Quantum Measurement Requires Entanglement; Measurement Takes Time

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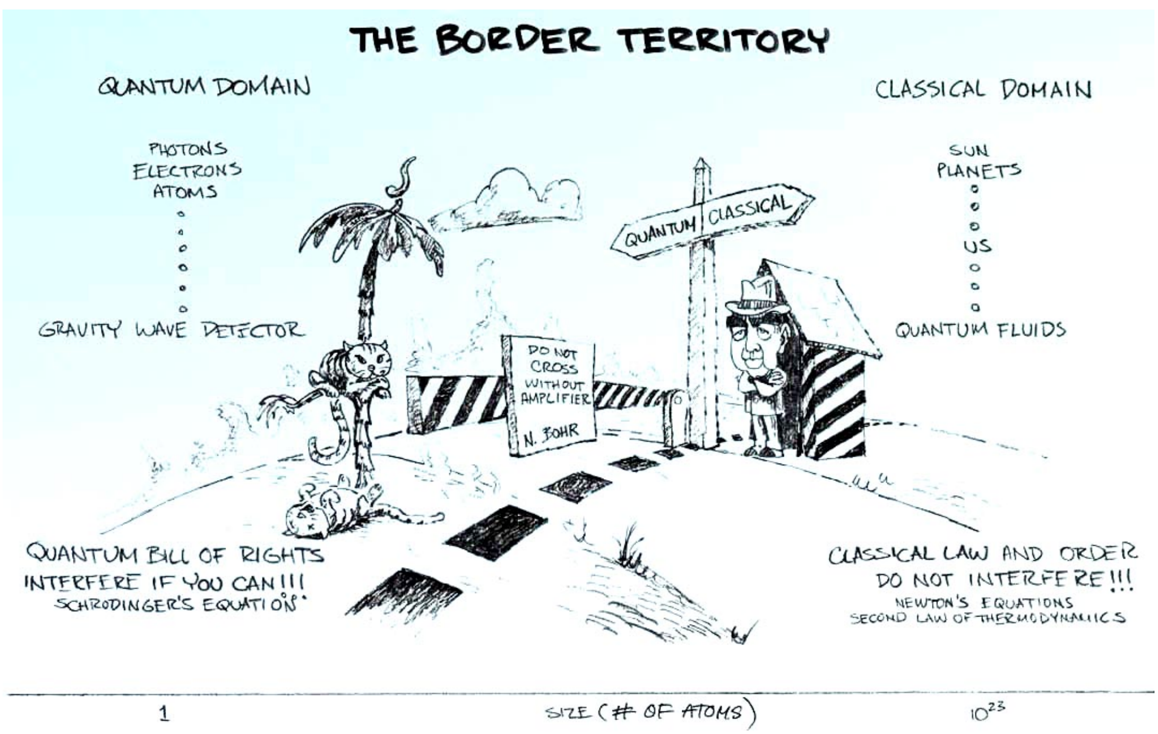
We follow an argument due to von Neumann[1,2] to get a sense of how a particular quantum system, say, one or more Qbits, might interact with a larger system to implement a measurement described by hermitian operator M that acts on the particular system.

We suppose that there exists an intermediary object, which we will call the pointer that can interact with the particular quantum system, and which is also very heavy so that the position of the pointer is "well defined." By the latter, we mean that the position of the pointer can be determined to sufficient accuracy, as defined below, by apparatus whose behavior is "classical" enough that we can leave the apparatus out of the quantum part of the analysis.

The goal is to establish a quantum correlation between the measurable property of the particular quantum state and the position of the pointer, and then to use a "classical" measurement of the position of the pointer to infer the result of the quantum correlation/measurement. Thus, the argument of von Neumann straddles the "quantum border" shown below.



[1]This note is transcribed from Problem 5 of my course on the Physics of Quantum Computation [1].

[2]The argument we give is based on the last few pages of *Mathematical Foundations of Quantum Mechanics*, J. von Neumann [2]; the German original was written in 1932, three years before Schrödinger coined the term "entanglement." See also sec. 3.1.1 of Preskill's Lectures [3].

