_r = \frac{d(\gamma m \rho \dot\phi)}{dt} + \gamma m \dot\rho \dot\phi = \frac{1}{\rho}\frac{d(\gamma m \rho^2 \dot\phi)}{dt} = \frac{1}{\rho}\frac{d(\rho P_\phi)}{dt} . \tag{41}$$

Combining eqs. (39)-(41), we have,

$$\frac{dL_z}{dt} = \frac{d(\rho P_\phi)}{dt} = -\frac{e}{c}\frac{d(\rho A_\phi)}{dt} . \tag{42}$$

Hence,

$$\frac{d}{dt}\left[\rho\left(P_\phi + \frac{e}{c}A_\phi\right)\right] = \frac{dl_z}{dt} = \frac{dp_\phi}{dt} = 0, \tag{43}$$

as found by the Lagrangian method as well.

A.2 Verification That $l_z = 0$ Throughout the Particle's Motion

We look for possible changes in the canonical angular momentum as the particle traverses the magnetic field.

If suffices to consider the case that the trajectory of the particle is in a plane perpendicular to the magnetic axis, and that the velocity of the particle is small compared to the speed of light.[6]

Inside the uniform solenoidal magnetic field $\mathbf{B} = B_0\hat{\mathbf{z}}$, the trajectory of the particle is a circle whose radius R can be obtained from the Lorentz force law $\mathbf{F} = m\mathbf{a} = e\mathbf{v}/c \times \mathbf{B}$. For motion in a plane perpendicular to the magnetic axis the acceleration is $a = v^2/R$,

$$\frac{mv^2}{R} = e\frac{v}{c}B_0, \tag{44}$$

so that the radius of the circular motion is,

$$R = \frac{cP}{eB_0} . \tag{45}$$

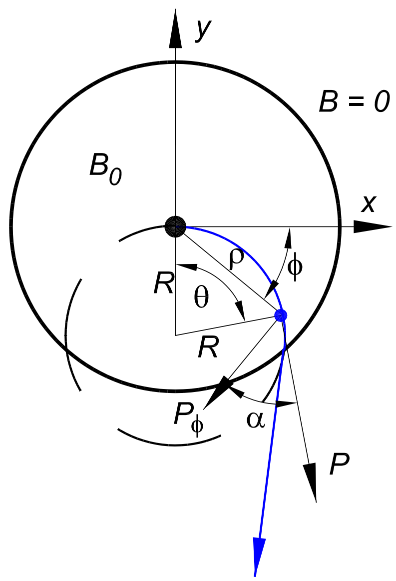
[6] See [20] for a discussion of the general case of motion of relativistic charged particles in a solenoid field.

When the charge particle has reached point (ρ, ϕ) its trajectory is an arc of angle θ on a circle of radius R, as shown in the figure below. The azimuthal component P_ϕ of the particle's mechanical momentum is negative in the righthanded coordinate system, and is given by,

$$P_\phi = -P \cos \alpha, \tag{46}$$

where angles α and ϕ are related by,

$$\alpha = \frac{\pi}{2} - \frac{\phi}{2}\,. \tag{47}$$



Thus,

$$P_\phi = -P \cos\left(\frac{\pi}{2} - \frac{\theta}{2}\right) = -P \sin \frac{\theta}{2}\,. \tag{48}$$

The lengths ρ and R are related by,

$$\rho = 2R \sin \frac{\theta}{2} = \frac{2cP}{eB_0} \sin \frac{\theta}{2}\,, \tag{49}$$

using eq. (45) for R. The vector potential is, $A_\phi = \rho B_0/2$ according to eqs. (7), so,

$$A_\phi = \frac{\rho B_0}{2} = \frac{cP}{e} \sin \frac{\theta}{2}\,, \tag{50}$$

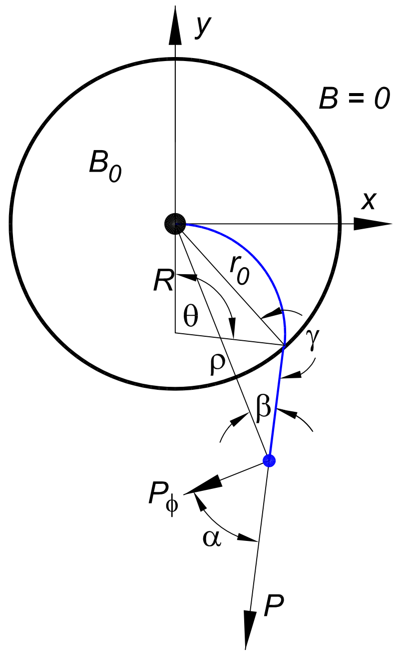
Finally, the z-component of the canonical angular momentum (8) of the particle + field is,

$$l_z = \rho \left(P_\phi + \frac{e}{c} A_\phi\right) = \rho \left(-P \sin \frac{\theta}{2} + \frac{e}{c}\frac{cP}{e} \sin \frac{\theta}{2}\right) = 0 \qquad (\rho < r_0). \tag{51}$$

We learn that the electromagnetic part of the particle's angular momentum exactly cancels its mechanical part, and the total (canonical) angular momentum of the particle in the field is zero at any point inside the magnet.[7]

For completeness, we verify that $l_z = 0$ for the particle's motion outside the magnet. Referring to the figure below, we again write,

$$P_\phi = -P \cos\alpha. \tag{52}$$



We now have,

$$r_0 = 2R \sin\frac{\theta}{2} = \frac{2cP}{eB_0} \sin\frac{\theta}{2}\,. \tag{53}$$

We relate the distances ρ and r_0 via the law of sines,

$$\frac{\rho}{\sin\gamma} = \frac{r_0}{\sin\beta}\,, \tag{54}$$

where,

$$\beta = \frac{\pi}{2} - \alpha, \qquad \gamma = \pi - \frac{\theta}{2}\,. \tag{55}$$

Hence,

$$\rho = \frac{r_0 \sin\frac{\theta}{2}}{\cos\alpha}\,, \tag{56}$$

[7]If the particle is created at a point not on the magnetic axis, its initial canonical angular momentum is nonzero. However, this value does not change during the particle's subsequent motion in the magnetic field, so there is again no transfer of angular momentum between the particle and the solenoid during the particle's motion. There is, of course, a tiny "kick" given to the solenoid at the moment of the particle's creation if this occurs away from the magnetic axis. In the case of radioactive decay, the direction of this initial "kick" is random, and the solenoid would not turn preferentially in one direction as is desirable for a motor.

For $\rho > r_0$, the vector potential is, using eqs. (7), (53) and (56),

$$A_\phi = \frac{r_0^2 B_0}{2\rho} = \frac{r_0 B_0 \cos\alpha}{2\sin\frac{\theta}{2}} = \frac{cP}{e}\cos\alpha. \tag{57}$$

Again, we find that the z-component of the canonical angular momentum of the particle + field vanishes,

$$l_z = \rho\left(P_\phi + \frac{e}{c}A_\phi\right) = \rho\left(-P\cos\alpha + \frac{e}{c}\frac{cP}{e}\cos\alpha\right) = 0 \qquad (\rho > r_0). \tag{58}$$

A.3 Equivalence of the Maxwell and Poynting Forms of Electromagnetic (Angular) Momentum for Static Fields

This Appendix is based on an argument by Vladimir Hnizdo. See also [21].

Static electromagnetic fields $\mathbf{E}$ and $\mathbf{B}$ can be characterized by the time-independent Maxwell equations,

$$\boldsymbol{\nabla}\cdot\mathbf{E} = 4\pi\varrho, \qquad \boldsymbol{\nabla}\times\mathbf{E} = 0, \tag{59}$$

where ϱ is the electric charge density, and,

$$\boldsymbol{\nabla}\cdot\mathbf{B} = 0, \qquad \boldsymbol{\nabla}\times\mathbf{B} = \frac{4\pi}{c}\mathbf{J}, \tag{60}$$

which implies that the time-independent current density $\mathbf{J}$ satisfies $\boldsymbol{\nabla}\cdot\mathbf{J} = 0$.