To describe von Neumann's argument we need to know something about the time evolution of a quantum system. Since the total probability of the quantum system to be in some state remains constant over time, the time evolution of a quantum state $|\Psi(t)\rangle$ is described by a unitary operator,

$$|\Psi(t')\rangle = \mathsf{U}(t,t')|\Psi(t)\rangle. \tag{1}$$

Over a short time interval, $t' - t = \delta t$, the unitary operator U cannot differ much from the identity operator,

$$\mathsf{U}(t, t+\delta t) \approx \mathbf{I} + \mathsf{u}(t)\delta t. \tag{2}$$

That is,

$$|\Psi(t+\delta t)\rangle = \mathsf{U}(t,t+\delta t)|\Psi(t)\rangle \approx |\Psi(t)\rangle + \mathsf{u}(t)|\Psi(t)\rangle\delta t, \tag{3}$$

which implies that,

$$\frac{\partial|\Psi\rangle}{\partial t} = \mathsf{u}|\Psi\rangle. \tag{4}$$

The famous insight of Schrödinger is that if we write,

$$\mathsf{u} = -\frac{i}{\hbar}\mathcal{H} = -i\mathsf{h}, \tag{5}$$

then the operator $\mathcal{H} = \hbar\mathsf{h}$ is not the Hadamard transformation but is related to the Hamiltonian of the system in a well-defined manner. Thus, eq. (4) becomes Schrödinger's equation,

$$i\frac{d|\Psi\rangle}{dt} = \mathsf{h}|\Psi\rangle. \tag{6}$$

Since the operator $\mathsf{U} \approx \mathbf{I} - i\mathsf{h}\,\delta t$ *is unitary,* $\mathsf{U}^{-1} = \mathsf{U}^\dagger \approx \mathbf{I} + i\mathsf{h}^\dagger\delta t$*. Then,*

$$1 = \mathsf{U}^{-1}\mathsf{U} \approx (\mathbf{I} + i\mathsf{h}^\dagger\delta t)(\mathbf{I} - i\mathsf{h}\delta t) \approx \mathbf{I} + i(\mathsf{h}^\dagger - \mathsf{h})\delta t, \tag{7}$$

so that we must have $\mathsf{h}^\dagger = \mathsf{h}$*, i.e., the Hamiltonian operator is hermitian.*

Returning to the case of a particular quantum system plus the pointer, we take the Hamiltonian of the combined system to be of the form,

$$\mathsf{h} = \mathsf{h}_0 + \frac{\mathsf{p}^2}{2m} + \lambda\mathsf{M}\mathsf{p} \approx \lambda\mathsf{M}\mathsf{p}, \tag{8}$$

where h_0 is the Hamiltonian of the particular system when in isolation, $\mathsf{p} = -i\partial/\partial x$ is the momentum operator of the pointer (which can move only in the x direction), m is the (large) mass of the pointer, λ is a coupling constant, and M is the measurement operator that applies to the particular quantum system. The approximate form of the Hamiltonian follows on noting that mass m is large, and that during the measurement the effect of the interaction term $\lambda\mathsf{M}\mathsf{p}$ is much larger than that of isolated Hamiltonian h_0 (otherwise the measurement could not produce a crisp result[3]

The state of the particular system to be measured is,

$$|\psi\rangle = \sum_j \psi_j|j\rangle, \tag{9}$$

[3] See, for example, [4].

and the initial state of the pointer is $|\phi(x)\rangle$, which is a Gaussian wave packet centered on, say, $x = 0$, normalized such that $\int |\phi(x)|^2\, dx = 1$. Since the pointer particle is heavy, its wave packet $|\phi(x)\rangle$ is narrow (but not so narrow that the wave packet spreads significantly during the measurement). The initial state of the combined system is the direct product,

$$|\Psi(0)\rangle = |\psi\rangle \otimes |\phi(x)\rangle = \left(\sum_j \psi_j |j\rangle\right) \otimes |\phi(x)\rangle. \tag{10}$$

The basis $[|j\rangle]$ for the particular system has been chosen so that the each basis state $|j\rangle$ has a well-defined value m_j of the measurement. That is, the measurement operator has the projective form,

$$\mathsf{M} = \sum_j m_j \cdot |j\rangle\langle j|. \tag{11}$$

The Hamiltonian $\mathsf{h} \approx \lambda \mathsf{M}\mathsf{p}$ is time independent, so Schrödinger's equation (6) has the formal solution,

$$|\Psi(t)\rangle = e^{-i\mathsf{h}t}|\Psi(0)\rangle. \tag{12}$$

Now,

$$\begin{aligned} e^{-i\mathsf{h}t} = \sum_{n=0}^{\infty} \frac{(-i\mathsf{h}t)^n}{n!} &= \sum_{n=0}^{\infty} \frac{1}{n!}\left[-i\lambda \sum_j m_j \cdot |j\rangle\langle j| \left(-i\frac{\partial}{\partial x}\right) t\right]^n \\ &= \sum_j |j\rangle\langle j| \sum_{n=0}^{\infty} \frac{1}{n!}\left(-\lambda m_j t \frac{\partial}{\partial x}\right)^n, \end{aligned} \tag{13}$$

recalling that $\langle j|k\rangle = \delta_{jk}$, so that the lengthy products of bras and kets all collapse back down to the projections $|j\rangle\langle j|$. Inserting eqs. (10) and (13) into (12), we obtain,

$$\begin{aligned} |\Psi(t)\rangle &= \sum_j |j\rangle\langle j| \sum_k \psi_k |k\rangle \otimes \sum_{n=0}^{\infty} \frac{1}{n!}\left(-\lambda m_j t \frac{\partial}{\partial x}\right)^n |\phi(x)\rangle \\ &= \sum_j \psi_j |j\rangle \otimes |\phi(x - \lambda m_j t)\rangle, \end{aligned} \tag{14}$$

noting that the Taylor expansion of $\phi(x - x_0)$ is,

$$\phi(x - x_0) = \sum_{n=0}^{\infty} \frac{1}{n!}\left(-x_0 \frac{\partial}{\partial x}\right)^n \phi(x). \tag{15}$$

The initial direct product state (10) has evolved into the entangled state (14) during the course of the measurement.

The supposition is that the position of the pointer at time t of the observation can be determined well enough to distinguish the j locations $\lambda m_j t$ from one another. This is more plausible for larger t: (accurate) measurements take time! If the pointer is found at (or near) position $\lambda m_j t$, the particular system $|\psi\rangle$ must be in state $|j\rangle$ and the value of the measurement is m_j. The probability that this is the outcome of the measurement is, of course, $\left|\psi_j\right|^2$ since $\int |\phi(x - \lambda m_j t)|\, dx = 1$.

This argument does a nice job of explaining how to entangle the state $|\psi\rangle = \sum \psi_j |j\rangle$ with a pointer such that different positions of the pointer are correlated with different basis states $|j\rangle$. However, it does not explain how the observation of the position of the pointer to be, say, $\lambda m_j t$ "collapses the wave function" of $|\psi\rangle$ to the basis state $|j\rangle$.[4]

Von Neumann's argument indicates that underlying every measurement process is the entanglement that bothered Einstein, Podolsky and Rosen (and Schrödinger, *etc.*) so much. This deserves further discussion, some of which is given in Prob. 20 of [1].

References

[1] K.T. McDonald, *Ph410: Physics of Quantum Computation* (2004),
http://kirkmcd.princeton.edu/examples/ph410problems.pdf

[2] J. von Neumann, *Mathematische Grundlagen der Qauntenmechanik* (Springer, 1932),
http://kirkmcd.princeton.edu/examples/QM/vonneumann_grundlagen_32.pdf
Mathematical Foundations of Quantum Mechanics (Princeton U. Press, 1955),
http://kirkmcd.princeton.edu/examples/QM/vonneumann_55.pdf

[3] J. Preskill, *Physics 219: Quantum Computation. Chapter 3. Foundations II: Measurement and Evolution*, (Oct. 2018), http://theory.caltech.edu/~preskill/ph219/chap3_15.pdf

[4] A. Peres and W.K. Wootters, *Quantum measurements of finite duration*, Phys. Rev. D **32**, 1968 (1985), http://kirkmcd.princeton.edu/examples/QM/peres_prd_32_1968_85.pdf

[4]The transformation from $|\Psi(0)\rangle$ to $|\Psi(t)\rangle$ is unitary/reversible as eq. (14) is valid for both increasing or decreasing t. The irreversible step in the measurement process is the "classical" reading of the position of the pointer at time $t_{\rm meas}$, which selects a value of $x \approx \lambda m_j t_{\rm meas}$ and leaves the system in the state,

$$|\Psi(t > t_{\rm meas})\rangle = \frac{|j\rangle \otimes |\phi(x - \lambda m_j t_{\rm meas})\rangle}{\sqrt{N}}\,, \qquad (16)$$

where N is the number of possible positions of the pointer.