For present purposes we can avoid use of the current density $\mathbf{J}$ and instead consider the vector potential $\mathbf{A}$, which has zero divergence in the Coulomb gauge (and also in the Lorentz gauge for static problems),

$$\mathbf{B} = \boldsymbol{\nabla}\times\mathbf{A}, \qquad \boldsymbol{\nabla}\cdot\mathbf{A} = 0. \tag{61}$$

To confirm that the electromagnetic momentum,

$$\mathbf{P}_{\mathrm{EM_M}} = \int \frac{\varrho\mathbf{A}}{c}\, d\mathrm{Vol} \tag{62}$$

is equal to,

$$\mathbf{P}_{\mathrm{EM_P}} = \int \frac{\mathbf{E}\times\mathbf{B}}{4\pi c}\, d\mathrm{Vol}, \tag{14}$$

and that the electromagnetic angular momentum,

$$\mathbf{L}_{\mathrm{EM_M}} = \int \mathbf{r}\times\mathbf{P}_{\mathrm{EM_M}}\, d\mathrm{Vol} = \int \mathbf{r}\times\frac{e\mathbf{A}}{c}\, d\mathrm{Vol} \tag{63}$$

is equal to,

$$\mathbf{L}_{\mathrm{EM_P}} = \int \frac{\mathbf{r}\times(\mathbf{E}\times\mathbf{B})}{4\pi c}\, d\mathrm{Vol}, \tag{15}$$

we show that $\mathbf{E}\times\mathbf{B}$ is equal to $\varrho\mathbf{A}$ plus the divergence of a vector field, and that $\mathbf{r}\times(\mathbf{E}\times\mathbf{B})$ is equal to $\mathbf{r}\times\varrho\mathbf{A}$ plus the divergence of another vector field. Then, we transform the volume integrals of the auxiliary vector field into surface integral using Gauss' law, and if the auxiliary field fall off sufficiently quickly with distance the equivalence of the various forms of electromagnetic momenta is established.

Also, we are cautioned that the equivalence may not hold in idealizations such as an infinite solenoid, in which the field does not fall off rapidly in all directions.

In addition to well-known vector calculus relations, it is useful to define a combined operation,

$$\boldsymbol{\nabla}\cdot\mathbf{ab} \equiv (\boldsymbol{\nabla}\cdot\mathbf{a})\mathbf{b} + (\mathbf{a}\cdot\boldsymbol{\nabla})\mathbf{b} = (\boldsymbol{\nabla}\cdot b_x\mathbf{a})\,\hat{\mathbf{x}} + (\boldsymbol{\nabla}\cdot b_y\mathbf{a})\,\hat{\mathbf{y}} + (\boldsymbol{\nabla}\cdot b_z\mathbf{a})\,\hat{\mathbf{z}}. \tag{64}$$

Then,

$$\begin{aligned}
\mathbf{E}\times\mathbf{B} &= \mathbf{E}\times(\boldsymbol{\nabla}\times\mathbf{A}) = \boldsymbol{\nabla}(\mathbf{A}\cdot\mathbf{E}) - (\mathbf{A}\cdot\boldsymbol{\nabla})\mathbf{E} - (\mathbf{E}\cdot\boldsymbol{\nabla})\mathbf{A} - \mathbf{A}\times(\boldsymbol{\nabla}\times\mathbf{E}) \\
&= (\boldsymbol{\nabla}\cdot\mathbf{E})\mathbf{A} + \boldsymbol{\nabla}(\mathbf{A}\cdot\mathbf{E}) - [(\boldsymbol{\nabla}\cdot\mathbf{E})\mathbf{A} + (\mathbf{A}\cdot\boldsymbol{\nabla})\mathbf{E}] - [(\boldsymbol{\nabla}\cdot\mathbf{A})\mathbf{E} + (\mathbf{E}\cdot\boldsymbol{\nabla})\mathbf{A}] \\
&= 4\pi\varrho\mathbf{A} + \boldsymbol{\nabla}(\mathbf{A}\cdot\mathbf{E}) - \boldsymbol{\nabla}\cdot\mathbf{EA} - \boldsymbol{\nabla}\cdot\mathbf{AE},
\end{aligned} \tag{65}$$

so that,

$$\begin{aligned}
\mathbf{P}_{\mathrm{EM_P}} &= \int \frac{\mathbf{E}\times\mathbf{B}}{4\pi c}\, d\mathrm{Vol} \\
&= \int \frac{\varrho\mathbf{A}}{c}\, d\mathrm{Vol} + \oint (\mathbf{A}\cdot\mathbf{E})\; d\mathbf{Area} - \oint \mathbf{E}(\mathbf{A}\cdot d\mathbf{Area}) - \oint \mathbf{A}(\mathbf{E}\cdot d\mathbf{Area}) \\
&= \int \frac{\varrho\mathbf{A}}{c}\, d\mathrm{Vol} = \mathbf{P}_{\mathrm{EM_M}}.
\end{aligned} \tag{66}$$

The surface integrals in eq. (66) are negligible when the charges and currents that create the electric field $\mathbf{E}$ and the vector potential $\mathbf{A}$ lie within a finite volume that is small compared to the volume of integration, and when radiation can be neglected.

We now evaluate eq. (15) by taking the cross product of eq. (65) with $\mathbf{r}$ and further transforming the various terms. Thus,

$$\mathbf{r}\times\boldsymbol{\nabla}(\mathbf{A}\cdot\mathbf{E}) = -\boldsymbol{\nabla}\times(\mathbf{A}\cdot\mathbf{E})\,\mathbf{r} + \boldsymbol{\nabla}(\mathbf{A}\cdot\mathbf{E})\boldsymbol{\nabla}\times\mathbf{r} = -\boldsymbol{\nabla}\times(\mathbf{A}\cdot\mathbf{E})\,\mathbf{r}. \tag{67}$$

Now,

$$\begin{aligned}
[\mathbf{r}\times\boldsymbol{\nabla}\cdot\mathbf{EA}]_x &= y\boldsymbol{\nabla}\cdot(A_z\mathbf{E}) - z\boldsymbol{\nabla}\cdot(A_y\mathbf{E}) \\
&= \boldsymbol{\nabla}\cdot(yA_z\mathbf{E}) - A_z\mathbf{E}\cdot\boldsymbol{\nabla}y - \boldsymbol{\nabla}\cdot(zA_y\mathbf{E}) + A_y\mathbf{E}\cdot\boldsymbol{\nabla}z \\
&= A_yE_z - A_zE_y + \boldsymbol{\nabla}\cdot(yA_z\mathbf{E}) - \boldsymbol{\nabla}\cdot(zA_y\mathbf{E}) \\
&= [\mathbf{A}\times\mathbf{E}]_x + \boldsymbol{\nabla}\cdot([\mathbf{r}\times\mathbf{A}]_x\mathbf{E}),
\end{aligned} \tag{68}$$

so,

$$[\mathbf{r}\times\boldsymbol{\nabla}\cdot\mathbf{AE}]_x = [\mathbf{E}\times\mathbf{A}]_x + \boldsymbol{\nabla}\cdot([\mathbf{r}\times\mathbf{E}]_x\mathbf{A}), \tag{69}$$

and,

$$[\mathbf{r}\times\boldsymbol{\nabla}\cdot\mathbf{EA}]_x + [\mathbf{r}\times\boldsymbol{\nabla}\cdot\mathbf{AE}]_x = \boldsymbol{\nabla}\cdot([\mathbf{r}\times\mathbf{A}]_x\mathbf{E}) + \boldsymbol{\nabla}\cdot([\mathbf{r}\times\mathbf{E}]_x\mathbf{A}). \tag{70}$$

Hence,

$$\begin{aligned}
\mathbf{L}_{\mathrm{EM_P}} &= \int \frac{\mathbf{r}\times(\mathbf{E}\times\mathbf{B})}{4\pi c}\, d\mathrm{Vol} \\
&= \int \mathbf{r}\times\frac{\varrho\mathbf{A}}{c}\, d\mathrm{Vol} - \oint (\mathbf{A}\cdot\mathbf{E})\, d\mathbf{Area}\times\mathbf{r} - \oint \mathbf{r}\times\mathbf{E}(\mathbf{A}\cdot d\mathbf{Area}) - \oint \mathbf{r}\times\mathbf{A}(\mathbf{E}\cdot d\mathbf{Area}) \\
&= \int \mathbf{r}\times\frac{\varrho\mathbf{A}}{c}\, d\mathrm{Vol} = \mathbf{L}_{\mathrm{EM_M}}. \qquad (71)
\end{aligned}$$

The arguments of the surface integrals in eq. (71) vanish less quickly with distance than those in eq. (66), so in some cases (such as McKenna's example) with sources of infinite extent we may find that $\mathbf{P}_{\mathrm{EM_P}} = \mathbf{P}_{\mathrm{EM_M}}$ but $\mathbf{L}_{\mathrm{EM_P}} \neq \mathbf{L}_{\mathrm{EM_M}}$.

The equivalence of $\mathbf{P}_{\mathrm{EM_P}}$ *and* $\mathbf{P}_{\mathrm{EM_M}}$ *extends to nonstatic systems in which the currents do not satisfy* $\boldsymbol{\nabla}\cdot\mathbf{J} = 0$, *so long as the velocities of all charges are low and radiation can be neglected [22, 23].*

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